5. cvičení – Limita posloupnosti

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Příklady

1. Určete limity

   (a) $\lim\limits_{n\to\infty} n = \infty$

   (b) $\lim\limits_{n\to\infty} n^2 = \infty$

   (c) $\lim\limits_{n\to\infty} \sqrt{n} = \infty$

   (d) $\lim\limits_{n\to\infty} \frac{1}{n} = 0$

   (e) $\lim\limits_{n\to\infty} \frac{1}{n^2} = 0$

   (f) $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

   (g) $\lim\limits_{n\to\infty} e^n = \infty$

   (h) $\lim\limits_{n\to\infty} e^{-n} = 0$

   (i) $\lim\limits_{n\to\infty} \ln n = \infty$

   (j) $\lim\limits_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$

   (k) $\lim\limits_{n\to\infty} 2^n = \infty$

   (l) $\lim\limits_{n\to\infty} n! = \infty$

2. Určete limity

   (a) $\lim\limits_{n\to\infty} (-1)^n \nexists$

   (b) $\lim\limits_{n\to\infty} (-1)^n n \nexists$

   (c) $\lim\limits_{n\to\infty} (-1)^n \frac{1}{n} = 0$

   (d) $\lim\limits_{n\to\infty} \cos(\pi n)\sqrt{n} = \lim\limits_{n\to\infty} (-1)^n \sqrt{n} \nexists$

3. Spočtěte limity

   (a)

$$\lim_{n\to\infty} \frac{20}{\sqrt{n}} \overset{VOAL}{=} \lim_{n\to\infty} 20 \cdot \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 20 \cdot 0 = 0$$

   (b) Podle aritmetiky limit a věty o odmocnině:

$$\lim_{n\to\infty} \frac{n^2}{\sqrt{n^3+1}} = \lim_{n\to\infty} \frac{n^{3/2}}{n^{3/2}} \cdot \frac{\sqrt{n}}{\sqrt{1+1/n^3}} \overset{VOAL}{=} \frac{\lim_{n\to\infty} \sqrt{n}}{\lim_{n\to\infty} \sqrt{1+1/n^3}} = \frac{\infty}{1+0} = \infty.$$

   (c)

$$\lim_{n\to\infty} -n^8 + 2n^3 - 4 = \lim_{n\to\infty} n^8(-1 + \frac{2}{n^5} - \frac{4}{n^8}) \overset{VOAL}{=\!=} \lim_{n\to\infty} n^8 \lim_{n\to\infty} (-1 + \frac{2}{n^5} - \frac{4}{n^8})$$
$$\overset{VOAL}{=} \lim_{n\to\infty} n^8 (\lim_{n\to\infty} -1 + \lim_{n\to\infty} \frac{2}{n^5} - \lim_{n\to\infty} \frac{4}{n^8}) = \infty(-1+0-0) = -\infty.$$

   (d)

$$\lim_{n\to\infty} \frac{2n^5+2n-7}{n^5-6n^2+4}$$

   Řešení: Užijeme opakovaně větu o aritmetice limit

$$\lim_{n\to\infty} \frac{2n^5+2n-7}{n^5-6n^2+4} = \lim_{n\to\infty} \frac{n^5(2+\frac{2}{n^4}-\frac{7}{n^5})}{n^5(1-\frac{6}{n^3}+\frac{4}{n^5})} = \frac{\lim_{n\to\infty}(2+\frac{2}{n^4}-\frac{7}{n^5})}{\lim_{n\to\infty}(1-\frac{6}{n^3}+\frac{4}{n^5})} =$$
$$= \frac{\lim_{n\to\infty} 2 + \lim_{n\to\infty} \frac{2}{n^4} - \lim_{n\to\infty} \frac{7}{n^5}}{\lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{6}{n^3} + \lim_{n\to\infty} \frac{4}{n^5}} = \frac{2+0-0}{1-0+0} = 2$$

(e)

$$\lim_{n\to\infty} \frac{5n^2+n-5}{n^3+8}$$

Řešení:

$$\lim_{n\to\infty} \frac{5n^2+n-5}{n^3+8} = \lim_{n\to\infty} \frac{n^3\left(\frac{5}{n}+\frac{1}{n}-\frac{5}{n^3}\right)}{n^3\left(1+\frac{8}{n^3}\right)} \overset{VOAL}{=} \frac{0+0-0}{1+0} = 0$$

(f)

$$\lim_{n\to\infty} \frac{\sqrt[3]{n^2}}{n+1} = \lim_{n\to\infty} \frac{n}{n}\cdot\frac{\frac{1}{n^{1/3}}}{1+\frac{1}{n}} \overset{VOAL}{=} \frac{\lim_{n\to\infty}\frac{1}{n^{1/3}}}{\lim_{n\to\infty}1+\lim_{n\to\infty}\frac{1}{n}} = \frac{0}{1+0} = 0$$

(g)

$$\lim_{n\to\infty} \frac{\sin(n)+\cos(n^2)}{n^2-3}$$

Řešení: Máme $|\sin n + \cos n^2| \leq 1+1 = 2$, tedy jde o omezenou posloupnost. Dále

$$\lim_{n\to\infty} \frac{1}{n^2-3} = 0.$$

Tedy dle věty o součinu omezené a mizející je výsledná limita rovna 0.

4. Spočtěte limitu

(a) $\lim\limits_{n\to\infty} \sqrt{n+2}+\sqrt{n}$

Řešení: Z aritmetiky limit

$$\lim_{n\to\infty} \sqrt{n+2}+\sqrt{n} = \infty+\infty = \infty$$

(b) $\lim\limits_{n\to\infty} \sqrt{n+2}-\sqrt{n}$

Řešení:

$$\lim_{n\to\infty} \sqrt{n+2}-\sqrt{n} = \lim_{n\to\infty} (\sqrt{n+2}-\sqrt{n})\frac{\sqrt{n+2}+\sqrt{n}}{\sqrt{n+2}+\sqrt{n}} = \lim_{n\to\infty} \frac{(\sqrt{n+2})^2-(\sqrt{n})^2}{\sqrt{n+2}+\sqrt{n}} =$$
$$\lim_{n\to\infty} \frac{n+2-n}{\sqrt{n+2}+\sqrt{n}} = \lim_{n\to\infty} \frac{2}{\sqrt{n+2}+\sqrt{n}} \overset{VOAL}{=} \frac{2}{\infty} = 0$$

(c) $\lim\limits_{n\to\infty} \frac{\sqrt{n^2+1}}{n}$

Řešení:

$$\lim_{n\to\infty} \frac{\sqrt{n^2+1}}{n} = \lim_{n\to\infty} \frac{n}{n}\frac{\sqrt{1+\frac{1}{n^2}}}{1} = \sqrt{1+0} = 1$$

(d) $\lim\limits_{n\to\infty} \frac{\sqrt{n-1}-\sqrt{n}}{\sqrt{n^2-3}-\sqrt{(n+2)^2}}$

Řešení:

$$\lim_{n\to\infty} \frac{\sqrt{n-1}-\sqrt{n}}{\sqrt{n^2-3}-\sqrt{(n+2)^2}} \cdot \frac{\sqrt{n-1}+\sqrt{n}}{\sqrt{n-1}+\sqrt{n}} \cdot \frac{\sqrt{n^2-3}+\sqrt{(n+2)^2}}{\sqrt{n^2-3}+\sqrt{(n+2)^2}}$$
$$= \lim_{n\to\infty} \frac{n-1-n}{n^2-3-(n+2)^2} \cdot \frac{\sqrt{n^2-3}+\sqrt{(n+2)^2}}{\sqrt{n-1}+\sqrt{n}}$$
$$= \lim_{n\to\infty} \frac{-1}{-4n-7} \cdot \frac{\sqrt{n^2-3}+\sqrt{n^2+4n+4}}{\sqrt{n-1}+\sqrt{n}}$$
$$= \lim_{n\to\infty} \frac{n}{n} \cdot \frac{-1}{-4-\frac{7}{n}} \cdot \frac{1}{\sqrt{n}} \cdot \frac{\sqrt{1-3/n^2}+\sqrt{1+4/n+4/n^2}}{\sqrt{1-1/n}+1}$$
$$\overset{VOAL}{=} \frac{-1}{-4-0} \cdot \frac{1}{\infty} \cdot \frac{\sqrt{1-0}+\sqrt{1+0+0}}{\sqrt{1-0}+\sqrt{1}} = 0$$

(e) $\lim\limits_{n\to\infty} \dfrac{\sqrt{n^2+1}-n}{n}$

Řešení:

$$\lim_{n\to\infty} \frac{\sqrt{n^2+1}-n}{n} = \lim_{n\to\infty} \frac{\sqrt{n^2+1}-n}{n} \frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n} = \lim_{n\to\infty} \frac{1}{n(\sqrt{n^2+1}+n)}$$
$$\overset{VOAL}{=} \frac{1}{\infty(\infty+\infty)} = 0$$

(f) $\lim\limits_{n\to\infty} \dfrac{\sqrt{n^4+3n-1}-n^2}{\sqrt[3]{n^3+1}-\sqrt{n^2-1}}$

Řešení: Jmenovatele rozšíříme podle vzorce A^6-B^6, čitatele podle A^2-B^2.

$$\lim_{n\to\infty} \frac{\sqrt{n^4+3n-1}-n^2}{\sqrt[3]{n^3+1}-\sqrt{n^2-1}}$$
$$= \lim_{n\to\infty} \frac{\sqrt{n^4+3n-1}-n^2}{\sqrt[3]{n^3+1}-\sqrt{n^2-1}} \cdot \frac{\sqrt{n^4+3n-1}+n^2}{\sqrt{n^4+3n-1}+n^2} \cdot$$
$$\cdot \frac{\sqrt[3]{(n^3+1)^5}+\sqrt[3]{(n^3+1)^4}\sqrt{n^2-1}+\cdots+\sqrt{(n^2-1)^5}}{\sqrt[3]{(n^3+1)^5}+\sqrt[3]{(n^3+1)^4}\sqrt{n^2-1}+\cdots+\sqrt{(n^2-1)^5}}$$
$$= \lim_{n\to\infty} \frac{n^4+3n-1-n^4}{(n^3+1)^2-(n^2-1)^3} \cdot \frac{\sqrt[3]{(n^3+1)^5}+\sqrt[3]{(n^3+1)^4}\sqrt{n^2-1}+\cdots+\sqrt{(n^2-1)^5}}{\sqrt{n^4+3n-1}+n^2}$$
$$= \lim_{n\to\infty} \frac{3n-1}{3n^4+2n^3-3n^2+2} \cdot \frac{\sqrt[3]{(n^3+1)^5}+\sqrt[3]{(n^3+1)^4}\sqrt{n^2-1}+\cdots+\sqrt{(n^2-1)^5}}{\sqrt{n^4+3n-1}+n^2}$$
$$= \lim_{n\to\infty} \frac{n\cdot n^5}{n^4\cdot n^2} \cdot \frac{3-\frac{1}{n}}{3+\frac{2}{n}-\frac{3}{n^2}+\frac{2}{n^4}} \cdot \frac{\sqrt[3]{(1+\frac{1}{n^3})^5}+\sqrt[3]{(1+\frac{1}{n^3})^4}\sqrt{1-\frac{1}{n^2}}+\cdots+\sqrt{(1-\frac{1}{n^2})^5}}{\sqrt{1+\frac{3}{n^3}-\frac{1}{n^4}}+1}$$
$$= \frac{3}{3} \cdot \frac{6}{2} = 3$$

5. Spočtěte limity

(a) $\lim_{n\to\infty} \left\{ \frac{1+2+\ldots+n}{n+2} - \frac{n}{2} \right\}$

Řešení: použitím vztahu $1+2+\cdots+n = n(n+1)/2$ máme

$$\lim_{n\to\infty} \left\{ \frac{1+2+\ldots+n}{n+2} - \frac{n}{2} \right\} = \lim_{n\to\infty} \left\{ \frac{n(n+1)}{n+2} - \frac{n}{2} \right\} =$$
$$= \lim_{n\to\infty} \left\{ \frac{n(n+1)-n(n+2)}{2(n+2)} \right\} = \lim_{n\to\infty} \left\{ \frac{-n}{2(n+2)} \right\} = -\frac{1}{2}$$

(b) $\lim_{n\to\infty} \left\{ \frac{1^2+2^2+\ldots+n^2}{n^3} \right\}$

Řešení: použitím vztahu $1^2+2^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$ máme

$$\lim_{n\to\infty} \left\{ \frac{1^2+2^2+\ldots+n^2}{n^3} \right\} = \lim_{n\to\infty} \left\{ \frac{n(n+1)(2n+1)}{6n^3} \right\} = \lim_{n\to\infty} \frac{(1+1/n)(2+1/n)}{6}$$
$$= \frac{(1+0)(2+0)}{6} = \frac{1}{3}.$$

(c)

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k(k+1)}$$

Řešení: Použijeme trik:

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

Nyní aplikováno na sumu získáme:

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Tedy

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k(k+1)} = \lim_{n\to\infty} 1 - \frac{1}{n} \overset{VOAL}{=} 1.$$

Bonus

6. Spočtěte limity

(a) $\lim_{n\to\infty} \frac{(n+4)^{100}-(n+3)^{100}}{(n+2)^{100}-n^{100}}$

Řešení: Roznásobíme závorky podle binomické věty:

$$\lim_{n\to\infty} \frac{(n^{100}+100\cdot 4n^{99}+\binom{100}{2}n^{98}4^2+\cdots+4^{100})-(n^{100}+100\cdot 3n^{99}+\cdots+3^{100})}{(n^{100}+100\cdot 2n^{99}+\cdots+2^{100})-n^{100}}$$
$$= \lim_{n\to\infty} \frac{n^{99}(100\cdot 4+\binom{100}{2}16\frac{1}{n}+\cdots+\frac{4^{100}}{n^{99}})}{n^{99}(200+\cdots+\frac{2^{100}}{n^{99}})} \overset{VOAL}{=} \frac{1}{2}$$

(b) $\lim_{n\to\infty} \frac{1+a+\dots+a^n}{1+b+\dots+b^n}$, kde$|a|, |b| < 1$

Řešení: Použijeme součet geometrické řady a větu o aritmetice limit: $\sum_{k=0}^n a^k = \frac{a^{n+1}-1}{a-1}$.

$$\lim_{n\to\infty} \frac{1+a+\dots+a^n}{1+b+\dots+b^n} = \lim_{n\to\infty} \frac{\frac{a^{n+1}-1}{a-1}}{\frac{b^{n+1}-1}{b-1}} = \lim_{n\to\infty} \frac{b-1}{a-1} \cdot \frac{a^{n+1}-1}{b^{n+1}-1}$$

$$\stackrel{VOAL}{=} \frac{b-1}{a-1} \cdot \frac{0-1}{0-1} = \frac{b-1}{a-1}.$$

(c)

$$\lim_{n\to\infty} \sqrt[3]{n+1} - \sqrt[3]{n}$$

Řešení: Rozšíříme dle vzorečků:

$$\lim_{n\to\infty} \sqrt[3]{n+1} - \sqrt[3]{n} = \lim_{n\to\infty} (\sqrt[3]{n+1} - \sqrt[3]{n}) \cdot \frac{(n+1)^{2/3} + \sqrt[3]{n+1}\sqrt[3]{n} + n^{2/3}}{(n+1)^{2/3} + \sqrt[3]{n+1}\sqrt[3]{n} + n^{2/3}} =$$

$$= \lim_{n\to\infty} \frac{n+1-n}{(n+1)^{2/3} + \sqrt[3]{n+1}\sqrt[3]{n} + n^{2/3}} = \lim_{n\to\infty} \frac{n^{2/3}}{n^{2/3}} \frac{\frac{1}{n^{2/3}}}{\sqrt[3]{1+\frac{2}{n}+\frac{1}{n^2}} + \sqrt[3]{1+\frac{1}{n}} + 1} =$$

$$\stackrel{AL}{=} \frac{0}{\sqrt[3]{1+0+0} + \sqrt[3]{1+0} + 1} = 0$$

(d) $\lim_{n\to\infty} \frac{\sqrt[3]{n^2+7} - \sqrt[3]{n^2+1}}{\sqrt[3]{n^2+6} - \sqrt[3]{n^2}}$

Řešení: Rozšíříme čitatel i jmenovatel tak, abychom podle vztahu $(A-B)(A^2 + AB + B^2) = A^3 - B^3$ odstranili odmocninu.

$$\lim_{n\to\infty} \frac{\sqrt[3]{n^2+7} - \sqrt[3]{n^2+1}}{\sqrt[3]{n^2+6} - \sqrt[3]{n^2}} \cdot \frac{\sqrt[3]{(n^2+7)^2} + \sqrt[3]{n^2+1}\sqrt[3]{n^2+7} + \sqrt[3]{(n^2+1)^2}}{\sqrt[3]{(n^2+7)^2} + \sqrt[3]{n^2+1}\sqrt[3]{n^2+7} + \sqrt[3]{(n^2+1)^2}} \cdot$$

$$\cdot \frac{\sqrt[3]{(n^2+6)^2} + \sqrt[3]{n^2}\sqrt[3]{n^2+6} + \sqrt[3]{(n^2)^2}}{\sqrt[3]{(n^2+6)^2} + \sqrt[3]{n^2}\sqrt[3]{n^2+6} + \sqrt[3]{(n^2)^2}} =$$

$$= \lim_{n\to\infty} \frac{6}{6} \cdot \frac{\sqrt[3]{(n^2+6)^2} + \sqrt[3]{n^2}\sqrt[3]{n^2+6} + \sqrt[3]{(n^2)^2}}{\sqrt[3]{(n^2+7)^2} + \sqrt[3]{n^2+1}\sqrt[3]{n^2+7} + \sqrt[3]{(n^2+1)^2}}$$

$$= \lim_{n\to\infty} \frac{n^{4/3}}{n^{4/3}} \cdot \frac{\sqrt[3]{(1+\frac{6}{n^2})^2} + 1 \cdot \sqrt[3]{1+\frac{6}{n^2}} + 1}{\sqrt[3]{(1+\frac{7}{n^2})^2} + \sqrt[3]{1+\frac{1}{n^2}}\sqrt[3]{1+\frac{7}{n^2}} + \sqrt[3]{(1+\frac{1}{n^2})^2}}$$

$$\stackrel{VOAL}{=} \frac{1+1+1}{1+1+1} = 1.$$

(e)

$$\lim_{n\to\infty} \left(1-\frac{1}{2^2}\right) \cdot \left(1-\frac{1}{3^2}\right) \cdots \left(1-\frac{1}{n^2}\right)$$

Řešení: Rozepíšeme

$$1-\frac{1}{k^2}=\frac{k^2-1}{k^2}=\frac{(k-1)(k+1)}{k^2}$$

Pak máme

$$\lim_{n\to\infty}\left(\frac{1\cdot 3}{2^2}\right)\left(\frac{2\cdot 4}{3^2}\right)\left(\frac{3\cdot 5}{4^2}\right)\cdots\left(\frac{(n-1)\cdot(n+1)}{n^2}\right)=\lim_{n\to\infty}\frac{1}{2}\cdot\frac{n+1}{n}\overset{VOAL}{=}\frac{1}{2}$$

(f)

$$\lim_{n\to\infty}\frac{1}{2}\frac{3}{4}\frac{5}{6}\cdots\frac{2n-1}{2n}$$

Řešení: Řešíme na základě znalosti nerovností:

$$0\le\frac{1}{2}\frac{3}{4}\frac{5}{6}\cdots\frac{2n-1}{2n}\le\frac{1}{\sqrt{2n+1}}$$

Máme tedy 2 policajty, navíc

$$\lim_{n\to\infty}\frac{1}{\sqrt{2n+1}}=0.$$

Dohromady tedy

$$\lim_{n\to\infty}\frac{1}{2}\frac{3}{4}\frac{5}{6}\cdots\frac{2n-1}{2n}=0$$