11. cvičení – Cylindrické a sférické souřadnice

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Teorie

Postup výpočtu $\int_M f(x)\,\mathrm{d}x$

1. načrtneme množinu
2. volba substituce
3. ověření předpokladů věty (hlavně regularita)
4. výpočet J_φ
5. určení $\phi^{-1}(M)$
6. výpočet integrálu $\int_{\varphi^{-1}(M)} f(\varphi(t))|J_\varphi(t)|\,\mathrm{d}t$

Hinty

$$\cos^2 t + \sin^2 t = 1$$
$$\cos^2 t - \sin^2 t = \cos 2t$$
$$\int \sin^2 t\,\mathrm{d}t = \frac{t}{2} - \frac{1}{2}\sin t\cos t$$
$$\int \cos^2 t\,\mathrm{d}t = \frac{t}{2} + \frac{1}{2}\sin t\cos t$$

Příklady

1. Za pomoci substitucí spočtěte integrály

   (a) $\int_M x^2 + y^2\,\mathrm{d}\lambda$, kde $M := \{[x,y,z] \in \mathbb{R}^3; 1 \le z \le 2; x^2+y^2 \le 1\}$

   (b) Spočtěte objem množiny M, kde $M := \{[x,y,z] \in \mathbb{R}^3; -1 < x < 1, z > 0, y^2+z^2 \le 1\}$

   (c) $\int_M 1\,\mathrm{d}\lambda$, kde $M := \{[x,y,z] \in \mathbb{R}^3; x^2+y^2+z^2 \le 1\}$

   (d) $\int_M \frac{1}{(x^2+y^2+z^2)^3}\,\mathrm{d}\lambda$, kde $M = \{[x,y,z] \in \mathbb{R}^3; 1 \le x^2+y^2+z^2 \le 4, z \le 0\}$

   (e) $\int_M \sqrt{x^2+y^2+z^2}\,\mathrm{d}\lambda$, kde $M = \{[x,y,z] \in \mathbb{R}^3; x,y,z \ge 0; x^2+y^2+z^2 \le 1\}$

   (f) Spočtěte objem tělesa, $M = \{[x,y,z] \in \mathbb{R}^3; x^2+4y^2+z^2 \le 4\}$

   (g) $\int_M z\,\mathrm{d}\lambda$, kde $M = \{[x,y,z] \in \mathbb{R}^3; x^2+y^2 \le z^2 \le 1, z \ge 0\}$

   (h) $\int_M \sqrt{x^2+y^2}\,\mathrm{d}\lambda$, kde $M = \{[x,y,z] \in \mathbb{R}^3; x^2+y^2 \le z \le 1\}$

   (i) $\int_M z\mathrm{d}A$, kde $M = \{[x,y,z] \in \mathbb{R}^3; 0 \le z \le 4 - 2\sqrt{x^2+y^2}\}$

   (j) $\int_M (x^2+y^2)z\mathrm{d}A$, kde $M := \{[x,y,z] \in \mathbb{R}^3; 1 \le x^2+y^2+z^2 \le 4, x^2+y^2 \le z^2, z \ge 0\}$

   (k) $\int_M (x^4+y^4)z\mathrm{d}A$,
   kde $M := \{[x,y,z] \in \mathbb{R}^3; x^2+y^2 \le 1, z \ge 0, x^2+y^2+z^2 \le 4\}$

(l) $\int_M z \mathrm{d}A$,

kde $M := \{[x,y,z] \in \mathbb{R}^3; \frac{x^2}{4} + \frac{y^2}{9} + z^2 \leq 2z\}$

(m) Spočtěte objem tělesa (anuloid - torus) určeného $M := \{[x,y,z] \in \mathbb{R}^3; (\sqrt{x^2+y^2} - a)^2 + z^2 \leq b^2\}$, $0 < b < a$.

(n) Spočtěte objem tělesa určeného vztahy $M := \{[x,y,z] \in \mathbb{R}^3; x^2+y^2+z^2 \leq 16; x^2 + y^2 \leq 4y\}$

Zdroje příkladů a řešení:

http://mat.fsv.cvut.cz/Sibrava/Vyuka/vic_int.pdf

https://math.fme.vutbr.cz/download.aspx?id_file=602492416

https://fix.prf.jcu.cz/~eisner/lock/UMB-566-materialy/matematika-sbirka-III-Krivkovy_integral.pdf

https://homel.vsb.cz/~bou10/archiv/ip2.pdf

https://is.muni.cz/el/1433/jaro2009/MB102/7448541/skripta4.pdf

https://math.fel.cvut.cz/en/people/habala/teaching/veci-ma2/ma2r4.pdf

http://www.matematika-lucerna.cz/matalyza/resene-matika3.pdf

2. Přiřaďte rovnici obrázku.

(a) $\{[x, y, z] \in \mathbb{R}^3; 1 \leq z \leq 2; x^2 + y^2 \leq 1\}$

(b) $\{[x, y, z] \in \mathbb{R}^3; x^2 + y^2 \leq z^2; 1 \leq x^2 + y^2 + z^2 \leq 4; z \geq 0\}$

(c) $\{[x, y, z] \in \mathbb{R}^3; x^2 + y^2 \leq z \leq 1\}$

(d) Ohraničeno plochami $z = 0$, $z = 3$, $x^2 + y^2 - 2x = 0$ a navíc $y \geq 0$

(e) $\{[x, y, z] \in \mathbb{R}^3; x^2 + y^2 + z^2 \leq 1; z \geq 0\}$

(f) $\{[x, y, z] \in \mathbb{R}^3; 0 \leq z \leq 4 - 2\sqrt{x^2 + y^2}\}$

(g) $\{[x, y, z] \in \mathbb{R}^3; 1 \leq x^2 + y^2 + z^2 \leq 4, z \leq 0\}$

(h) $\{[x, y, z] \in \mathbb{R}^3; x^2 + 4y^2 + z^2 \leq 4\}$

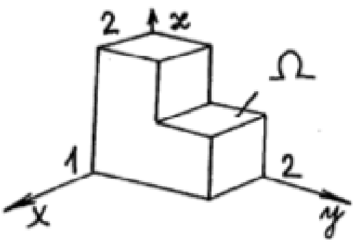
(i) $\{[x, y, z] \in \mathbb{R}^3; x^2 + y^2 + z^2 \leq z\}$

(j) $\{[x, y, z] \in \mathbb{R}^3; \sqrt{x^2 + y^2} \leq z \leq 6 - (x^2 + y^2)\}$

(k) $M = M_1 \cup M_2$, kde $M_1 = [0, 1] \times [0, 1] \times [0, 2]$ a $M_2 = [0, 1] \times [1, 2] \times [0, 1]$

(l) $\{[x, y, z] \in \mathbb{R}^3; -1 \leq x \leq 1; z \geq 0; y^2 + z^2 \leq 1\}$

Zdroj: https://math.fme.vutbr.cz/download.aspx?id_file=602492416



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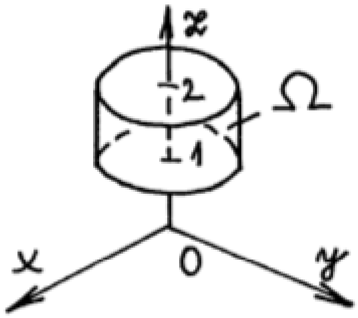

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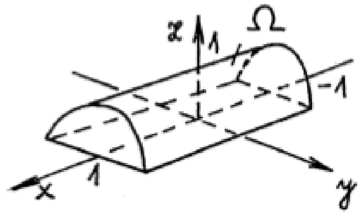

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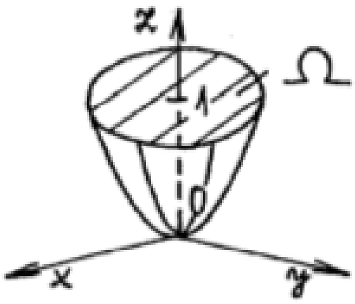

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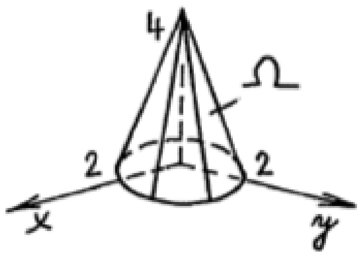

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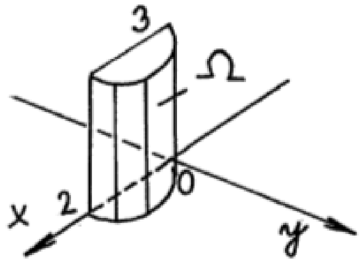

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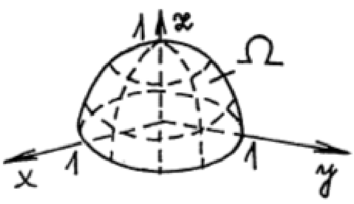

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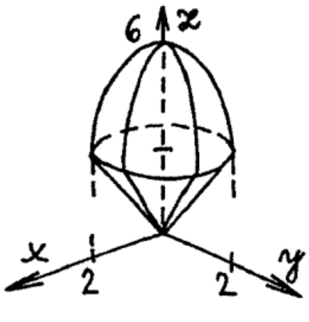

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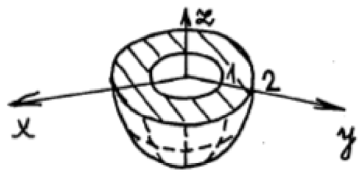

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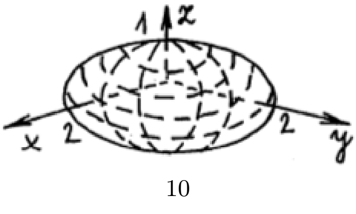

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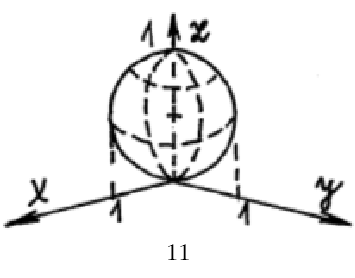

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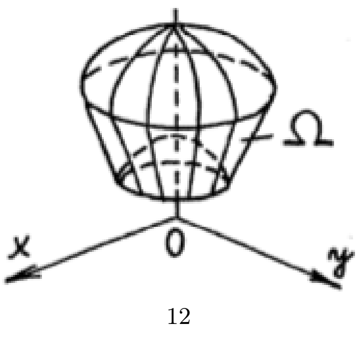

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