



## 8. cvičení – Derivace integrálu závislého na parametru

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### Teorie

**Věta 1** (Derivace integrálu závislého na parametru). *Nechť  $I \subset \mathbf{R}$  je otevřený interval. Nechť funkce  $f : X \times I \rightarrow \mathbf{R}$  má následující vlastnosti:*

(De-1) *pro všechna  $\alpha \in I$  je funkce  $x \rightarrow f(x, \alpha)$  měřitelná,*

(De-2) *pro všechna  $x \in X$  a  $\alpha \in I$  existuje vlastní  $\frac{\partial f(x, \alpha)}{\partial \alpha}$ ,*

(De-3) *existuje funkce  $g \in L^1(X, \mu)$  tak, že pro všechna  $x \in X$  a  $\alpha \in I$  je*

$$\left| \frac{\partial f}{\partial \alpha}(x, \alpha) \right| \leq g(x),$$

(De-4) *existuje  $\alpha_0 \in I$  tak, že  $F(\alpha_0) = \int_X f(x, \alpha_0) d\mu(x) \in \mathbf{R}$  (je konečný).*

*Potom pro všechna  $\alpha \in I$  je  $F(\alpha) = \int_X f(x, \alpha) d\mu(x) \in \mathbf{R}$ , existuje derivace této funkce a*

$$F'(\alpha) = \int_X \frac{\partial f(x, \alpha)}{\partial \alpha} d\mu(x).$$

**Věta 2** (Heineova). *Nechť  $a \in \mathbf{R}^*$ ,  $A \in \mathbf{R}^*$  a nechť funkce  $f$ , je definována na nějakém prstencovém okolí bodu  $a$ . Potom jsou následující dva výroky ekvivalentní:*

(i)

$$\lim_{x \rightarrow a} f(x) = A;$$

(ii) *Pro každou posloupnost  $\{x_n\}_{n \in \mathbf{N}}$ , splňující  $x_n \in D_f, \forall n \in \mathbf{N} : x_n \neq a$  a  $\lim_{n \rightarrow \infty} x_n = a$  platí  $\lim_{n \rightarrow \infty} f(x_n) = A$ .*

### Limita

1. Určete  $\lim_{\alpha \rightarrow \infty} F(\alpha)$ , kde  $F(\alpha) = \int_0^{\infty} e^{-\alpha x} dx, \alpha \in (0, \infty)$ .

2.  $\clubsuit$  Určete  $\lim_{\alpha \rightarrow 0^+} F(\alpha)$ , kde  $F(\alpha) = \int_0^{\pi} \frac{\ln(1 + \alpha \sin x)}{x} dx, \alpha \in (0, \infty)$ .

3.  $\spadesuit$  Určete  $\lim_{\alpha \rightarrow \infty} F(\alpha)$  a  $\lim_{\alpha \rightarrow 0} F(\alpha)$ , kde  $F(\alpha) = \int_0^1 \frac{1}{\sqrt{x^2 + \alpha^2}} dx, \alpha \in (-\infty, 0) \cup (0, \infty)$ .

## Derivace

4. Spočtěte

$$(a) \quad F(\alpha) = \int_0^\infty \frac{1 - e^{-\alpha x^2}}{x e^{x^2}} dx \quad \alpha \in (-1, \infty)$$

$$(b) \quad F(\alpha) = \int_0^\infty \frac{1 - e^{-\alpha x}}{x e^x} dx \quad \alpha \in (-1, \infty)$$

$$(c) \quad F(\alpha, \beta) = \int_0^\infty \frac{e^{-\alpha x^2} - e^{-\beta x^2}}{x} dx \quad (\alpha = \beta) \vee (\alpha, \beta > 0)$$

$$(d) \heartsuit \quad F(\alpha, \beta) = \int_0^\infty \frac{e^{-\alpha x^2} - e^{-\beta x^2}}{x^2} dx \quad (\alpha = \beta) \vee (\alpha, \beta \geq 0)$$

$$\text{Hint: } \int_0^\infty -e^{-\alpha x^2} dx = -\frac{1}{2} \sqrt{\pi/\alpha}$$

$$(e) \quad F(\alpha) = \int_0^\infty \frac{1 - e^{-\alpha x^2}}{x^2 e^{x^2}} dx \quad \alpha \in [-1, \infty)$$

$$(f) \quad F(\alpha, \beta) = \int_0^\infty \frac{\arctan \alpha x - \arctan \beta x}{x} dx \quad (\alpha = \beta) \vee (\alpha, \beta > 0) \vee (\alpha, \beta < 0)$$

$$(g) \quad F(\alpha, \beta) = \int_0^\infty e^{-\beta x} \frac{\sin \alpha x}{x} dx \quad \beta \in (0, \infty), \alpha \in \mathbb{R}$$

$$\text{Hint: dce dle } \alpha, \int_0^\infty e^{-\beta x} \cos \alpha x dx = \beta/(\alpha^2 + \beta^2).$$

$$(h) \quad F(\alpha) = \int_0^{\pi/2} \frac{\arctan(\alpha \operatorname{tg} x)}{\operatorname{tg} x} dx, \quad \alpha \in \mathbb{R}$$

$$\text{Hint: } \int_0^{\pi/2} \frac{1}{1 + \alpha^2 \operatorname{tg}^2 x} dx = \frac{\pi}{2} \frac{1}{1 + \alpha^2}.$$

$$(i) \quad F(\alpha, \beta) = \int_0^{\pi/2} \ln(\alpha^2 \sin^2 x + \beta^2 \cos^2 x) dx, \quad (\alpha, \beta) \in \mathbb{R}^2 \setminus (0, 0)$$

$$\text{Hint: dce dle } \alpha, \int_0^{\pi/2} \frac{2}{\alpha} \frac{\alpha^2 \sin^2 x}{\alpha^2 \sin^2 x + \beta^2 \cos^2 x} dx = \frac{\pi}{\alpha + \beta}$$

## Bonus

5. Ukažte, že funkce  $F(\alpha) = \int_0^\infty \frac{e^{-\alpha x}}{1+x^2} dx$  konverguje pro  $\alpha \geq 0$  a pro  $\alpha \in (0, \infty)$  splňuje diferenciální rovnici  $F'' + F = \frac{1}{\alpha}$ .

## Bonus teorie

6. Existuje posloupnost funkcí  $f_n \in L^1(\mathbb{R})$  taková, že konverguje k nulové funkci na každém kompaktu a zároveň platí, že  $\int_{\mathbb{R}} f_n = 1$ ?

(7)  $\log(1+t) \leq t$   
(8)  $\lim_{t \rightarrow 0} \frac{\log(1+t)}{t} = 1$   
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