



18. cvičení – Určitý integrál 2

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Příklady

Spočtěte Newtonovy integrály:

1. $\int_4^{\infty} \frac{x}{(x-1)(x-2)(x-3)} dx$

Řešení:

Rozkladem na parciální zlomky dostaneme

$$\int_4^{\infty} \frac{x}{(x-1)(x-2)(x-3)} dx = \frac{1}{2} \int_4^{\infty} \frac{1}{x-1} + \frac{-4}{x-2} + \frac{3}{x-3} dx$$

Po integraci

$$= \frac{1}{2} \left[\ln|x-1| - 4 \ln|x-2| + 3 \ln|x-3| \right]_4^{\infty} = \frac{1}{2} \left[\ln \frac{(x-1)(x-3)^3}{(x-2)^4} \right]_4^{\infty} = 0 - \frac{1}{2} \ln \frac{3 \cdot 1^3}{2^4} = \frac{1}{2} \ln \frac{16}{3}$$

2. $\int_{-\infty}^0 \frac{x}{x^3-1} dx$

Řešení:

$$\int_{-\infty}^0 \frac{x}{x^3-1} dx = \int_{-\infty}^0 \frac{x}{(x-1)(x^2+x+1)} dx$$

Po rozkladu na parciální zlomky

$$\begin{aligned} &= \frac{1}{3} \int_{-\infty}^0 \frac{1}{x-1} - \frac{x-1}{x^2+x+1} dx = \frac{1}{3} \int_{-\infty}^0 \frac{1}{x-1} - \frac{1}{2} \frac{2x+1-1-2}{x^2+x+1} dx \\ &= \frac{1}{3} \int_{-\infty}^0 \frac{1}{x-1} - \frac{1}{2} \frac{2x+1}{x^2+x+1} + \frac{1}{2} \frac{3}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} dx \end{aligned}$$

Po integraci

$$\begin{aligned} &\frac{1}{3} \left[\ln|x-1| - \frac{1}{2} \ln|x^2+x+1| + \sqrt{3} \arctan \frac{2x+1}{\sqrt{3}} \right]_{-\infty}^0 \\ &= \frac{1}{3} \left[\ln \frac{|x-1|}{\sqrt{|x^2+x+1|}} + \sqrt{3} \arctan \frac{2x+1}{\sqrt{3}} \right]_{-\infty}^0 = 0 + \frac{\sqrt{3}}{3} \cdot \frac{\pi}{6} - \left(0 + \frac{\sqrt{3}}{3} \cdot \frac{-\pi}{2} \right) \\ &= \frac{2\pi}{9} \sqrt{3}. \end{aligned}$$

3. $\int_0^{\pi} \frac{\sin x}{\cos^2 x + 1} dx$

Řešení: Zvolíme substituci $t = \cos x$.

$$\int_1^{-1} \frac{-1}{t^2+1} dt = \int_{-1}^1 \frac{1}{t^2+1} dt = [\arctan t]_{-1}^1 = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}.$$

$$4. \int_{-\infty}^{\infty} \frac{e^x}{e^{2x} - 3e^x + 3} dx$$

Řešení:

Substituce $t = e^x$. Pak

$$\begin{aligned} \int_0^{\infty} \frac{1}{t^2 - 3t + 3} dt &= \int_0^{\infty} \frac{1}{\left(t - \frac{3}{2}\right)^2 + \frac{3}{4}} dt = \left[\frac{2}{\sqrt{3}} \arctan \frac{2t - 3}{\sqrt{3}} \right]_0^{\infty} \\ &= \frac{2}{\sqrt{3}} \left(\frac{\pi}{2} - \left(-\frac{\pi}{3}\right) \right) = \frac{5\pi}{3\sqrt{3}} \end{aligned}$$

$$5. \int_0^{\pi} \sin^2 x \cos^2 x dx$$

Řešení:

Nejprve upravme na

$$\int_0^{\pi} \sin^2 x \cos^2 x dx = \int_0^{\pi} \frac{1}{4} \sin^2(2x) dx$$

Po substituci $t = 2x$ máme

$$\int_0^{2\pi} \frac{1}{8} \sin^2(t) dt = \int_0^{2\pi} \frac{1}{16} (1 - \cos(2t)) dt = \left[\frac{1}{16} \left(t - \frac{1}{2} \sin(2t) \right) \right]_0^{2\pi} = \frac{\pi}{8}$$

$$6. \int_0^{\frac{\pi}{4}} \sqrt{\cos x - \cos^3 x} dx$$

Řešení: Protože jsme na $(0, \frac{\pi}{4})$, můžeme psát

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \sqrt{\cos x - \cos^3 x} dx &= \int_0^{\frac{\pi}{4}} \sqrt{\cos x} \sqrt{1 - \cos^2 x} dx = \int_0^{\frac{\pi}{4}} \sqrt{\cos x} |\sin x| dx \\ &= \int_0^{\frac{\pi}{4}} \sqrt{\cos x} \sin x dx \end{aligned}$$

Substituce $t = \cos x$:

$$\int_1^{\sqrt{2}/2} -\sqrt{t} dt = \left[\frac{t^{3/2}}{3/2} \right]_{\sqrt{2}/2}^1 = \frac{2}{3} \left(1 - \frac{\sqrt{2}}{\sqrt{8}} \right)$$

$$7. \int_{-1}^1 x^2 e^{-x} dx$$

Řešení:

Aplikujeme per partes

$$\begin{aligned} \int_{-1}^1 x^2 e^{-x} dx &= [-x^2 e^{-x}]_{-1}^1 - \int_{-1}^1 -2xe^{-x} dx = [-x^2 e^{-x}]_{-1}^1 + [-2xe^{-x}]_{-1}^1 - \int_{-1}^1 -2e^{-x} dx \\ &= [-x^2 e^{-x}]_{-1}^1 + [-2xe^{-x}]_{-1}^1 + [-2e^{-x}]_{-1}^1 \\ &= -e^{-1} + e - 2e^{-1} - 2e - 2e^{-1} + 2e = e - \frac{5}{e} \end{aligned}$$

8. $\int_0^1 \arccos^2 x \, dx$

Řešení:

Substituce $x = \cos t$, pak $dx = -\sin t \, dt$ a

$$\int_0^{\frac{\pi}{2}} t^2 \sin t \, dt$$

Dvakrát per partes:

$$\begin{aligned} \int_0^{\frac{\pi}{2}} t^2 \sin t \, dt &= [t^2 \cos t]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -2t \cos t \, dt = [t^2 \cos t]_0^{\frac{\pi}{2}} + [2t \sin t]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2 \sin t \, dt \\ &= [t^2 \cos t]_0^{\frac{\pi}{2}} + [2t \sin t]_0^{\frac{\pi}{2}} + [2 \cos t]_0^{\frac{\pi}{2}} = 0 - 0 + \pi - 0 + 0 - 2 = \pi - 2 \end{aligned}$$

9. $\int_0^1 x \arcsin x \, dx$

Řešení: Substituce $x = \sin t$, $dx = \cos t \, dt$:

$$\int_0^{\frac{\pi}{2}} t \sin t \cos t \, dt = \int_0^{\frac{\pi}{2}} t \frac{1}{2} \sin(2t) \, dt = \frac{1}{4} \int_0^{\frac{\pi}{2}} 2t \sin(2t) \, dt$$

Substituce $u = 2t$, $du = 2 \, dt$

$$\frac{1}{8} \int_0^{\pi} u \sin u \, du$$

Per partes

$$= \frac{1}{8} [-u \cos u]_0^{\pi} + \frac{1}{8} \int_0^{\pi} \cos u \, du = \frac{1}{8} [-u \cos u]_0^{\pi} + \frac{1}{8} [\sin u]_0^{\pi} = \frac{1}{8} \pi$$

10. $\int_0^1 x^2 \sqrt{1-x^2} \, dx$

Řešení:

Substituce $x = \sin t$, $dx = \cos t \, dt$:

$$\int_0^{\frac{\pi}{2}} \sin^2 t \cos^2 t \, dt = \int_0^{\frac{\pi}{2}} \frac{1}{4} \sin^2(2t) \, dt$$

Substituce $u = 2t$, $du = 2 \, dt$:

$$\begin{aligned} \int_0^{\pi} \frac{1}{8} \sin^2(u) \, du &= \int_0^{\pi} \frac{1}{16} (1 - \cos(2u)) \, du = \left[\frac{1}{16} \left(u - \frac{1}{2} \sin(2u) \right) \right]_0^{\pi} \\ &= \frac{1}{16} \pi \end{aligned}$$

11. $\int_0^1 \sqrt{\frac{x+1}{x}} \, dx$

Řešení:

Substituce $t = \sqrt{\frac{x+1}{x+0}}$, pak $x = \frac{-1}{1-t^2}$ a $dx = \frac{-2t}{(t^2-1)^2} dt$:

$$\int_{\sqrt{2}}^{\infty} t \frac{2t}{(t^2-1)^2} dt$$

Rozklad na parciální zlomky

$$\begin{aligned} &= \frac{1}{2} \int_{\sqrt{2}}^{\infty} \frac{1}{t-1} + \frac{1}{(t-1)^2} + \frac{-1}{t+1} + \frac{1}{(t+1)^2} dt = \frac{1}{2} \left[\ln|t-1| - \ln|t+1| - \frac{1}{t-1} - \frac{1}{t+1} \right]_{\sqrt{2}}^{\infty} \\ &= \frac{1}{2} \left[\ln \frac{|t-1|}{|t+1|} - \frac{1}{t-1} - \frac{1}{t+1} \right]_{\sqrt{2}}^{\infty} = -\frac{1}{2} \left(\ln \frac{\sqrt{2}-1}{\sqrt{2}+1} - \frac{1}{\sqrt{2}-1} - \frac{1}{\sqrt{2}+1} \right) \\ &= \frac{1}{2} \left(\ln \frac{\sqrt{2}+1}{\sqrt{2}-1} + 2\sqrt{2} \right) \end{aligned}$$

Podmínky věty o substituci: $f(x) = \sqrt{\frac{x+1}{x}}$, $\omega(t) = \frac{-1}{1-t^2}$. Interval $(\alpha, \beta) = (\sqrt{2}, \infty)$, $(a, b) = (0, 1)$. Platí $\omega((\alpha, \beta)) = (a, b)$ a navíc $\omega' = \frac{-2t}{(t^2-1)^2} \neq 0$ na celém (α, β) .

(Plyne z: obrázku <https://www.geogebra.org/calculator/cb3k7uxu>)

12. $\int_4^{\infty} \frac{1}{x^2} \sqrt{\frac{x-2}{x-4}} dx$

Řešení:

Substituce: $t = \sqrt{\frac{x-2}{x-4}}$, $x = \frac{-4t^2+2}{1-t^2}$, $dx = \frac{-4t}{(1-t^2)^2} dt$.

$$\int_1^{\infty} \frac{(1-t^2)^2}{(-4t^2+2)^2} t \frac{4t}{(1-t^2)^2} dt = \int_1^{\infty} \frac{4t^2}{(4t^2-2)^2} dt = \int_1^{\infty} \frac{4t^2}{((2t-\sqrt{2})(2t+\sqrt{2}))^2} dt$$

Rozkladem na parciální zlomky

$$\int_1^{\infty} -\frac{1}{8(\sqrt{2}t+1)} + \frac{1}{8(\sqrt{2}t+1)^2} + \frac{1}{8(\sqrt{2}t-1)} + \frac{1}{8(\sqrt{2}t-1)^2} dt$$

Po integraci

$$\begin{aligned} &\left[-\frac{1}{8\sqrt{2}} \ln|\sqrt{2}t+1| + \frac{1}{8\sqrt{2}} \ln|\sqrt{2}t-1| - \frac{1}{8\sqrt{2}} \frac{1}{\sqrt{2}t+1} - \frac{1}{8\sqrt{2}} \frac{1}{\sqrt{2}t-1} \right]_1^{\infty} \\ &= \left[\frac{1}{8\sqrt{2}} \ln \frac{|\sqrt{2}t-1|}{|\sqrt{2}t+1|} - \frac{1}{8\sqrt{2}} \frac{1}{\sqrt{2}t+1} - \frac{1}{8\sqrt{2}} \frac{1}{\sqrt{2}t-1} \right]_1^{\infty} \\ &= -\frac{1}{8\sqrt{2}} \left(\ln \frac{|\sqrt{2}-1|}{|\sqrt{2}+1|} - \frac{1}{\sqrt{2}+1} - \frac{1}{\sqrt{2}-1} \right) \\ &= \frac{1}{8\sqrt{2}} \left(\ln \frac{\sqrt{2}+1}{\sqrt{2}-1} + 2\sqrt{2} \right) \end{aligned}$$

Podmínky věty o substituci: $f(x) = \frac{1}{x^2} \sqrt{\frac{x-2}{x-4}}$, $\omega(t) = \frac{-4t^2+2}{1-t^2}$. Interval $(\alpha, \beta) = (1, \infty)$, $(a, b) = (4, \infty)$. Platí $\omega((\alpha, \beta)) = (a, b)$ a navíc $\omega' = \frac{-4t}{(1-t^2)^2} \neq 0$ na celém (α, β) .

(Plyne z:

$$\omega(t) = \frac{-4t^2 + 2}{1 - t^2} = \frac{-4t^2 + 4}{1 - t^2} + \frac{-2}{1 - t^2} = 4 + \frac{-2}{1 - t^2},$$

což už lze načrtnout: <https://www.geogebra.org/calculator/cb3k7uxu>)

13. $\int_0^{4\pi} \frac{1}{\cos x + 2 \sin x + 3} dx$

Řešení:

Protože funkce je 2π periodická, můžeme psát

$$\int_0^{4\pi} \frac{dx}{\cos x + 2 \sin x + 3} = \int_{-\pi}^{3\pi} \frac{dx}{\cos x + 2 \sin x + 3}$$

Což rozepíšeme na

$$\int_{-\pi}^{3\pi} \frac{dx}{\cos x + 2 \sin x + 3} = \int_{-\pi}^{\pi} \frac{dx}{\cos x + 2 \sin x + 3} + \int_{\pi}^{3\pi} \frac{dx}{\cos x + 2 \sin x + 3}$$

Na intervalech $(-\pi, \pi)$ a $(\pi, 3\pi)$ pak můžeme substituovat $t = \tan \frac{x}{2}$. Po aplikaci vzorců dostaneme

$$\begin{aligned} \int_{-\pi}^{\pi} \frac{dx}{\cos x + 2 \sin x + 3} &= \int_{-\infty}^{\infty} \frac{2}{2t^2 + 4t + 4} dt = \int_{-\infty}^{\infty} \frac{2}{2t^2 + 4t + 4} dt \\ &= \int_{-\infty}^{\infty} \frac{1}{(t+1)^2 + 1} dt = [\arctan(t+1)]_{-\infty}^{\infty} = \pi \end{aligned}$$

Na intervalu $(\pi, 3\pi)$ dostaneme stejný výsledek, celkem tedy máme

$$\int_{-\pi}^{3\pi} \frac{dx}{\cos x + 2 \sin x + 3} = 2\pi.$$

Podmínky věty o substituci: $f(t) = \frac{2}{2t^2+4t+4}$, $\omega(x) = \tan \frac{x}{2}$. Interval $(\alpha, \beta) = (-\pi, \pi)$, $(a, b) = (-\infty, \infty)$. Platí $\omega((\alpha, \beta)) = (a, b)$ a navíc $\omega' = \frac{1}{2 \cos^2 \frac{x}{2}} \neq 0$ na celém (α, β) .

Pro interval $(-\pi, 3\pi)$ analogicky.

14. $\int_0^{\frac{\pi}{2}} \frac{dx}{1 + \operatorname{tg} x}$

Řešení:

Substituujeme za $t = \tan x$. Dostaneme

$$\begin{aligned} \int_0^{\infty} \frac{1}{(1+t)(1+t^2)} dt &= \int_0^{\infty} \frac{\frac{1}{2}}{1+t} - \frac{1}{2} \frac{t-1}{(1+t^2)} dt = \int_0^{\infty} \frac{\frac{1}{2}}{1+t} - \frac{1}{4} \frac{2t}{(1+t^2)} + \frac{1}{2} \frac{1}{(1+t^2)} dt \\ &= \left[\frac{1}{2} \ln|1+t| - \frac{1}{4} \ln(t^2+1) + \frac{1}{2} \arctan t \right]_0^{\infty} \\ &= \left[\frac{1}{4} \ln \frac{(1+t)^2}{t^2+1} + \frac{1}{2} \arctan t \right]_0^{\infty} = \frac{\pi}{4} \end{aligned}$$

Podmínky věty o substituci: $f(t) = \frac{1}{(1+t)(1+t^2)}$, $\omega(x) = \tan x$. Interval $(\alpha, \beta) = (0, \frac{\pi}{2})$, $(a, b) = (0, \infty)$. Platí $\omega((\alpha, \beta)) = (a, b)$ a navíc $\omega' = \frac{1}{\cos^2 x} \neq 0$ na celém (α, β) .

Zkouškové příklady

doc. Rokyty: <https://www2.karlin.mff.cuni.cz/~rokyta/vyuka/index.html>

prof. Spurného: <https://www2.karlin.mff.cuni.cz/~spurny/pages/ma2.php#>

15. $\int_{-\pi}^{\pi} \frac{2 + \cos x}{3 + \sin x + \cos x} dx$

Řešení: Substituujeeme $y = \tan \frac{x}{2}$, pak dostaneme

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{y^2 + 3}{(y^2 + 1)(y^2 + y + 2)} dy &= \int_{-\infty}^{\infty} \frac{y + 1}{y^2 + y + 2} + \frac{-y + 1}{y^2 + 1} dy \\ &= \int_{-\infty}^{\infty} \frac{1}{2} \cdot \frac{2y + 1}{y^2 + y + 2} + \frac{1}{2} \cdot \frac{1}{y^2 + y + 2} - \frac{1}{2} \cdot \frac{2y}{y^2 + 1} + \frac{1}{y^2 + 1} dy \\ &= \int_{-\infty}^{\infty} \frac{1}{2} \cdot \frac{2y + 1}{y^2 + y + 2} + \frac{1}{2} \cdot \frac{1}{(y + \frac{1}{2})^2 + \frac{7}{4}} - \frac{1}{2} \cdot \frac{2y}{y^2 + 1} + \frac{1}{y^2 + 1} dy \\ &= \left[\frac{1}{2} \log(y^2 + y + 2) + \frac{1}{\sqrt{7}} \arctan \frac{2y + 1}{\sqrt{7}} - \frac{1}{2} \log(y^2 + 1) + \arctan y \right]_{-\infty}^{\infty} \\ &= \left[\frac{1}{2} \log \frac{(y^2 + y + 2)}{y^2 + 1} + \frac{1}{\sqrt{7}} \arctan \frac{2y + 1}{\sqrt{7}} + \arctan y \right]_{-\infty}^{\infty} \\ &= \pi \left(1 + \frac{1}{\sqrt{7}} \right). \end{aligned}$$

16. $\int_0^1 \frac{\sqrt{2x+1}}{(x+2)^2} dx$

Řešení: Substituujeeme $y = \sqrt{2x+1}$. Pak $dy = \frac{1}{\sqrt{2x+1}} dx$, $x = \frac{1}{2}(y^2 - 1)$. Dostaneme

$$\begin{aligned} \int_0^1 \frac{\sqrt{2x+1}}{(x+2)^2} dx &= \int_0^1 \frac{\sqrt{2x+1}}{(x+2)^2} \cdot \frac{\sqrt{2x+1}}{\sqrt{2x+1}} dx \rightarrow \int_1^{\sqrt{3}} \frac{4y^2}{(y^2+3)^2} dy \\ &= \int_1^{\sqrt{3}} \frac{4}{y^2+3} - \frac{12}{(y^2+3)^2} dy = \left[\frac{4}{\sqrt{3}} \arctan \frac{y}{\sqrt{3}} - \frac{2y}{y^2+3} - \frac{2}{\sqrt{3}} \arctan \frac{y}{\sqrt{3}} \right]_1^{\sqrt{3}} \\ &= \frac{2}{\sqrt{3}} \cdot \frac{\pi}{4} - \frac{\sqrt{3}}{3} - \left(\frac{2}{\sqrt{3}} \frac{\pi}{6} - \frac{1}{2} \right) = \frac{1}{\sqrt{3}} \frac{\pi}{6} + \frac{1}{2} - \frac{\sqrt{3}}{3} \end{aligned}$$

17. $\int_{-\infty}^{\infty} \frac{e^{3x}}{(e^x + 2)^2 (e^x + 1)^2} dx$

Řešení: Substituce $y = e^x$, $dy = e^x dx$. Pak

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^{2x} \cdot e^x}{(e^x + 2)^2(e^x + 1)^2} dx &\rightarrow \int_0^{\infty} \frac{y^2}{(y + 2)^2(y + 1)^2} dy \\ &= \int_0^{\infty} \frac{4}{(y + 2)^2} + \frac{4}{y + 2} + \frac{1}{(y + 1)^2} - \frac{4}{y + 1} dy \\ &= \left[\frac{-4}{y + 2} + 4 \log(y + 2) - \frac{1}{y + 1} - 4 \log(y + 1) \right]_0^{\infty} \\ &= \left[\frac{-4}{y + 2} + 4 \log \frac{(y + 2)}{y + 1} - \frac{1}{y + 1} \right]_0^{\infty} = 3 - 4 \log 2 \end{aligned}$$

18. (R) $\int_0^1 \frac{\sqrt{x} + 2\sqrt[4]{x}}{(\sqrt{x} + 1)(\sqrt[4]{x} + 1)} dx$

Řešení: Funkce $g(x) = \frac{\sqrt{x} + 2\sqrt[4]{x}}{(\sqrt{x} + 1)(\sqrt[4]{x} + 1)}$ je spojitá na $[0, 1]$, tedy existuje Riemannův integrál a rovná se Newtonovu.

Dále tedy počítáme Newtonův integrál. Substituce $y = \sqrt[4]{x}$, $dy = \frac{1}{4\sqrt[4]{x^3}} dx$. Dostaneme

$$\begin{aligned} \int_0^1 \frac{(y^2 + 2y)4y^3}{(y^2 + 1)(y + 1)} dy &= 4 \int_0^1 y^2 + y - 2 + \frac{y + 2}{(y^2 + 1)(y + 1)} dy \\ &= \int_0^1 4y^2 + 4y - 8 + \frac{6 - 2y}{(y^2 + 1)} + \frac{2}{(y + 1)} dy \\ &= \int_0^1 4y^2 + 4y - 8 + \frac{-2y}{(y^2 + 1)} + \frac{6}{(y^2 + 1)} + \frac{2}{(y + 1)} dy \\ &= \left(\frac{4y^3}{3} + 2y^2 - 8y - \log(y^2 + 1) + 6 \arctan y + 2 \log |y + 1| \right)_0^1 \\ &= -\frac{14}{3} + \log 2 + \frac{3\pi}{2} \end{aligned}$$

19. $\int_0^1 \frac{e^x}{e^x + \sqrt{e^{2x} + e^x + 1}} dx$

Řešení: Substituce $y = e^x$, $dy = e^x dx$. Dostaneme

$$\int_1^e \frac{1}{y + \sqrt{y^2 + y + 1}} dy$$

Aplikujeme Eulerovu substituci

$$\begin{aligned} \sqrt{y^2 + y + 1} &= s - y \\ y^2 + y + 1 &= s^2 - 2sy + y^2 \\ y &= \frac{s^2 - 1}{1 + 2s} \\ dy &= \frac{2(s^2 + s + 1)}{(1 + 2s)^2} ds \end{aligned}$$

Pak

$$\begin{aligned} \int_{\sqrt{3}+1}^{e+\sqrt{e^2+e+1}} \frac{2(s^2+s+1)}{s(1+2s)^2} ds &= \int_{\sqrt{3}+1}^{e+\sqrt{e^2+e+1}} \left(\frac{2}{s} - \frac{3}{2s+1} - \frac{3}{(1+2s)^2} \right) ds \\ &= \left[2 \log s - \frac{3}{2} \log(2s+1) + \frac{3}{2} \cdot \frac{1}{1+2s} \right]_{\sqrt{3}+1}^{e+\sqrt{e^2+e+1}} \\ &= 2 \log(\sqrt{3}+1) - \frac{3}{2} \log(2(\sqrt{3}+1)+1) + \frac{3}{2} \cdot \frac{1}{1+2(\sqrt{3}+1)} \\ &\quad - \left(2 \log(e+\sqrt{e^2+e+1}) - \frac{3}{2} \log(2(e+\sqrt{e^2+e+1})+1) \right. \\ &\quad \left. + \frac{3}{2} \cdot \frac{1}{1+2(e+\sqrt{e^2+e+1})} \right) \end{aligned}$$