



15. cvičení – Goniometrické substituce + lepení

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Příklady

Najděte primitivní funkce

$$1. g(x) = \frac{1}{2 \sin x - \cos x + 5}.$$

Řešení:

Funkce g je spojitá na $(-\infty, \infty)$, má tam tedy PF.

Použijeme substituci $t = \tan \frac{x}{2} = \varphi(x)$. Funkce $f(t) = \frac{1}{3t^2 + 2t + 2}$.

Intervaly: $(\alpha_k, \beta_k) = (-\pi + 2k\pi, \pi + 2k\pi)$, $(a, b) = (-\infty, \infty)$. Navíc $\varphi(\alpha_k, \beta_k) = (-\infty, \infty) \subseteq (a, b)$.

Zasubstituuje

$$\begin{aligned} \int \frac{1}{2 \sin x - \cos x + 5} dx &\rightarrow \int \frac{2}{4t - 1 + t^2 + 5 + 5t^2} dt = \int \frac{1}{3t^2 + 2t + 2} dt \stackrel{C}{=} \frac{1}{\sqrt{5}} \arctan \frac{3t + 1}{\sqrt{5}} \\ &\rightarrow \int g(x) \stackrel{C}{=} \frac{1}{\sqrt{5}} \arctan \frac{3 \operatorname{tg} \frac{x}{2} + 1}{\sqrt{5}}. \end{aligned}$$

$$x \in (-\pi + 2k\pi, \pi + 2k\pi)$$

Tedy

$$G(x) = \frac{1}{\sqrt{5}} \arctan \frac{3 \operatorname{tg} \frac{x}{2} + 1}{\sqrt{5}} + c_k, \quad x \in (-\pi + 2k\pi, \pi + 2k\pi)$$

V bodech $\pi + 2k\pi$ je potřeba funkci slepit. Limity:

$$\begin{aligned} \lim_{x \rightarrow (\pi + 2\pi k)^-} \frac{1}{\sqrt{5}} \arctan \frac{3 \operatorname{tg} \frac{x}{2} + 1}{\sqrt{5}} + c_k &= \frac{\pi}{2\sqrt{5}} + c_k, \\ \lim_{x \rightarrow (\pi + 2\pi k)^+} \frac{1}{\sqrt{5}} \arctan \frac{3 \operatorname{tg} \frac{x}{2} + 1}{\sqrt{5}} + c_{k+1} &= -\frac{\pi}{2\sqrt{5}} + c_{k+1}. \end{aligned}$$

tedy

$$c_{k+1} = \frac{\pi}{\sqrt{5}} + c_k.$$

Odtud

$$c_k = \frac{\pi k}{\sqrt{5}} + c_0, \quad k \in \mathbb{Z}.$$

Protože G i g jsou spojité na $(-\infty, \infty)$, můžeme použít lemma o lepení.

Závěr:

$$G(x) = \begin{cases} \frac{1}{\sqrt{5}} \arctan \frac{3 \operatorname{tg} \frac{x}{2} + 1}{\sqrt{5}} + \frac{\pi k}{\sqrt{5}} + c_0, & x \in (-\pi + 2\pi k, \pi + 2\pi k) \\ \frac{(2k+1)\pi}{2\sqrt{5}} + c_0, & x = \pi + 2\pi k \end{cases}$$

$$2. g(x) = \frac{1}{2 - \sin x}$$

Řešení:

Funkce g je spojitá na $(-\infty, \infty)$, má tam tedy PF.

Použijeme substituci $t = \tan \frac{x}{2} = \varphi(x)$. Funkce $f(t) = \frac{1}{t^2 - t + 1}$.

Intervaly: $(\alpha_k, \beta_k) = (-\pi + 2k\pi, \pi + 2k\pi)$, $(a, b) = (-\infty, \infty)$. Navíc $\varphi(\alpha_k, \beta_k) = (-\infty, \infty) \subseteq (a, b)$.

Zasubstituuje

$$\begin{aligned} \int \frac{1}{2 - \sin x} dx &\rightarrow \int \frac{2}{t^2 + 1} \cdot \frac{1}{2 - \frac{2t}{t^2 + 1}} dt = \int \frac{1}{t^2 - t + 1} dt = \int \frac{4}{3 \left(\frac{t - \frac{1}{2}}{\frac{\sqrt{3}}{2}} \right)^2 + 1} dt \\ &\stackrel{C}{=} \frac{2}{\sqrt{3}} \arctan \frac{2t - 1}{\sqrt{3}} \\ &\rightarrow \int g(x) \stackrel{C}{=} \frac{2}{\sqrt{3}} \arctan \frac{2 \operatorname{tg} \frac{x}{2} - 1}{\sqrt{3}} \end{aligned}$$

$$x \in (-\pi + 2k\pi, \pi + 2k\pi)$$

Tedy

$$G(x) = \frac{2}{\sqrt{3}} \arctan \frac{2 \operatorname{tg} \frac{x}{2} - 1}{\sqrt{3}} + c_k, \quad x \in (-\pi + 2k\pi, \pi + 2k\pi)$$

V bodech $\pi + 2k\pi$ je potřeba funkci slepit. Limity:

$$\begin{aligned} \lim_{x \rightarrow (\pi + 2\pi k)^-} \frac{2}{\sqrt{3}} \arctan \frac{2 \operatorname{tg} \frac{x}{2} - 1}{\sqrt{3}} + c_k &= \frac{\pi}{\sqrt{3}} + c_k, \\ \lim_{x \rightarrow (\pi + 2\pi k)^+} \frac{2}{\sqrt{3}} \arctan \frac{2 \operatorname{tg} \frac{x}{2} - 1}{\sqrt{3}} + c_{k+1} &= -\frac{\pi}{\sqrt{3}} + c_{k+1}. \end{aligned}$$

tedy

$$c_{k+1} = \frac{2\pi}{\sqrt{3}} + c_k.$$

Odtud

$$c_k = \frac{2\pi k}{\sqrt{3}} + c_0, \quad k \in \mathbb{Z}.$$

Protože G i g jsou spojité na $(-\infty, \infty)$, můžeme použít lemma o lepení.

Závěr:

$$G(x) = \begin{cases} \frac{2}{\sqrt{3}} \arctan \frac{2 \operatorname{tg} \frac{x}{2} - 1}{\sqrt{3}} + k\pi \frac{2}{\sqrt{3}} + c_0 & x \in (-\pi + 2k\pi; \pi + 2k\pi) \\ \frac{\pi}{\sqrt{3}} + k\pi \frac{2}{\sqrt{3}} + c_0 & x = \pi + 2k\pi \end{cases}$$

$$3. g(x) = \frac{1}{1 + \sin x}$$

Řešení: Podmínky: $\sin x \neq -1$, tedy $x \neq -\frac{\pi}{2} + 2k\pi$.

Funkce g je spojitá na $(-\frac{\pi}{2} + 2k\pi, \frac{3\pi}{2} + 2k\pi)$, má tam tedy PF.

Použijeme substituci $t = \tan \frac{x}{2} = \varphi(x)$. Funkce $f(t) = \frac{2}{(1+t)^2}$.

Intervaly: $(\alpha_k, \beta_k) = (-\frac{\pi}{2} + 2k\pi, \pi + 2k\pi)$, $(a, b) = (-1, \infty)$. Navíc $\varphi(\alpha_k, \beta_k) = (-1, \infty) \subseteq (a, b)$.

Intervaly podruhé: $(\bar{\alpha}_k, \bar{\beta}_k) = (\pi + 2k\pi, \frac{3\pi}{2} + 2k\pi)$, $(\bar{a}, \bar{b}) = (-\infty, -1)$. Navíc $\varphi(\bar{\alpha}_k, \bar{\beta}_k) = (-\infty, -1) \subseteq (\bar{a}, \bar{b})$.

Zasubstituujeme

$$\begin{aligned} \int \frac{dx}{1 + \sin x} &\rightarrow \int \frac{2}{t^2 + 1} \frac{1}{1 + \frac{2t}{t^2 + 1}} dt = \int \frac{2}{t^2 + 2t + 1} dt = \int \frac{2}{(t + 1)^2} dt \stackrel{C}{=} -2 \frac{1}{t + 1} \\ &\rightarrow \int g(x) \stackrel{C}{=} -2 \frac{1}{\tan \frac{x}{2} + 1} \end{aligned}$$

$$x \in (-\frac{\pi}{2} + 2k\pi, \pi + 2k\pi), (\pi + 2k\pi, \frac{3\pi}{2} + 2k\pi)$$

Tedy

$$G(x) = \begin{cases} -2 \frac{1}{\tan \frac{x}{2} + 1} + c_k, & x \in (-\frac{\pi}{2} + 2k\pi, \pi + 2k\pi), \\ -2 \frac{1}{\tan \frac{x}{2} + 1} + d_k, & x \in (\pi + 2k\pi, \frac{3\pi}{2} + 2k\pi). \end{cases}$$

V bodech $\pi + 2k\pi$ je potřeba funkci slepit. Limity:

$$\lim_{x \rightarrow (\pi + 2k\pi)^-} -2 \frac{1}{\tan \frac{x}{2} + 1} + c_k = 0 + c_k$$

$$\lim_{x \rightarrow (\pi + 2k\pi)^+} -2 \frac{1}{\tan \frac{x}{2} + 1} + d_k = 0 + d_k$$

tedy

$$c_k = d_k$$

Protože G i g jsou spojité na $(-\frac{\pi}{2} + 2k\pi, \frac{3\pi}{2} + 2k\pi)$, můžeme použít lemma o lepení.

Závěr:

$$G(x) = \begin{cases} -2 \frac{1}{\tan \frac{x}{2} + 1} + c_k, & x \in (-\frac{\pi}{2} + 2k\pi, \pi + 2k\pi), \\ c_k, & x = \pi + 2k\pi, \\ -2 \frac{1}{\tan \frac{x}{2} + 1} + c_k, & x \in (\pi + 2k\pi, \frac{3\pi}{2} + 2k\pi). \end{cases}$$

Pozn.: V bodech $-\pi/2 + 2k\pi$ nelepíme, původní funkce g tam není definovaná, tedy tam nemůže mít PF.

$$4. g(x) = \frac{\sin^2 x}{1 + \sin^2 x}$$

Řešení:

Funkce g je spojitá na $(-\infty, \infty)$, má tam tedy PF.

Použijeme substituci $t = \tan x = \varphi(x)$. Funkce $f(t) = \frac{t^2}{(t^2 + 1)(2t^2 + 1)}$.

Intervaly: $(\alpha_k, \beta_k) = (-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi)$, $(a, b) = (-\infty, \infty)$. Navíc $\varphi(\alpha_k, \beta_k) = (-\infty, \infty) \subseteq (a, b)$.

Zasubstituuje

$$\begin{aligned} \int \frac{\sin^2 x}{1 + \sin^2 x} dx &\rightarrow \int \frac{\frac{t^2}{t^2+1}}{1 + \frac{t^2}{t^2+1}} \cdot \frac{1}{t^2+1} dt = \int \frac{t^2}{(1+t^2)(2t^2+1)} dt \\ &= \int \frac{1}{1+t^2} dt - \int \frac{1}{2t^2+1} dt \stackrel{C}{=} \arctan t - \frac{1}{\sqrt{2}} \arctan \sqrt{2}t \end{aligned}$$

$$x \in \left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right)$$

Tedy

$$G(x) = \arctan(\tan x) - \frac{1}{\sqrt{2}} \arctan(\sqrt{2} \tan x) + c_k, \quad x \in \left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right)$$

V bodech $\frac{\pi}{2} + k\pi$ je potřeba funkci slepit. Limity:

$$\lim_{x \rightarrow (\frac{\pi}{2} + \pi k)^-} \arctan(\tan x) - \frac{1}{\sqrt{2}} \arctan(\sqrt{2} \tan x) + c_k = \left(\frac{\sqrt{2}-1}{\sqrt{2}}\right) \frac{\pi}{2} + c_k$$

$$\lim_{x \rightarrow (\frac{\pi}{2} + \pi k)^+} \arctan(\tan x) - \frac{1}{\sqrt{2}} \arctan(\sqrt{2} \tan x) + c_{k+1} = -\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right) \frac{\pi}{2} + c_{k+1}.$$

tedy

$$c_{k+1} = \left(\frac{\sqrt{2}-1}{\sqrt{2}}\right) \pi + c_k.$$

Odtud

$$c_k = \left(\frac{\sqrt{2}-1}{\sqrt{2}}\right) k\pi + c_0, \quad k \in \mathbb{Z}.$$

Protože G i g jsou spojité na $(-\infty, \infty)$, můžeme použít lemma o lepení.

Závěr:

$$G(x) = \begin{cases} \arctan \tan x - \frac{1}{\sqrt{2}} \arctan \sqrt{2} \tan x + k\pi \frac{\sqrt{2}-1}{\sqrt{2}} + c_0, & x \in (-\pi/2 + k\pi; \pi/2 + 2\pi) \\ \frac{\pi}{2} \frac{\sqrt{2}-1}{\sqrt{2}} + k\pi \frac{\sqrt{2}-1}{\sqrt{2}} + c_0, & x = \frac{\pi}{2} + k\pi \end{cases}$$

$$5. \quad g(x) = \frac{1}{(1 - \cos^2 x)(1 + \cos^2 x)}$$

Řešení:

Podmínky: $\cos x \neq \pm 1, x \neq 0 + k\pi$.

Funkce g je spojitá na $(0 + k\pi, \pi + k\pi)$, má tam tedy PF.

Použijeme substituci $t = \cot x = \varphi(x)$. Funkce $f(t) = -\frac{1+t^2}{1+2t^2}$.

Intervaly: $(\alpha_k, \beta_k) = (0 + k\pi, \pi + k\pi)$, $(a, b) = (-\infty, \infty)$. Navíc $\varphi(\alpha_k, \beta_k) = (-\infty, \infty) \subseteq (a, b)$.

Zasubstituujeme

$$\begin{aligned} \int \frac{1}{(1 - \cos^2 x)(1 + \cos^2 x)} dx &\rightarrow \int \frac{1}{\frac{1}{1+t^2} \left(1 + \frac{t^2}{1+t^2}\right)} \frac{-1}{1+t^2} dt \\ &= - \int \frac{1+t^2}{1+2t^2} dt = - \int \frac{1}{2} + \frac{1}{2} \frac{1}{1+2t^2} dt \\ &\stackrel{C}{=} -\frac{1}{2}t - \frac{1}{2\sqrt{2}} \arctan(\sqrt{2}t) \\ &\rightarrow \int g(x) \stackrel{C}{=} -\frac{1}{2} \cot x - \frac{1}{2\sqrt{2}} \arctan(\sqrt{2} \cot x) \end{aligned}$$

$$x \in (0 + k\pi, \pi + k\pi)$$

Tedy

$$G(x) = -\frac{1}{2} \cot x - \frac{1}{2\sqrt{2}} \arctan(\sqrt{2} \cot x) + c_k, \quad x \in (0 + k\pi, \pi + k\pi)$$

Funkce se při této substituci nelepí.

$$6. \quad g(x) = \frac{1 + \sin x}{2 + \cos x}$$

Řešení: Zdroj příkladu: <https://matematika.cuni.cz/ikalkulus.html>

Funkce g je spojitá na $(-\infty, \infty)$, má tam tedy PF.

Použijeme substituci $t = \tan \frac{x}{2} = \varphi(x)$. Funkce $f(t) = \frac{2(1+t)^2}{(t^2+1)(t^2+3)}$.

Intervaly: $(\alpha_k, \beta_k) = (-\pi + 2k\pi, \pi + 2k\pi)$, $(a, b) = (-\infty, \infty)$. Navíc $\varphi(\alpha_k, \beta_k) = (-\infty, \infty) \subseteq (a, b)$.

Zasubstituujeme

$$\begin{aligned} \int \frac{1 + \sin x}{2 + \cos x} dx &= \int \frac{2(1+t)^2}{(t^2+1)(t^2+3)} dt = \int \frac{2}{3+t^2} + \frac{2t}{1+t^2} - \frac{2t}{3+t^2} dt \\ &\stackrel{C}{=} \frac{2}{\sqrt{3}} \arctan \frac{t}{\sqrt{3}} + \log(1+t^2) - \log(3+t^2) \\ &= \frac{2}{\sqrt{3}} \arctan \frac{t}{\sqrt{3}} + \log \frac{(1+t^2)}{(3+t^2)} \\ &\rightarrow \int g(x) \stackrel{C}{=} \frac{2}{\sqrt{3}} \arctan \frac{\operatorname{tg} \frac{x}{2}}{\sqrt{3}} + \log \frac{(1 + (\operatorname{tg} \frac{x}{2})^2)}{(3 + (\operatorname{tg} \frac{x}{2})^2)} \end{aligned}$$

$$x \in (-\pi + 2k\pi, \pi + 2k\pi)$$

Tedy

$$G(x) = \frac{2}{\sqrt{3}} \arctan \frac{\operatorname{tg} \frac{x}{2}}{\sqrt{3}} + \log \frac{(1 + (\operatorname{tg} \frac{x}{2})^2)}{(3 + (\operatorname{tg} \frac{x}{2})^2)} + c_k, \quad x \in (-\pi + 2k\pi, \pi + 2k\pi)$$

V bodech $\pi + 2k\pi$ je potřeba funkci slepit. Limity:

$$\begin{aligned} \lim_{x \rightarrow (\pi + 2\pi k)^-} \frac{2}{\sqrt{3}} \arctan \frac{\operatorname{tg} \frac{x}{2}}{\sqrt{3}} + \log \frac{(1 + (\operatorname{tg} \frac{x}{2})^2)}{(3 + (\operatorname{tg} \frac{x}{2})^2)} + c_k &= \frac{\pi}{\sqrt{3}} + c_k, \\ \lim_{x \rightarrow (\pi + 2\pi k)^+} \frac{2}{\sqrt{3}} \arctan \frac{\operatorname{tg} \frac{x}{2}}{\sqrt{3}} + \log \frac{(1 + (\operatorname{tg} \frac{x}{2})^2)}{(3 + (\operatorname{tg} \frac{x}{2})^2)} + c_{k+1} &= -\frac{\pi}{\sqrt{3}} + c_{k+1}. \end{aligned}$$

tedy

$$c_{k+1} = \frac{2\pi}{\sqrt{3}} + c_k.$$

Odtud

$$c_k = \frac{2\pi k}{\sqrt{3}} + c_0, \quad k \in \mathbb{Z}.$$

Protože G i g jsou spojité na $(-\infty, \infty)$, můžeme použít lemma o lepení.

Závěr:

$$G(x) = \begin{cases} \frac{2}{\sqrt{3}} \arctan \frac{\operatorname{tg} \frac{x}{2}}{\sqrt{3}} + \log \frac{(1+(\operatorname{tg} \frac{x}{2})^2)}{(3+(\operatorname{tg} \frac{x}{2})^2)} + k\pi \frac{2}{\sqrt{3}} + c_0 & x \in (-\pi + 2k\pi; \pi + 2k\pi) \\ \frac{\pi}{\sqrt{3}} + k\pi \frac{2}{\sqrt{3}} + c_0 & x = \pi + 2k\pi \end{cases}$$

7. $g(x) = \frac{\sin^2 x - \cos^2 x}{\sin^2 x + 4 \cos^2 x}$

Řešení: Zdroj příkladu: <https://matematika.cuni.cz/ikalkulus.html>

Funkce g je spojitá na $(-\infty, \infty)$, má tam tedy PF.

Použijeme substituci $t = \tan x = \varphi(x)$. Funkce $f(t) = \frac{t^2-1}{(t^2+1)(t^2+4)}$.

Intervaly: $(\alpha_k, \beta_k) = (-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi)$, $(a, b) = (-\infty, \infty)$. Navíc $\varphi(\alpha_k, \beta_k) = (-\infty, \infty) \subseteq (a, b)$.

Zasubstituujeme

$$\begin{aligned} \int \frac{\sin^2 x - \cos^2 x}{\sin^2 x + 4 \cos^2 x} dx &\rightarrow \int \frac{t^2 - 1}{(t^2 + 1)(t^2 + 4)} dt = \frac{5}{3} \frac{1}{4 + t^2} - \frac{2}{3} \frac{1}{1 + t^2} dt \\ &\stackrel{C}{=} \frac{5}{6} \arctan \frac{t}{2} - \frac{2}{3} \arctan t \\ &\rightarrow \int g(x) \stackrel{C}{=} \frac{5}{6} \arctan \frac{\tan x}{2} - \frac{2}{3} \arctan(\tan x) \end{aligned}$$

$$x \in (-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi)$$

Tedy

$$G(x) = \frac{5}{6} \arctan \frac{\tan x}{2} - \frac{2}{3} \arctan(\tan x) + c_k, \quad x \in (-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi)$$

V bodech $\frac{\pi}{2} + k\pi$ je potřeba funkci slepit. Limity:

$$\begin{aligned} \lim_{x \rightarrow (\frac{\pi}{2} + \pi k)^-} \frac{5}{6} \arctan \frac{\tan x}{2} - \frac{2}{3} \arctan(\tan x) + c_k &= \frac{\pi}{12} + c_k \\ \lim_{x \rightarrow (\frac{\pi}{2} + \pi k)^+} \frac{5}{6} \arctan \frac{\tan x}{2} - \frac{2}{3} \arctan(\tan x) + c_{k+1} &= -\frac{\pi}{12} + c_{k+1} \end{aligned}$$

tedy

$$c_{k+1} = \frac{\pi}{6} + c_k.$$

Odtud

$$c_k = \frac{\pi}{6}k + c_0, \quad k \in \mathbb{Z}.$$

Protože G i g jsou spojité na $(-\infty, \infty)$, můžeme použít lemma o lepení.

Závěr:

$$G(x) = \begin{cases} \frac{5}{6} \arctan \frac{\tan x}{2} - \frac{2}{3} \arctan(\tan x) + c_k, & x \in (-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi) \\ \frac{\pi}{12} + \frac{\pi}{6}k + c_0, & x = \frac{\pi}{2} + k\pi \end{cases}$$

8. $g(x) = \frac{1}{\sin x + 2}$

Řešení: Zdroj příkladu: Petr Holický, Ondřej F.K. Kalenda : Metody řešení vybraných úloh z matematické analýzy pro 2. - 4. semestr

Funkce g je spojitá na $(-\infty, \infty)$, má tam tedy PF.

Použijeme substituci $t = \tan \frac{x}{2} = \varphi(x)$. Funkce $f(t) = \frac{1}{t^2+t+1}$

Intervaly: $(\alpha_k, \beta_k) = (-\pi + 2k\pi, \pi + 2k\pi)$, $(a, b) = (-\infty, \infty)$. Navíc $\varphi(\alpha_k, \beta_k) = (-\infty, \infty) \subseteq (a, b)$.

Zasubstituuje

$$\begin{aligned} \int \frac{1}{\sin x + 2} dx &\rightarrow \int \frac{1}{t^2+t+1} dt \stackrel{C}{=} \frac{2}{\sqrt{3}} \arctan \left(\frac{2}{\sqrt{3}}t + \frac{1}{\sqrt{3}} \right) \\ &\rightarrow \int g(x) \stackrel{C}{=} \frac{2}{\sqrt{3}} \arctan \left(\frac{2}{\sqrt{3}} \tan \frac{x}{2} + \frac{1}{\sqrt{3}} \right) \end{aligned}$$

$$x \in (-\pi + 2k\pi, \pi + 2k\pi)$$

Tedy

$$G(x) = \frac{2}{\sqrt{3}} \arctan \left(\frac{2}{\sqrt{3}} \tan \frac{x}{2} + \frac{1}{\sqrt{3}} \right) + c_k, \quad x \in (-\pi + 2k\pi, \pi + 2k\pi)$$

V bodech $\pi + 2k\pi$ je potřeba funkci slepit. Limity:

$$\begin{aligned} \lim_{x \rightarrow (\pi+2\pi k)^-} \frac{2}{\sqrt{3}} \arctan \left(\frac{2}{\sqrt{3}} \tan \frac{x}{2} + \frac{1}{\sqrt{3}} \right) + c_k &= \frac{\pi}{\sqrt{3}} + c_k, \\ \lim_{x \rightarrow (\pi+2\pi k)^+} \frac{2}{\sqrt{3}} \arctan \left(\frac{2}{\sqrt{3}} \tan \frac{x}{2} + \frac{1}{\sqrt{3}} \right) + c_{k+1} &= -\frac{\pi}{\sqrt{3}} + c_{k+1}. \end{aligned}$$

tedy

$$c_{k+1} = \frac{2\pi}{\sqrt{3}} + c_k.$$

Odtud

$$c_k = \frac{2\pi k}{\sqrt{3}} + c_0, \quad k \in \mathbb{Z}.$$

Protože G i g jsou spojité na $(-\infty, \infty)$, můžeme použít lemma o lepení.

Závěr:

$$G(x) = \begin{cases} \frac{2}{\sqrt{3}} \arctan \left(\frac{2}{\sqrt{3}} \tan \frac{x}{2} + \frac{1}{\sqrt{3}} \right) + k\pi \frac{2}{\sqrt{3}} + c_0 & x \in (-\pi + 2k\pi; \pi + 2k\pi) \\ \frac{\pi}{\sqrt{3}} + k\pi \frac{2}{\sqrt{3}} + c_0 & x = \pi + 2k\pi \end{cases}$$