(54) $\sum_{n=1}^{\infty} (n^{n^{\alpha}} - 1)$

$$a_n = \exp(n^{\alpha} \ln n) - 1 = \frac{\ln n}{n^{-\alpha}} + o\left(\frac{\ln n}{n^{-\alpha}}\right)$$

$$b_n = \frac{\ln n}{n^{-\alpha}}$$

$$\lim \frac{a_n}{b_n} = 1$$

$$\text{z} \iff \alpha < -1$$

ukázka

(5'2) $\displaystyle\sum_{n=1}^{\infty} \left(n^{\frac{1}{n^2+1}} - 1 \right)$

$$a_n = \exp\left\{ \frac{\ln n}{n^2+1} \right\} - 1 = \frac{\ln n}{n^2+1} + o\left(\frac{\ln n}{n^2} \right)$$

srovnání s $b_n = \dfrac{\ln n}{n^2}$ $\quad \underset{0}{\nabla}$

$$\lim \frac{a_n}{b_n} = \lim \frac{n^2}{n^2+1} + \frac{o\left(\frac{\ln n}{n^2}\right)}{\frac{\ln n}{n^2}} = 1$$

K

(3) $\sum_{n=1}^{\infty} \sin\frac{1}{n} - \arcsin\frac{1}{n}$

$$a_n = \cancel{\frac{1}{n}} - \frac{1}{n^3}\cdot\frac{1}{6} + o\left(\frac{1}{n^3}\right)$$
$$- \left(\cancel{\frac{1}{n}} + \frac{2}{4\cdot 3}\,\frac{1}{n^3} + o\left(\frac{1}{n^3}\right)\right)$$
$$= -\left(\frac{1}{6} + \frac{2}{\cancel{4}6}\right)\cdot\frac{1}{n^3} + o\left(\frac{1}{n^3}\right)$$

AZ: $b_n = \frac{1}{n^3}$

$\underline{\underline{\sum\ \mathcal{L}}}$

$$\lim \frac{\left|-\frac{3}{6}\frac{1}{n^3} + o\left(\frac{1}{n^3}\right)\right|}{\frac{1}{n^3}}$$

$\lim a_n = A \Rightarrow \lim |a_n| = |A|$

(37) $a_n = \log_{b^n}\left(1 + \frac{\sqrt[n]{a}}{n}\right) \qquad a>0,\ b>0,\ b\neq 1$

$$\log_{b^n} x = \frac{\ln_e x}{\ln b^n} = \frac{1}{\ln b^n}\cdot\frac{\ln x}{n}$$

$$a_n = \frac{\ln\left(1+\frac{\sqrt[n]{a}}{n}\right)}{n}\cdot\frac{1}{\ln b}$$

$$= \frac{1}{\ln b}\cdot\frac{1}{n}\left[\left(\frac{\sqrt[n]{a}}{n}\right) - \cancel{\frac{1}{2}\frac{(\sqrt[n]{a})^2}{n^2}} + o\left(\left(\frac{\sqrt[n]{a}}{n}\right)^{*}\right)\right]$$

$$\frac{1}{\ln b}\sum_{n=0}^{\infty} \frac{\sqrt[n]{a}}{n^2} + \frac{1}{n}\,o\left(\frac{\sqrt[n]{a}}{n}\right)$$

LSK

sroomaime s $b_n = \frac{1}{n^2}$

$$\lim \underset{\substack{\downarrow\\1}}{\sqrt[n]{a}} + \frac{\frac{1}{n}\cdot o\left(\frac{\sqrt[n]{a}}{n}\right)}{\frac{1}{n^2}} = 1$$

$$\lim \frac{n^2}{n^4}\,\frac{o\left(\frac{\sqrt[n]{a}}{n}\right)}{\underset{\substack{\downarrow\\0}}{\frac{\sqrt[n]{a}}{n}}}\cdot \underset{\substack{\downarrow\\1}}{\sqrt[n]{a}}$$

(Z)

$$(1)\ \sum_{n=1}^{\infty} \left(\sin\frac{1}{n} - \frac{1}{n}\right)\frac{1}{n^{\alpha}}$$

$$\sin\frac{1}{n} = \frac{1}{n} - \frac{1}{n^3}\cdot\frac{1}{6} + o\left(\frac{1}{n^3}\right)$$

tedy

$$a_n = \left[-\frac{1}{6n^3} + o\left(\frac{1}{n^3}\right)\right]\cdot\frac{1}{n^{\alpha}}$$

vytkneme " – " $\qquad b_n = \frac{1}{n^{\alpha}\cdot n^3}$

$$\lim \frac{a_n}{b_n} = \underline{\underline{\frac{1}{6}}} + 0$$

$$\sum k \iff \alpha + 3 > 1$$

$$\boxed{\alpha > -2}$$

$$\lim_{n\to\infty} \frac{\left(\frac{1}{6}\cdot\frac{1}{n^3} + o\left(\frac{1}{n^2}\right)\right)\cdot\frac{1}{n^{\alpha}}}{\frac{1}{n^3}\cdot\frac{1}{n^{\alpha}}} = \frac{1}{6} + 0$$

(36) $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt{n+1}-\sqrt{n}\right)^p \ln \frac{n-1}{n+1}$

$$a_n = \frac{1}{\left(\sqrt{n+1}+\sqrt{n}\right)^p} \cdot \ln\left(1-\frac{2}{n+1}\right)$$

$$= \frac{1}{\sqrt{n+1}+\sqrt{n}} \cdot \left(-\frac{2}{n+1} + o\left(-\frac{2}{n+1}\right)\right)$$

$$b_n = \frac{1}{(n^{1/2})^p} \cdot \frac{1}{n+1}$$

vytknout "–"

$$\lim \frac{a_n}{b_n} = \frac{\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right)^p}{\left(\frac{1}{\sqrt{n}}\right)^p} \left[\frac{+\frac{2}{n+1}}{1/n+1} + \frac{o\left(-\frac{2}{n+1}\right)}{\frac{1}{n+1}}\right]$$

$$= \frac{1}{2^p}\left(2+0\right) = 1$$

$$\sum \rightsquigarrow \iff \sum \frac{1}{(n+1)\cdot n^{\frac{p}{2}}} \rightsquigarrow$$

$$\iff \underline{\underline{p > 0}}$$

(56) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{1}{n^\alpha}\right) - \ln\left(\sin\frac{1}{n^\alpha}\right)$

$$a_n = -\ln\left(n^\alpha \sin\frac{1}{n^\alpha}\right) =$$

$$= -\ln n^\alpha\left(\frac{1}{n^\alpha} - \frac{1}{6n^{3\alpha}} + o\left(\frac{1}{n^{3\alpha}}\right)\right)$$

~~[illegible]~~

$$= -\ln\left(1 - \frac{1}{6n^{2\alpha}} + n^\alpha o\left(\frac{1}{n^{3\alpha}}\right)\right)$$

$$= -\left[-\frac{1}{6}\cdot\frac{1}{n^{2\alpha}} + n^\alpha o\left(\frac{1}{n^{3\alpha}}\right) + o\left(-\frac{1}{6n^{2\alpha}} + n^\alpha o\left(\frac{1}{n^{3\alpha}}\right)\right)\right]$$

LSK $\quad b_n = \frac{1}{n^{2\alpha}}$

$$\lim \frac{a_n}{b_n} = \lim \underbrace{-\frac{1}{6}}_{-\frac{1}{6}} + \underbrace{\frac{\cancel{n^\alpha}\, o\left(\frac{1}{n^{3\alpha}}\right)}{\cancel{n^\alpha}\,\frac{1}{n^{3\alpha}}}}_{0} + \cancel{o\left(\frac{(\quad)}{(\quad)}\right)}$$

$$\bullet \quad \cancel{\frac{-\frac{1}{6n^{2\alpha}} + n^\alpha o\left(\frac{1}{n^{3\alpha}}\right)}{\frac{1}{n^{2\alpha}}}}$$

z toho máme, že $o\left(-\frac{1}{6n^{2\alpha}} + n^\alpha o\left(\frac{1}{n^{3\alpha}}\right)\right)$

je též $o\left(\frac{1}{n^{2\alpha}}\right)$.

Tedy $\sum$ k $\iff \alpha > \frac{1}{2}$.

(2) $\sum_{n=1}^{\infty} 2\,\mathrm{tg}\left(\frac{1}{n^{1/5}}\right) - \sin\left(\frac{1}{n^{1/5}}\right) - \frac{1}{n^{3/5}}$

$$a_n = 2\left(\frac{1}{n^{1/5}} + \frac{1}{3}\cdot\frac{1}{n^{3/5}} + o\left(\frac{1}{n^{3/5}}\right)\right)$$
$$- \left(\frac{1}{n^{1/5}} - \frac{1}{6}\cdot\frac{1}{n^{3/5}} + o\left(\frac{1}{n^{3/5}}\right)\right)$$
$$- \frac{1}{n^{3/5}}$$

$$= \frac{1}{n^{1/5}} + \frac{1}{n^{3/5}}\underbrace{\left(\frac{2}{3} + \frac{1}{6} - 1\right)}_{-1/6} + o\left(\frac{1}{n^{3/5}}\right)$$

$b_n = \frac{1}{n^{1/5}}$ Ak:

$$\lim \frac{a_n}{b_n} = \lim 1 + \frac{1}{6}\,\frac{n^{1/5}}{n^{3/5}} + \frac{o\left(\frac{1}{n^{3/5}}\right)}{\frac{1}{n^{3/5}}}\cdot\frac{\frac{1}{n^{3/5}}}{\frac{1}{n^{1/5}}}$$

$$= \lim 1 + \frac{1}{6}\underbrace{\frac{1}{n^{2/5}}}_{\to 0} + \underbrace{\cancel{a_n}}_{0}\cdot\underbrace{\quad}_{0} = 1$$

$\sum k \iff \sum \cancel{\frac{1}{n^{1/5}}}\, b_n\, k$

$$\sum \frac{1}{n^{1/5}} \; D$$