Computational mathematics
Master's degree - State Final Examination

This document summarizes everything relevant to the state final exams in the Master's degree programme Computational mathematics. The state final exam consists of two parts:

- Master's thesis defense
- Oral exam

Thesis defense

First, the thesis defenses will take place. Each student will give a presentation about their work of approximately 15 minutes, either from their own laptop or they can send their presentation to the committee chair. Presentations will be given in alphabetical order. After each presentation, the reports of the supervisor and the opponent will be read, with the student answering their questions or comments. Then there will be a general discussion, followed by a closed committee meeting. Its chairman then informs the student of the grade.

Oral exam

The defenses are followed by an oral exam. The student is given three questions selected from the five topics below (each question from a different topic). The content of these topics is covered by compulsory lectures, which are listed for each topic. The student has 45 minutes to prepare all three questions. Subsequently, each question will be examined by a pair of examiners, approximately 20 minutes per question. After all three questions have been examined, there is a closed meeting of the committee, after which the grades are communicated to the students.

The purpose of the exam is not to demonstrate detailed technical knowledge, complete proofs, etc. The students have already demonstrated this in the exams for individual subjects. Here, students are expected to demonstrate a broader overview, an understanding of basic ideas and, a broader context and connections to other areas. The list of topics is attached below.

1. Partial differential equations

- NMMA405 – Partial differential equations 1,
- NMNV401 – Functional analysis,
- NMNV406 – Nonlinear differential equations

Sobolev spaces: Weak derivative, Sobolev spaces – definition and basic properties (reflexivity, density of smooth functions). Trace theorem, theorems on continuous and compact embeddings.

Linear Elliptic Equations: Weak problem formulation for a linear elliptic equation with various boundary conditions, ellipticity, Lax-Milgram theorem. Equivalence of the problem with the minimization of the quadratic functional. Regularity of the solution.

Linear evolution equation: Bochner spaces - definition, duals, weak time derivative. Weak formulation of second-order parabolic and hyperbolic equations. The basic idea of the existence and uniqueness of solutions using the Galerkin approximation.

Functional analysis: Spectral analysis of symmetric linear operators in Hilbert spaces. Compact symmetric operators (spectral properties, expression in the form of a series, decomposition of the identity). Spectral analysis of continuous linear operators in Banach spaces (openness of the resolvent set, resolvent operator in the form of a series and its derivatives).

Nonlinear differential equations: Monotone, coercive and potential operators, basics of the theory of existence and uniqueness of solutions. Němycky operators. Application to partial differential equations in divergence form.

2. Finite element method

- NMNV405 – Finite element method 1
- NMNV406 – Nonlinear differential equations

Theory of finite elements: Galerkin method for solving linear elliptic equation, Céa's lemma. Definition of an abstract finite element, simple examples of finite elements of Lagrangian and Hermitian type. Affine equivalence. Approximation theory in Sobolev spaces: approximation properties of operators preserving polynomials, Bramble-Hilbert lemma.

Application to linear elliptic equations: Estimation of the finite element method error in H1 and L2 norms, Aubin-Nitsche lemma. Nonhomogeneous boundary conditions.

Application to non-linear equations: Application of the finite element method to the solution of non-linear equations in divergence form.

3. Numerical linear algebra

- NMNV411 – Algorithms for matrix iterative methods,
- NMNV412 – Analysis of matrix iterative methods - principles and interconnections

Principles and algorithms: Krylov subspaces and the projection process, methods for solving systems of linear algebraic equations with a symmetric matrix (CG, MINRES, SYMMLQ) and nonsymmetric matrix (GMRES, FOM, BiCG). Methods for solving the least squares problem (CGLS, LSQR).

Analysis and interconnections: Convergence behavior of methods for symmetric, normal and general matrices. The effect of finite-precision arithmetic. Stopping criteria. Projection process and the problem of moments. Connections with orthogonal polynomials and Gauss-Christoffel quadrature. Conjugate gradients for functional equations in Hilbert spaces.

4. Adaptive discretization methods

- NMNV403 – Numerical software 1

Adaptive methods for numerical integration, half-step error estimation, Gauss-Kronrod quadrature formulas, local and global adaptive strategies.

Adaptive methods for ordinary differential equations, global and local errors, local error estimates, Runge-Kutta-Fehlberg methods, adaptive choice of time step.

5. Numerical optimization methods

- NMNV503 – Numerical optimization methods 1

Nonlinear equations and their systems: Iteration of functions, order of convergence, conditions for convergence to a fixed point and conditions for higher order convergence. Newton's method, Kantorovich local convergence theorem. Approximation of the derivative in Newton's method, secant method, convergence theorem (without proof). Quasi-Newton methods (Broyden), construction and properties. Continuation (homotopy) methods, basic idea.

Minimization of a functional: The idea of descent direction methods, the task of finding a suitable approximation in a given direction (Armijo and Wolfe conditions), the steepest descent method, Newton's method, conjugate direction methods and the nonlinear conjugate gradient method, quasi-Newtonian methods. The idea of trust region methods, the task of finding a suitable approximation on the trust region using the dogleg strategy or using the Steihaug version of the conjugate gradient method. Convergence theorems for the considered methods (without proofs).