FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

MASTER THESIS

Štěpán Hudeček

Symplectic Dirac Operators on $Gr_2(\mathbb{C}^4)$

Mathematical Institute of Charles University

Supervisor of the master thesis: doc. RNDr. Svatopluk Krýsl, Ph.D.
Study programme: Mathematics
Study branch: Mathematical structures

Prague 2022

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In Prague date 5.5.2022 Štěpán Hudeček

I would like to thank doc. RNDr. Svatopluk Krýsl, Ph.D. for his great advice and for his patience.
I would also like to thank my parents, for their continuous support during my studies.

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Author: Štěpán Hudeček

Institute: Mathematical Institute of Charles University

Supervisor: doc. RNDr. Svatopluk Krýsl, Ph.D., Mathematical Institute of Charles University

Abstract: In this thesis we are presenting a construction of the symplectic Dirac operators as done by Katharina Habermann in 1995. We emphasize the differences with the classical Dirac operators.
We are then computing the associated second order operator to the symplectic Dirac operators on the Kähler symmetric space $Gr_2(\mathbb{C}^4)$. We have also managed to find a way of inductive computing of its spectrum and we are presenting explicitly a part of the spectrum.

Keywords: Spectrum symplectic Dirac operator Grassmanian symmetric space Weyl algebra differential operator Lie groups associated bundles

Contents

Introduction

The study of Spin structures and classical Riemannian and pseudo-Riemannian Dirac operators acting on them is a well-established area of mathematical research. It was mainly motivated by physics but it soon gained its popularity within the mathematical community. It has been shown multiple times that the Dirac operators, respectively their spectrum, contain information about the Riemannian geometry and topology of the underlying manifold see [Friedrich, 2000] or Baum [1981]

The symplectic Dirac operators are analogues of the classical Riemannian Dirac operators, but constructed in the framework of symplectic geometry. They have been firstly introduced by Katharina Habermann in 1995 [Habermann, 1995] with the idea that they may be another tool for studying symplectic geometry/topology. However, there are several differences to the classical case.

In the classical case, it is required for the manifold to have a Spin structure, that is a lift of the structure group of the tangent bundle to a Spin group which is a connected double cover of the group $SO(n, \mathbb{R})$. The reduction of the structure group to the group $O(n, \mathbb{R})$ is, of course, always present. In the symplectic case, one needs to start with the symplectic manifold (i.e. reduction to $Sp(2n, \mathbb{R})$ and some integrability condition) and then one requires a lift to the Metaplectic group which is the connected double covering of the symplectic group. A small difference is that the Spin group is actually simply-connected while the metaplectic group has $\mathbb{Z}$ as its fundamental group. However, when manifolds admit a symplectic structure, the conditions on existence of metaplectic and spin structures are equivalent.

Next step is to construct the space of spinors - space on which the Dirac operators act. In the classical case, the choice is clear, it is the associated bundle via the Spin representation - representation that does not exist for $SO(n, \mathbb{R})$. In the symplectic case, one takes the metaplectic representation, which is (under some constrains) a unique representation of the metaplectic group. Here the difference is much more noticeable since the Riemannian spinor bundle is a bundle with finite-dimensional fibers while the symplectic spinor bundle is a bundle with the space $L^2(\mathbb{R}^n)$ as the typical fiber.

Lastly one constructs the Dirac operators themselves using the connection compatible with the Riemannian metric and the Clifford multiplication of the Clifford algebra on the space of spinors. In the symplectic case, one usually does not have a clear choice of a connection since there are several torsion free connections compatible with the symplectic structure. However, sometimes there are other circumstances that can make one's choice of a connection more privileged. Last difference is the discrepancy in the Clifford algebrae. The symplectic Clifford algebra (the Weyl algebra) is constructed using the symplectic form and thus, leads again to an infinite-dimensional algebra in contrary to the standard Clifford algebra that is finite-dimensional.

Computing the spectrum of the Dirac operators on symmetric spaces G/H has been done using the Parthasarathy formula [Bourguignon et al., 2015, Proposition

15.7.] which states that

$$D^2(A) = A \circ \Omega + \frac{\text{Scal}}{8} A$$

where Scal is the scalar curvature of the symmetric space and Ω is a representation of a Casimir element of the Lie algebra of G.

Nevertheless, such a general formula probably does not hold for the symplectic Dirac operators. Habermann has computed the associated second order operator $\mathcal{P}$ (similar to D^2 in Riemanninan case) on a Riemannian sphere [Habermann and Habermann, 2006] and showed that it is

$$\mathcal{P}(\varphi) = -\Omega(\varphi) - 12H_0^2(\varphi)$$

where H_0 is quantum Hamiltonian of the harmonic oscillator and Ω is again a representation of the Casimir element. Wyss then computed $\mathcal{P}$ generally on the odd-dimensional complex projective spaces $\mathbb{C}P^n$ [Wyss, 2003], where it holds that

$$\mathcal{P}(\varphi) = -\Omega(\varphi) - 12H_0^2(\varphi) - \frac{3n(n-1)}{2}\varphi.$$

In this thesis we try to continue the effort and compute the operator $\mathcal{P}$ on the Grassmannian $Gr_2(\mathbb{C}^4)$ where we found that

$$\mathcal{P}(\varphi) = -\Omega(\varphi) - 12H_0^2(\varphi) + 12\varphi - \Phi \cdot \varphi$$

where the operator $\Phi\cdot$ is a bit more complicated to define. The proper definition is given at the chapter 4 of this thesis.

Next we have computed a part of the spectra of this operator but we have not yet found a nice expression (in contrary to the work of Habermann and Wyss) that describes the whole spectrum. However, we have found an iterative method of finding eigenvalues one by one. Our hope is to continue this work and find the all encompassing expressions.

In the first chapter we present basic definitions and properties of the objects of interests. Namely, symplectic vector spaces, Weyl algebra, symplectic Clifford multiplication, the Metaplectic group and its representation and symplectic manifold with connections on them.

The second chapter consists of providing more details on the construction of the symplectic Dirac operators hinted above.

The third chapter serves as a small introduction to homogeneous and symmetric spaces. Then we present the Grassmannian $Gr_2(\mathbb{C}^4)$ as a Kähler symmetric space and we prove that it has a unique metaplectic structure.

The last chaper is the main computation of the associated second order operator $\mathcal{P}$. It shows the splitting of the operator into the expression written above and the chapter concludes with the computation of a part of the point spectrum of the operator. Throughout the computation of the spectrum, it is hinted how the general inductive procedure works.

1. Symplectic Clifford Algebra and Symplectic manifolds

1.1 Symplectic Clifford algebra

Let us start by making some basic definitions. We follow Crumeyrolle [1990]

Definition 1.1.1 (Symplectic vector space)**.** *Given a vector space V over a field $\mathbb{F}$ where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$ and a bilinear form on this vector space $\omega : V \times V \to \mathbb{F}$. We say that (V, ω) is a symplectic vector space if ω is skew-symmetric and non-degenerate. We say that ω is the symplectic form.*

Remark. Let us note that this implies that if V is finite-dimensional then it must be even dimensional. Consider vector space V of dimension n and let us choose its basis B. With respect to this basis the symplectic form must have a skew-symmetric matrix A. Now let us look at the determinant $\det(A) = \det(A^T) = \det(-A) = (-1)^n \det(A)$. This clearly yields $\det(A) = -\det(A)$ and thus $\det(A) = 0$ for odd-dimensional vector space violates the non-degeneracy condition.

Definition 1.1.2 (Symplectic basis)**.** *Consider a symplectic vector space (V, ω) and its basis $(a_1, \ldots, a_n, b_1 \ldots, b_n)$. We say that this basis is symplectic if*

$$\omega(a_i, a_j) = \omega(b_i, b_j) = 0 \qquad \textit{and} \qquad \omega(a_i, b_i) = 1 \qquad \textit{for } 0 \leq i, j \leq n.$$

Example 1.1.1 (Standard symplectic vector space)*.* Consider the vector space $\mathbb{R}^{2n}$ and a matrix

$$J_0 := \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

then the standard symplectic structure is defined as $\omega_0(v, w) := \langle J_0 v, w \rangle$ where $\langle \cdot, \cdot, \rangle$ is the standard Euclidean inner product.

This example shows how does the symplectic form look like with respect to the symplectic basis.

Given two symplectic vector spaces (V_1, ω_1) and (V_2, ω_2) we can define a notion of symplectomorphism as a linear isomorphism $f : V_1 \to V_2$ such that $\omega_1 = f^* \omega_2$ where the last equation means that for all $v, w \in V_1$ it holds that $\omega_1(v, w) = \omega_2(f(v), f(w))$. Thus finding a symplectic basis of (V, ω) is actually equivalent to finding a symplectomorphism $f : (V, \omega) \to (\mathbb{R}^{2n}, \omega_0)$. Moreover, it can be shown that each finite-dimensional symplectic vector space is symplectomorphic to the standard symplectic vector space, i.e., each symplectic vector space has a symplectic basis.

Now we shall define a symplectic Clifford algebra, sometimes also called the Weyl algebra and describe some of its basic properties.

Definition 1.1.3 (Generalised Symplectic Clifford algebra)**.** *Let V be a vector space over a field $\mathbb{F}$ and let $\eta : V \times V \to \mathbb{F}$ be a skew-symmetric form. Let us* *consider the tensor algebra $\mathcal{T}(V)$ and its two-sided ideal $\mathcal{I}$ generated by the set $\{x \otimes y - y \otimes x + \eta(x, y) | x, y \in V\}$. We define the symplectic Clifford algebra as the quotient of the tensor algebra by this ideal, i.e. $Cl(V, \eta) := \mathcal{T}(V)/\mathcal{I}$.*

Remark. Note that in Crumeyrolle [1990] the ideal $\mathcal{I}$ is defined with a different sign, i.e. it is generated by the set $\{x \otimes y - y \otimes x - \eta(x, y) | x, y \in V\}$. But these constructions are the same, it is enough to define $\eta' = -\eta$.

Compare this to the orthogonal Clifford algebra where the two-sided ideal is generated by the set $x \otimes x = B(x, x)$ where B is a symmetric form. If we choose $B = 0$ we obtain the algebra of skew-symmetric tensors. However, if in the symplectic case we choose $\eta = 0$ we obtain the algebra of symmetric tensors on V, i.e. $Cl(V, 0) = \mathcal{S}(V)$. In general, it holds that the standard Clifford algebra is finite-dimensional while the symplectic one is infinite dimensional.

Since there is a canonical injection $i : V \to \mathcal{T}(V)$, we can compose it with the projection $\pi : \mathcal{T}(V) \to Cl(V, \eta)$ to obtain a morphism $\iota : V \to Cl(V, \eta)$. Since $i(V)$ generates $\mathcal{T}(V)$ as an algebra, we have that $\iota(V)$ generates $Cl(V, \eta)$ as an algebra.

Symplectic Clifford algebra can be also defined via the universal property as is shown in the next proposition.

Proposition 1.1.1 (Universal property of symplectic Clifford algebra)**.** *Let (V, η) be a vector space over $\mathbb{F}$ with a skew-symmetric form η and let A be an associative $\mathbb{F}$-algebra with $\cdot$ as a multiplication. Then for a morphism $F : V \to A$ satisfying*

$$F(x) \cdot F(y) - F(y) \cdot F(x) = -\eta(x, y), \qquad \text{for all } x, y \in V$$

there exists a unique algebra homomorphism $\hat{F} : Cl(V, \eta) \to A$ such that $\hat{F} \circ \iota = F$

Proof. Existence. We can use the universal property of the tensor algebra to obtain a unique algebra homomorphism $F_0 : \mathcal{T}(V) \to A$ such that $F_0 \circ i = F$ where i is the canonical injection of vector space into its tensor algebra. Then from $F_0(x \otimes y - y \otimes x + \eta(x, y)) = 0$, we get that $F_0(\mathcal{I}) = 0$. Thus we can factor F_0 through the projection map $\pi : \mathcal{T}(V) \to Cl(V, \eta)$ to obtain $F_0 = \hat{F} \circ \pi$ where $\hat{F}$ is the map we had to construct.

Uniqueness. Because of the relation $\hat{F} \circ \iota = F$, we have at most one choice on how to define $\hat{F}$ on $\iota(V)$ and since $\iota(V)$ generates $Cl(V, \eta)$ it gives the uniqueness of $\hat{F}$.

□

It can be shown Crumeyrolle [1990] that the symplectic Clifford algebra is isomorphic as a vector space to the symmetric algebra $\mathcal{S}(V)$. Thus the canonical morphism $\iota : V \to Cl(V, \eta)$ is actually injective. Furthermore also the composed morphism $\mathbb{F} \to \mathcal{T}(V) \xrightarrow{\pi} Cl(V, \eta)$ is injective as well. Therefore, we will identify elements of the field $\mathbb{F}$ and of the vector space V with appropriate elements from the Clifford algebra.

From now onward we will work only with the real symplectic Clifford algebra with a symplectic form as the skew-symmetric form. If V is a symplectic vector space we will write simply $Cl(V)$ instead of $Cl(V, \omega)$.

Let us make a remark about the Heisenberg group and its Lie algebra.

Suppose that we have a symplectic vector space (V, ω). Let us define a Heisenberg group $H(2n)$ as $V \times \mathbb{R}$ with multiplication defined by

$$(x, t) \cdot (y, s) := (x + y, t + s + \frac{1}{2}\omega(x, y)),$$

where $(x, s), (y, t) \in V \times \mathbb{R}$. It can be shown Folland [1989] that its Lie algebra can be identified with $\mathfrak{h}(2n) := V \times \mathbb{R}$ with the obvious addition and the Lie bracket given by

$$[(x, t), (y, s)] = \omega(x, y)(0, 1),$$

for any $(x, t), (y, s) \in V \times \mathbb{R}$. Define a map $f : V \times \mathbb{R} \to \mathcal{T}(V)$ by $f(x, t) = x - t1$ where $x \in V, t \in \mathbb{R}$ and the 1 is the unit of the tensor algebra. This morphism is clearly linear and since $\mathcal{T}(V)$ is an algebra there is a unique morphism $\varphi : \mathcal{T}(\mathfrak{h}) \to \mathcal{T}(V)$ such that $\varphi \circ i = f$ where i is the canonical inclusion $i : \mathfrak{h} \to \mathcal{T}(\mathfrak{h})$. For $(x, t), (y, s) \in \mathfrak{h} = V \times \mathbb{R}$, $(x, t) \otimes (y, s) - (y, s) \otimes (x, t) - [(x, t), (y, s)] = (x, 0) \otimes (y, 0) + (ty, 0) + (sx, 0) + st1 - (y, 0) \otimes (x, 0) - (sx, 0) - (ty, 0) - st1 - \omega(x, y)1 = (x, 0) \otimes (y, 0) - (y, 0) \otimes (x, 0) - \omega(x, y)1 \overset{\varphi}{\mapsto} x \otimes y - y \otimes x + \omega(x, y)$.

That is the ideal K generated by the elements $a \otimes b - b \otimes a - [a, b]$ where $a, b \in \mathfrak{h}$, is sent into the ideal $\mathcal{I}$. Therefore, the morphism φ factors to the universal enveloping algebra [see Dixmier and Society, 1996] : $\hat{\varphi} : U(\mathfrak{h}) \to Cl(V)$. Since φ is surjective so is the $\hat{\varphi}$ and thus we see that the symplectic Clifford algebra is just the factor of the enveloping algebra of the Heisenberg algebra.

It can actually be shown that the universal enveloping algebra $U(\mathfrak{h})$ is, up to identification of the new coordinate with the negative unit of the algebra, isomorphic as a real associative algebra to the symplectic Clifford algebra $Cl(V)$, but we will not need that.

If (V, ω) is a $2n$-dimensional symplectic vector space with a symplectic basis $(a_1, \ldots, a_n, b_1, \ldots, b_n)$, then $Cl(V)$ has basis Habermann and Habermann [2006]

$$a_1^{\alpha_1} \cdot \ldots \cdot a_n^{\alpha_n} b_1^{\beta_1} \cdot \ldots \cdot b_n^{\beta_n},$$

where α_i, β_i are non-negative integers. Here we consider a_i, b_i as elements of $V \subseteq Cl(V)$ which is included via the monomorphism ι. Furthermore for $v, w \in Cl(V)$ we can define the bracket $[v, w] := v \cdot w - w \cdot v$ which turns the real algebra $Cl(V)$ into a real Lie algebra.

1.2 Symplectic and Metaplectic group

Definition 1.2.1 (Symplectic group)**.** *Given a standard symplectic vector space $(\mathbb{R}^{2n}, \omega_0)$, we define the group of all linear automorphisms that preserve the symplectic structure as the symplectic group $Sp(2n, \mathbb{R})$.*

This group is thus the set of all automorphisms $A \in \mathrm{Aut}(\mathbb{R}^{2n}) \cong GL(2n, \mathbb{R})$ such that for all $v, w \in \mathbb{R}^{2n}$ it holds that

$$\omega_0(Av, Aw) = \omega_0(w, v).$$

If we consider a matrix J_0 from the example 1.1.1 the symplectic group is also the set of matricies A such that

$$AJ_0A^T = J_0.$$

If we choose any $2n$-dimensional real symplectic vector space (V, ω), its automorphism group (subset of all linear automorphisms preserving the symplectic structure) will be denoted by $Sp(V)$. It is isomorphic to the symplectic group $Sp(2n, \mathbb{R})$.

Note that the symplectic group can be identified with the set of all symplectic bases. This is due to the fact that since the elements of the symplectic group preserves the symplectic structure we can choose a symplectic basis B and obtain all others as $f(B)$ where $f \in Sp(V)$.

A known result, whose proof can be found for example in Folland [1989] states that

Proposition 1.2.1 (Topological properties of Symplectic group). *(i) $U(n)$ is a maximal compact subgroup of $Sp(2n, \mathbb{R})$*

(ii) $Sp(2n, \mathbb{R})$ is homeomorphic to the product $U(n) \times \mathbb{R}^{n^2+n}$

Since it is well-known that the fundamental group of $U(n)$ is isomorphic to $\mathbb{Z}$, we have the following corollary.

Corollary 1.2.2 (Fundamental group of symplectic group). *$Sp(2n, \mathbb{R})$ has a $\mathbb{Z}$ as its fundamental group.*

From this we can see that the connected double cover exists and it is a unique Lie group and so we set

Definition 1.2.2 (Metaplectic group). *We define the Metaplectic group to be the unique connected double covering space of the symplectic group. We denote the covering homomorphism by $\rho : Mp(2n, \mathbb{R}) \to Sp(2n, \mathbb{R})$*

It can be shown that the Metaplectic group does not admit a faithful finite-dimensional representation see theorem 1 in Krýsl [2017] and thus it is a Lie group, which cannot be realised as a subgroup of $Gl(m, \mathbb{R})$ for any $m \in \mathbb{N}$.

Let us look at Lie algebrae of Symplectic and Metaplectic groups. They are of course isomorphic because of the double cover. However, we will realise them differently in order to separate them and make the later computation of the differential of the double cover easier and more organised. Our description will be the same as in Habermann and Habermann [2006]. We will identify the symplectic Lie algebra with the space $S^2(\mathbb{R}^{2n})$ of symmetric 2-tensors on $\mathbb{R}^{2n}$ and the metaplectic Lie algebra with a Lie subalgebra of the symplectic Clifford Lie algebra.

If we are given a Lie group $Sp(2n, \mathbb{R})$ then its Lie algebra $\mathfrak{sp}(2n, \mathbb{R})$ is by definition a set of endomorphsim $X \in \mathrm{End}(\mathbb{R}^{2n}) \cong M_{2n}(\mathbb{R})$ such that for vectors $v, w \in \mathbb{R}^{2n}$ holds

$$\omega_0(Xv, w) + \omega_0(v, Xw) = 0$$

or in matricies

$$XJ_0 + J_0X^T = 0.$$

Let us define a morphism $\varphi : \mathcal{S}^2(\mathbb{R}^{2n}) \to \mathfrak{sp}(2n, \mathbb{R})$ as follows, for $v, v_1, v_2 \in \mathbb{R}^{2n}$

$$\varphi(v_1 \odot v_2)(v) = \omega_0(v, v_1)v_2 + \omega_0(v, v_2)v_1. \tag{1.1}$$

We denote the symmetric product by $\odot$. The morphism is well defined since for $v, w \in \mathbb{R}^{2n}$ it holds that

$$\begin{aligned}&\omega_0(\varphi(v_1 \odot v_2)v, w) + \omega_0(v, \varphi(v_1 \odot v_2)w) = \omega_0(\omega_0(v, v_1)v_2 + \omega_0(v, v_2)v_1, w)+\\&+ \omega_0(v, \omega_0(w, v_1)v_2 + \omega_0(w, v_2)v_1) = \omega_0(v, v_1)\omega_0(v_2, w) + \omega_0(v, v_2)\omega_0(v_1, w)+\\&+ \omega_0(v, v_2)\omega_0(w, v_1) + \omega_0(v, v_1)\omega_0(w, v_2) = 0\end{aligned}$$

where the last equality follows by the skew-symmetry of ω_0. This is the defining property of the symplectic Lie algebra and so the morphism is well defined. On the other hand, we can define $\psi : \mathfrak{sp}(2n, \mathbb{R}) \to \mathcal{S}^2(\mathbb{R}^{2n})$ for $X \in \mathrm{End}(\mathbb{R}^{2n})$ and a symplectic basis $(a_1, \ldots, a_n, b_1, \ldots, b_n)$ as

$$\psi(X) = \frac{1}{2}\sum_{j=1}^{n}(Xa_j \odot b_j - a_j \odot Xb_j).$$

Let us denote $v_1 = \sum_i^n(\alpha_i^1 a_i + \beta_i^1 b_i)$ and $v_2 = \sum_i^n(\alpha_i^2 a_i + \beta_i^2 b_i)$. Now we can compute the composition

$$\begin{aligned}&\psi(\varphi(v_1 \odot v_2)) = \frac{1}{2}\sum_{j=1}^{n}((\omega_0(a_j, v_1)v_2 + \omega_0(a_j, v_2)v_1) \odot b_j - a_j \odot (\omega_0(b_j, v_1)v_2+\\&+ \omega_0(b_j, v_2)v_1)) = \frac{1}{2}\sum_{j=1}^{n}(\beta_j^1 v_2 + \beta_j^2 v_1) \odot b_j + a_j \odot (\alpha_j^1 v_2 + \alpha_j^2 v_1) = \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\\&(\beta_j^1\alpha_k^2 + \alpha_k^2\beta_j^1)a_k \odot b_j + (\beta_j^1\beta_k^2 + \beta_j^2\beta_k^1)b_k \odot b_j + (\beta_j^2\alpha_k^1 + \alpha_k^1\beta_j^2)a_k \odot b_j+\\&+ (\alpha_k^1\alpha_j^2 + \alpha_j^2\alpha_k^1)a_k \odot a_j = v_1 \odot v_2.\end{aligned}$$

Where in the third equality we have expanded the v_1 and v_2 as written above and in the second summand we have switched k and j. A proof of the second possible composition is Lemma 1.1.4 in Habermann and Habermann [2006].

We return to the symplectic Clifford algebra $Cl(\mathbb{R}^{2n})$. It can be shown [see Habermann and Habermann, 2006, Lemma 1.1.6] that the vector subspace generated by $\{v \cdot w + w \cdot v |\, v, w \in \mathbb{R}^{2n}\}$ is a Lie subalgebra of $Cl(\mathbb{R}^{2n})$. Furthermore, it is isomorphic to the Lie algebra $\mathfrak{sp}(2n, \mathbb{R})$. We will thus identify it with the Lie algebra $\mathfrak{mp}(2n, \mathbb{R})$. We will think of the differential of ρ as

$$d\rho(v \cdot w + w \cdot v) = 2v \odot w \tag{1.2}$$

Let us proceed with definitions of particular representations of groups and show that there exists a unique unitary representation of the metaplectic group with a certain property. This is essential for the later definition of the symplectic Dirac operator. For proofs and a more elaborate description of the constructions we refer to Wallach [2018].

If we are given a separable complex Hilbert vector space H, we endow the space $GL(H)$ of invertible bounded operators on H with the strong topology. That is a sequence of operators T_k converges to T if and only if the sequence $T_k(h)$ converges to $T(h)$ for every $h \in H$.

Definition 1.2.3 (Group representation)**.** *Let G be a Lie group and H a separable complex Hilbert vector space. By a representation of G on H we mean a continuous group homomorphism $\varphi : G \to GL(H)$. We will denote it by (φ, H).*

It is called a unitary representation if it maps into the group of unitary operators, i.e. $\varphi(G) \subseteq U(H)$.

A closed subspace $V \subseteq H$ is called an invariant subspace with respect to this representation (sometimes called φ-invariant) if for all $g \in G$ it holds that $\varphi(g)(V) \subseteq V$.

A representation is said to be irreducible if there are precisely two invariant subspaces. Those are $\{0\}$ and H.

Given two representations $(\varphi, H_1), (\psi, H_2)$ of the same group G we say that they are equivalent if there exists a linear homeomorphism $T : H_1 \to H_2$ such that $T \circ \varphi = \psi \circ T$.

The are called unitary equivalent if T is an isometric isomorphism.

Let $V = (\mathbb{R}^{2n}, \omega_0)$ be a standard symplectic vector space as in example 1.1.1. Consider the Heisenberg group $H(2n)$ on V. We can define the Schrödinger representations $\psi_{S_\lambda} : H(2n) \to GL(L^2(\mathbb{R}^n))$ for $\lambda \in \mathbb{R}\backslash\{0\}$ by

$$(\psi_{S_\lambda}(\begin{pmatrix} x \\ y \end{pmatrix}, s)(f))(t) = e^{-i\lambda(s+\langle x, t-\frac{1}{2}y\rangle}f(t-y),$$

where $(\begin{pmatrix} x \\ y \end{pmatrix}, s) \in H(2n)$, $f : \mathbb{R}^n \to \mathbb{C}$ and $t \in \mathbb{R}^n$. Note that this definition relies on the fact, that the symplectic structure on the vector space is the standard one. Also it clearly holds that for $f, g : \mathbb{R}^n \to \mathbb{C}$

$$\int_{\mathbb{R}^n} \psi_{S_\lambda}(\begin{pmatrix} x \\ y \end{pmatrix}, s)(f)(t)\overline{\psi_{S_\lambda}(\begin{pmatrix} x \\ y \end{pmatrix}, s)(g)(t)}dt = \int_{\mathbb{R}^n} e^{-i\lambda(s+\langle x,t-\frac{1}{2}y\rangle}f(t-y)$$
$$e^{i\lambda(s+\langle x,t-\frac{1}{2}y\rangle}\overline{g(t-y)}dt = \int_{\mathbb{R}^n} f(t-y)\overline{g(t-y)} = \int_{\mathbb{R}^n} f(t')\bar{g}(t')dt',$$

where the last equality follows from the translation invariance of the Lebesgue measure. This shows that the representation is unitary. Furthermore, it can be shown that these representations are irreducible Folland [1989]. It holds that

$$\psi_{S_\lambda}(0, s)(f) = e^{-i\lambda s}f.$$

If $\lambda = 1$, the representation is denoted by ψ_S. It turns out, that these are all irreducible unitary representations of the Heisenberg group up to equivalence.

Theorem 1.2.3 (Stone-von Neumann)**.** *Let (φ, H) be an irreducible unitary representation of the Heisenberg group on a separable complex Hilbert space. Then there is $\lambda \in \mathbb{R}$ such that $\varphi(0, s)h = e^{-i\lambda s}h$ where $h \in H$ and $s \in \mathbb{R}$ and either*

(i) $\lambda = 0$ *and* $\dim(H) = 1$ *and* $\varphi(x, s) = e^{i\alpha(x)}\mathrm{Id}$*, where* $\alpha \in V^*$ *or*

(ii) $\lambda \neq 0$ *and then* φ *is unitary equivalent to* ψ_{S_λ}*.*

Let us outline a construction of the metaplectic representation which can be found in Weil [1964].

Consider the standard symplectic vector space $(\mathbb{R}^{2n}, \omega_0)$ and the associated Heiseneberg group $H(2n)$. There is an action of the symplectic group $Sp(2n, \mathbb{R})$ on $H(2n)$ given by $\tau_A(x, s) := (Ax, s)$ where $A \in Sp(2n, \mathbb{R})$ and $(x, s) \in H(2n)$. Since this action is an automorphism of the Heisenberg group, the composition $\psi_S \circ \tau_A : H(2n) \to U(L^2(\mathbb{R}^n))$ is again an irreducible representation of the Heisenberg group. Because $(\psi_S \circ \tau_A)(0, s)(f) = \psi_S(0, s)(f) = e^{-is} f$, we see that it is the irreducible representation with the same factor $\lambda = 1$ with the notation from the theorem above. Therefore, it is unitary equivalent to the Schrödinger representation ψ_S. In other word for each $A \in Sp(2n, \mathbb{R})$ there exist an element $T(A) \in U(L^2(\mathbb{R}^n))$ such that

$$T(A) \circ \psi_S = \psi_S \circ \tau_A \circ T(A).$$

Thanks to the Schur lemma for Hilbert spaces [see Deitmar and Echterhoff, 2014] and the fact that the Schrödinger representation is irreducible, we deduce that the operator $T(A)$ is unique up to a complex multiple. This means that the operator T gives rise to a projective unitary representation of $Sp(2n, \mathbb{R})$, i.e. there exists a function $c : Sp(2n, \mathbb{R}) \times Sp(2n, \mathbb{R}) \to S^1$ such that

$$T(AB) = c(A, B) T(A) \circ T(B).$$

It was shown in Weil [1964] that this representation lifts to a unitary representation of the metaplectic group.

Proposition 1.2.4 (Segal-Shale-Weil representation)**.** *There exists a unique unitary representation*

$$\mathbf{m} : Mp(2n, \mathbb{R}) \to U(L^2(\mathbb{R}^n)$$

which satisfies

$$\mathbf{m}(q) \circ \psi_S(x, s) = \psi_S(\rho(q)x, s) \circ \mathbf{m}(q)$$

for all $q \in Mp(2n, \mathbb{R})$ *and* $(x, s) \in H(2n)$

Elements of $L^2(\mathbb{R}^n)$ are called the symplectic spinors and they present an analogue of the classical spinors in the orthogonal case.

Let us note that this representation is faithful, but not irreducible. It decomposes into the sum of two inequivalent irreducible representations - even and odd functions [see Robinson and Rawnsley, 1989]. Its space of smooth vectors (those are the elements $f \in L^2(\mathbb{R})$ such that the map $Mp(2n, \mathbb{R}) \ni A \mapsto \mathbf{m}(A)f$ is $\mathcal{C}^\infty$-Fréchet differentiable) is exactly the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, i.e. the space of rapidly decreasing smooth functions, which is dense in $L^2(\mathbb{R}^n)$ [see Borel et al., 2000].

To the end of this chapter we shall define the Symplectic Clifford multiplication. We shall see that these operators are up to a constant factor just the differentiation of the Segal-Shale-Weil representation [see Habermann and Habermann, 2006, p. 11].

First consider the position and momentum operators from Physics, i.e. $Q_j, P_j : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ for $j \in \{1, \dots, n\}$, defined by $(Q_j f)(x) := i x_j f(x)$ and $P_j f := \frac{\partial f}{\partial x_j}$. Consider the symplectic vector space (V, ω) and its symplectic basis $(a_1, \dots, a_n, b_1, \dots, b_n)$. We define $\sigma : V \to \mathrm{End}(\mathcal{S}(\mathbb{R}^n))$ by extending the map $\sigma(a_j) := Q_j$ and $\sigma(b_j) := P_j$ linearly.

We define a map $\hat{\sigma} : \mathcal{T}(V) \to \mathrm{End}(\mathcal{S}(\mathbb{R}^n))$ as the linear extension from the generators where it is defined by

$$\hat{\sigma}(1) = i\mathrm{Id}$$
$$\hat{\sigma}(v_1) = \sigma(v_1)$$
$$\hat{\sigma}(v_1 \otimes \ldots \otimes v_m) = \hat{\sigma}(v_1) \circ \ldots \circ \hat{\sigma}(v_m),$$

where $v_1, \ldots, v_m \in V$. This map is clearly linear but it is not an algebra homomorphism ($\hat{\sigma}(1) \neq \mathrm{Id}$).

Since $Q_j \circ Q_k = Q_k \circ Q_j$ and $P_j \circ P_k = P_k \circ P_j$ and also

$$P_k \circ Q_j - Q_j \circ P_k = \frac{\partial}{\partial x_k} i(x_j \cdot) - i(x_j \cdot) \frac{\partial}{\partial x_k} = i\delta_{jk}.$$

By linearity we obtain that

$$\sigma(v) \circ \sigma(w) - \sigma(w) \circ \sigma(v) = -i\omega(v, w) \tag{1.3}$$

for any $v, w \in V$ and thus $\hat{\sigma}(\mathcal{I}) = 0$. Hence we can factor it down and obtain a well-defined linear map $Cl(V) \to \mathrm{End}(\mathcal{S}(\mathbb{R}^n))$ which we will also denote σ. Note that this map is also not an algebra homomorphism, however it is a Lie algebra homomorphism.

Definition 1.2.4 (Symplectic Clifford multiplication)**.** *The symplectic Clifford multiplication is the map*

$$\mu_0 : \mathbb{R}^{2n} \otimes \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$$

defined by

$$\mu_0(v \otimes f) = \sigma(v)(f),$$

where $f \in \mathcal{S}(\mathbb{R}^n)$ *and* $v \in \mathbb{R}^{2n}$.

In the following, we will omit the σ symbol and write just the dot $\cdot$ instead, i.e. $\sigma(v)(f) = v \cdot f$. There are two results about this multiplication that will be needed later. We state them and the proofs can be found in Habermann and Habermann [2006] as Lemma 1.4.4 and Proposition 1.4.5.

Proposition 1.2.5 (Equivariance of symplectic Clifford multiplication)**.**
The symplectic Clifford multiplication is $Mp(2n, \mathbb{R})$*-equivariant. That is*

$$\mu_0(\rho(g)v \otimes \mathbf{m}(g)f) = \mathbf{m}(g)\mu_0(v \otimes f),$$

where $g \in Mp(n, \mathbb{R}), v \in \mathbb{R}^{2n}$ *and* $f \in \mathcal{S}(\mathbb{R}^n)$.

Proposition 1.2.6 (Differential of the metaplectic representation)**.** *The differential of the metaplectric representation is the symplectic Clifford multiplication up a factor of* $-i$. *That is*

$$d\mathbf{m}(v)f = -iv \cdot f,$$

where $v \in \mathbb{R}^{2n}$ *and* $f \in \mathcal{S}(\mathbb{R}^n)$.

Example 1.2.1. One important example of symplectic Clifford multiplication is the Hamiltonian of the Harmonic oscillator H_0 defined by

$$H_0(f) := \frac{1}{2}\sum_{j=1}^{2n} e_j^2 \cdot f = \frac{1}{2}\sum_{j=1}^{n}(P_j^2 + Q_j^2)(f) = \frac{1}{2}\sum_{j=1}^{n}(\frac{\partial^2}{\partial x_j^2} - x_j^2)(f),$$

where $(e_1, \ldots, e_{2n})$ is some symplectic basis of $\mathbb{R}^{2n}$.

1.3 Symplectic manifolds

Definition 1.3.1. *We say that a pair (M, ω) is a symplectic manifold if M is a manifold and ω is an everywhere non-degenerate closed 2-form. That is $\omega \in \Omega^2(M)$ such that (ω_p, T_pM) is a symplectic vector space for every $p \in M$ and $d\omega = 0$.*

We say that a diffeomorphism between two symplectic manifolds $f : (M_1, \omega_1) \to (M_2, \omega_2)$ is a symplectomorphism if $f^*\omega_2 = \omega_1$.

Remark. It is also possible to define a symplectic structure by saying that there exists a reduction of the structure group of the tangent bundle from $GL(2n, \mathbb{R})$ to $Sp(2n, \mathbb{R})$ (which is called an almost symplectic manifold) and the so called integrability condition, i.e. that $d\omega = 0$.

Since each tangent space must be a symplectic vector space and we know that these can only be even dimensional. We obtain that also the symplectic manifold must be even dimensional because the dimension of the manifold is the same as the dimension of its tangent spaces.

The non-degeneracy condition implies that $\omega^{\wedge n}$ is a nowhere zero volume form, where $2n$ is the dimension of the manifold. Thus each symplectic manifold is orientable.

Note also that the closedness condition, i.e. $d\omega = 0$ gives a constrain on the topology of the manifold. If the manifold is closed, i.e. compact and without a boundary, ω cannot be exact because of the following computation. Suppose that $\omega = d\eta$ for the contradiction. Then

$$\int_M \omega^n = \int_M d(\omega^{n-1} \wedge \eta) = \int_{\partial M} \omega^{n-1} \wedge \eta = 0,$$

where in the second equality the Stokes theorem is used and the last equality follows because M has no boundary, i.e. $\partial M = 0$. Thus the second deRham cohomology of such a manifold cannot be zero. An interesting consequence of this is that the 2-sphere is actually the only sphere which posses a symplectic structure.

Example 1.3.1 (symplectic manifolds). The standard symplectic vecor space (see 1.1.1) is a typical example of a symplectic manifold. The closedness can be seen via a simple computation.

Another example is the already mentioned 2-sphere with any volume form. Non-degeneracy follows by definition and the closedness is clear since there are no 3-forms on a 2-dimensional manifold.

Similarly as in the Riemannian case, there is a canonical isomorphism between the tangent and cotangent bundle of the underlying manifold $\psi_\omega : TM \to T^*M$ defined by $X \mapsto \omega(X, -)$. Note that this isomorphism can be used to relate differentials of functions with vector fields, so called Hamiltonian vector fields.

The following is a well-known result, whose proof can be found for example in Berndt [2001]. It states that symplectic manifolds of a fixed dimension look locally the same.

Theorem 1.3.1 (Darboux theorem)**.** *Let (M, ω) be a symplectic manifold and let $p \in M$ be a point. Then there exists an open neighborhood of $p \in U \subseteq M$ such that $(U, \omega|_U)$ is symplectomorphic to $(V \subseteq \mathbb{R}^{2n}, \omega_0|_V)$.*

This essentially means that there are no local invariants in symplectic geometry (except of the dimension), since the theorem guarantees that around each point we can choose such a coordinate chart (U, x^i) that the tangent vectors $(\frac{\partial}{\partial x^i})$ form a symplectic basis at each point of U.

Although there are no local invariants, we can define a notion of a connection for symplectic manifolds.

Definition 1.3.2. *Let (M, ω) be a symplectic manifold and $\nabla : \Gamma(TM) \to \Gamma(T^*M \otimes TM)$ an affine connection. We say that the connection is symplectic if $\nabla \omega = 0$, i.e. $\nabla_X(\omega(Y, Z)) = \omega(\nabla_X Y, Z) + \omega(Y, \nabla_X Z)$ for all $X, Y, Z \in \Gamma(TM)$.*

Furthermore the connection is called a Fedosov connection if it is in addition torsion free, i.e. $\nabla_X Y - \nabla_Y X = [X, Y]$ for all $X, Y \in \Gamma(TM)$.

Remark. Note that every symplectic manifold admits a Fedosov connection. We can choose a cover by coordinate charts that we obtain from the Darboux theorem. On these charts the coordinate vector fields form local symplectic frames $\frac{\partial}{\partial x^i}$. Thus if we choose the connection to be trivial (i.e. $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$) it can be easily checked that this connection is symplectic and torsion free within our coordinate chart. Note that this cannot be done on general Riemanian manifold since we do not have these coordinate charts with canonical structure ("Riemannian geometry has local invariant"). Now we can choose a partition of unity subordinated to this cover and glue these connections together.

As one may expect this notion is not unique. For example suppose that we have a Fedosov connection ∇ and choose any $(2,1)$-tensor field A. Now we can compute a torsion of the connection $\nabla' := \nabla + A$

$$\begin{aligned} T^{\nabla'}(X, Y) &= \nabla'_X Y - \nabla'_Y X - [X, Y] = T^{\nabla}(X, Y) + A(X, Y) - A(Y, X) = \\ &= A(X, Y) - A(Y, X). \end{aligned}$$

Hence ∇' is torsion free if and only if A is symmetric, i.e. $A(X, Y) = A(Y, X)$. Similarly we can compute the symplecticity of ∇'

$$\begin{aligned} (\nabla'_X \omega)(Y, Z) &= (\nabla_X \omega)(Y, Z) + \omega(A(X, Y), Z) + \omega(Y, A(X, Z)) = \\ &= \omega(A(X, Y), Z) + \omega(Y, A(X, Z)). \end{aligned}$$

Thus the new connection ∇' is symplectic if and only if $\omega(A(X, Y), Z) = = \omega(A(X, Z), Y)$ for all $X, Y, Z \in \Gamma(TM)$. This is equivalent to saying that A is a symplectic Lie algebra valued 1-form, i.e. $A \in \Omega^1(M, \mathfrak{sp}(TM))$.

If we put these conditions together and use the isomorphism between tangent and cotangent bundle ψ_ω, we see that A is actually a symmetric 3-tensor, i.e. $A \in \Gamma(S^3(TM))$ where S denotes the symmetric product. So we obtain that the space of Fedosov connections is an affine space modeled on symmetric 3-tensor fields.

However we can put on some extra constrains to get a unique connection. We shall describe one important example here.

Definition 1.3.3 (Compatible triple)**.** *Let V be $2n$-dimensional real vector space and let (ω, g, J) be a symplectic structure, an inner product and a complex structure on V, respectively. We say that they form a compatible triple if $\omega(-,-) = g(J(-), -)$.*

Note that each 2 of these 3 structures determines the third one. Thus we can say for example that a pair (ω, J) is compatible if the map $V \times V \to \mathbb{R}$ defined as $(v, w) \mapsto \omega(v, Jw)$ is an inner product.

Remark. This exactly corresponds to the so-called 2 out of 3 property of the unitary group. That is that the unitary group $U(n)$ is the intersection of any of 2 different groups from the set $\{Sp(2n, \mathbb{R}), O(2n), GL(n, \mathbb{C})\}$.

Now we translate these conditions to a manifolds.

Definition 1.3.4 (Kähler manifold)**.** *Let (M, ω) be a symplectic manifold, $J \in \Gamma(End(TM))$ be a complex structure and $g \in \Gamma(S^2(TM))$ be a Riemannian metric such that at each point $p \in M$ we have a compatible triple (ω_p, g_p, J_p) on T_pM. Then M is called an almost Kähler manifold.*

If furthermore the complex structure is integrable. We say that the manifold is Kähler.

Remark. Note that this corresponds to the reduction of the structure group of the tangent bundle from $GL(2n, \mathbb{R})$ to $U(n, \mathbb{C})$ (this is called an almost Hermitian structure) and two integrability conditions, the known $d\omega = 0$ (producing almost kähler structure) and integrability of J, i.e. vanishing of the Nijenhuis tensor yielding Kähler structure.

If we dropped the first integrability condition ($d\omega = 0$) we would obtain Hermitian manifolds, which are complex analogue of Riemannian manifolds, where the complex inner product is $h = g + i\omega$.

Note also that these conditions on integrability can be rephrased by saying that there exists a torsion-free connection ∇ such that $\nabla g = 0$, $\nabla J = 0$ and $\nabla\omega = 0$. From the first equation it is visible that the connection must be unique since it must be a Levi-Civita connection. Thus Kähler manifolds poses unique compatible connection which is by definition also symplectic.

As in the case of Riemannian geometry we can define the curvature tensor.

Definition 1.3.5. *Let (M, ω) be a symplectic manifold and ∇ a symplectic connection. We define the curvature tensor as $(3, 1)$-tensor field given by the equation*

$$R(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \qquad \text{for all } X, Y, Z \in \Gamma(TM)$$

and the curvature 4-tensor field by

$$S(X, Y, Z, W) = \omega(R(X, Y, Z), W) \qquad \text{for all } X, Y, Z, W \in \Gamma(TM).$$

Note that the 4-tensor has different symmetries from its Riemannian analogue since ω is skew-symmetric. Namely it holds that $S(X, Y, Z, W) = S(X, Y, W, Z)$.

Similarly as in the Riemannian case we can define different contractions resulting in Ricci curvature and symplectic Ricci curvature - the third contraction is zero. If the connection is also torsion-free we get that these Ricci curvatures are the same symmetric tensor. Since ω is skew-symmetric the contraction of the symmetric Ricci tensor would be zero. Therefore, there is no notion of scalar curvature on symplectic manifolds. However, it is possible to choose compatible almost complex structure J and define a scalar curvature using it. It is obvious that in the Kähler case we would obtain the same notion as in the Riemannian case since the connection is compatible with all of the three structures. For this see Gelfand et al. [1998]

2. Symplectic spinor bundle and symplectic Dirac operators

In this section we are going to construct a symplectic spinor bundle which is a Hilbert space bundle obtained via the Segal-Shale-Weil representation. Then we are also going to construct the symplectic Dirac operators which operate on the sections of the symplectic spinor bundle and show some of its basic properties.

2.1 Symplectic spinor bundle

We follow Habermann and Habermann [2006]

Let (M, ω) be a $2n$-dimensional symplectic manifold. Consider a space $\mathbf{R}_p$ of all symplectic bases of the space (T_pM, ω_p), elements of which are called symplectic frames at p.

Recall that all symplectic bases of the space (T_pM, ω_p) can be identified with the symplectic group $\mathbf{R}_p \cong Sp(2n, \mathbb{R})$ but not canonically. Also, note that each symplectic frame $F \in \mathbf{R}_p$ at p can be considered as a symplectomorphism $F : (\mathbb{R}^{2n}, \omega_0) \to (T_pM, \omega_p)$, since it produces coordinates and preserves the symplectic structure.

We can take the disjoint union of all symplectic frames through all points on the manifolds M and denote it by $\mathbf{R} := \dot{\bigcup}_{p\in M}\mathbf{R}_p$. We have the projection map $\pi : \mathbf{R} \to M$ defined by $\mathbf{R}_p \ni F \mapsto p \in M$.

Because of the Darboux theorem we obtain local trivialization of this space. In more detail, let $p \in M$ be a point and consider a Darboux chart $\varphi : U \ni p \to V \subseteq \mathbb{R}^{2n}$, its differential is the symplectomorphism $d\varphi : (TU, \omega|_U) \to (TV \cong V \times \mathbb{R}^{2n}, \omega_0|_V)$ and so we have a map

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{(\pi, d\varphi\circ -)} & U \times Sp(2n,\mathbb{R}) \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle pr_1} & \\ & U & \end{array},$$

where pr_1 is the projection onto the first coordinate. It is obvious that the symplectic group $Sp(2n, \mathbb{R})$ acts simply transitively on each fiber by the precomposition. Therefore we have obtained a principal $Sp(2n, \mathbb{R})$-bundle. Local sections of this bundle are called local symplectic frames.

Remark. Note that this bundle can also be obtained via the reduction of the structure group from $GL(2n, \mathbb{R})$ to $Sp(2n, \mathbb{R})$ of the frame bundle of the tangent bundle. This reduction is possible due to the symplectic structure of the manifold.

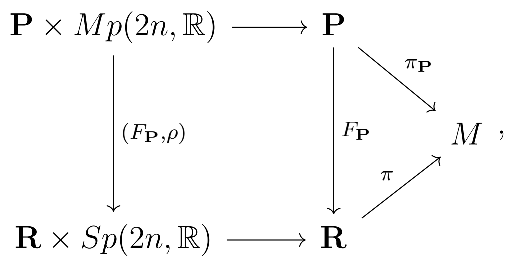
Also, because of this the tangent bundle can be identified with the associated bundle to the symplectic frame bundle using the defining representation $\nu : Sp(2n, \mathbb{R}) \to \mathrm{Aut}(\mathbb{R}^{2n})$, i.e. $\nu(A)(v) = Av$ where $A \in Sp(2n, \mathbb{R})$ and $v \in \mathbb{R}^{2n}$. The isomorphism $\mathbf{R} \times_\nu \mathbb{R}^{2n} \cong TM$ can be explicitly written as $[F, v] \mapsto F(v)$. It is well-defined since $[F, v] = [F \circ A, A^{-1}v] \mapsto F \circ \nu(A) \circ \nu(A^{-1})(v) = F(v)$, where

$F \in \mathbf{R}, v \in \mathbb{R}^{2n}, A \in Sp(2n, \mathbb{R})$. This can be seen also from the construction of the local trivialization which gives also a trivialization of the tangent bundle.

Now we define metaplectic structure which is an analogue to the spin structure. It is a double cover of the symplectic frame bundle compatible with the double cover $\rho : Mp(2n, \mathbb{R}) \to Sp(2n, \mathbb{R})$.

Definition 2.1.1 (Metaplectic structure)**.** *Given a symplectic manifold* (M, ω) *and its symplectic frame bundle* $\boldsymbol{R}$ *we say that* $\pi_{\boldsymbol{P}} : \boldsymbol{P} \to M$ *is the metaplectic structure if it is a principal* $Mp(2n, \mathbb{R})$*-bundle and there exists a bundle map* $F_{\boldsymbol{P}} : \boldsymbol{P} \to \boldsymbol{R}$ *such that* $F_{\boldsymbol{P}}(pA) = F_{\boldsymbol{P}}(p)\rho(A)$*, where* $p \in \boldsymbol{P}$ *and* $A \in Mp(2n, \mathbb{R})$*.*

We can also say that the metaplectic structure is an equivariant lift of the symplectic frame bundle with respect to the double cover ρ. Good illustration is the following commutative diagram.



where the horizontal arrows represent the action of the particular group.

Of course such a lift does not need to exist in general. The topological obstructions are the same as in the case of spin structure [see Bourguignon et al., 2015, Proposition 3.6].

The following proposition is the proposition 3.1.2 from Habermann and Habermann [2006].

Proposition 2.1.1 (Existence of metaplectic structure)**.** *A symplectic manifold* (M, ω) *admits a metaplectic structure if and only if the second Stiefel-Whitney class* $w_2(M) \in H^2(M; \mathbb{Z}_2)$ *vanishes.*

In this case the isomorphism classes of metaplectic structures on (M, ω) *are classified by the first cohomology group* $H^1(M; \mathbb{Z}_2)$*.*

Note that the first Stiefel-Whitney class vanishes since every symplectic manifold is orientable.

Easy consequence of this is the following corollary.

Corollary 2.1.2 (Uniqueness of Metaplectic structure)**.** *A simply connected symplectic manifold* (M, ω) *admits at most one metaplectic structure.*

Proof. From the universal coefficient theorem we obtain an isomorphism $H^1(M; \mathbb{Z}_2) \cong Hom_{\mathbb{Z}}(H_1(M), \mathbb{Z}_2)$ and since $H_1(M) \cong \pi_1(M)^{ab} = \{1\}$ we see that $H^1(M; \mathbb{Z}_2) \cong Hom_{\mathbb{Z}}(\{1\}, \mathbb{Z}_2) = \{1\}$.

□

Remark. Note that a metaplectic structure can be thought of as a reduction of the structure group along the double covering $\rho : Mp(2n, \mathbb{R}) \to Sp(2n, \mathbb{R})$. In this case the term reduction may be somewhat misleading since we have actually enlarged the structure group.

Consequence of this process is that the symplectic frame bundle $\mathbf{R}$ can be realised as the associated bundle to the metaplectic structure, i.e. $\mathbf{R} = \mathbf{P} \times_\rho Sp(2n, \mathbb{R})$.

Since the tangent bundle of the manifold is associated to the principal $Sp(2n, \mathbb{R})$-bundle and associated bundle of an associated bundle can be realised as associated bundle to the original one, we obtain a tangent bundle associated to the principal $Mp(2n, \mathbb{R})$-bundle. That is $TM \cong \mathbf{P} \times_{\nu \circ \rho} \mathbb{R}^{2n}$ where ν is the defining representation and ρ is the double covering between metaplectic and symplectic group.

Given a metaplectic structure we may now define the spinor bundle.

Definition 2.1.2 (Symplectic spinor bundle)**.** *Let (M, ω) be a symplectic manifold with metaplectic structure $\boldsymbol{P}$. The symplectic spinor bundle $\boldsymbol{Q}$ is the bundle associated to the principal $Mp(2n, \mathbb{R})$-bundle via the Segal-Shale-Weil representation 1.2.4 $\boldsymbol{m} : Mp(2n, \mathbb{R}) \to U(L^2(\mathbb{R}))$*

$$\boldsymbol{Q} := \boldsymbol{P} \times_{\boldsymbol{m}} L^2(\mathbb{R}^n).$$

Recall that given a principal G-bundle P and its associated bundle $E := P \times_\nu V$, where $\nu : G \to \mathrm{Aut}(V)$ is a representation, we have a correspondence between continuous sections of the associated bundle E and the G-equivariant continuous functions $f : P \to V$. Here by the G-equivariant functions we mean functions such that $f(pg) = \nu(g^{-1})f(p)$. In other words there is a bijection $\Gamma(E) \to \mathcal{C}(P, V)^G$.

Here is an overview of how the correspondence work. Given a section $s \in \Gamma(E)$ we can write $s(\pi(p)) = [p, v]$, where $p \in P$ and $v \in V$. Such $v \in V$ always exists and is unique, this is exactly due to the fact that on the principal bundle the group G acts freely (unique) an transitively (exists). We define the map $f_s \in \mathcal{C}(P, V)^G$ to be the map $f_s(p) = v$ where v is the element which was described above. On the other hand if we are given a map $f : P \to V$ which is G-equivariant we may construct a section $s_f : M \to E$ such that $s_f(m) := [p, f(p)]$ this map is well-defined because of the G-equivariance of f. And easy computation shows that these mappings are mutually inverse.

If we apply this construction to our case, i.e. $P = \mathbf{P}$ and $E = \mathbf{Q}$ we obtain a correspondence $\Gamma(\mathbf{Q}) \to \mathcal{C}(\mathbf{P}, L^2(\mathbb{R}^n))$. In the latter space we know what a smooth function is, thus we may define a smooth sections of $\mathbf{Q}$ as such that under this correspondence the appropriate $Mp(2n, \mathbb{R})$-equivariant function is smooth. From now onwards the symbol $\Gamma(\mathbf{Q})$ will denote the space of smooth sections which are called symplectic spinor fields.

We can do the same procedure with the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and obtain an associated bundle $\mathbf{S} = \mathbf{P} \times_{\mathbf{m}} \mathcal{S}(\mathbb{R}^n)$. Again the smooth sections will be denoted by $\Gamma(\mathbf{S})$. It can be proved [Habermann and Habermann, 2006, Lemma 3.2.3] that the smooth sections of $\mathbf{Q}$ are in fact sections of $\mathbf{S}$. However, this does not necessarily mean that $\Gamma(\mathbf{S}) = \Gamma(\mathbf{Q})$.

2.2 Multiplication and the connection on the symplectic spinor bundle

Recall that we have defined the symplectic Clifford multiplication 1.2.4 as the map $\mu_0 : \mathbb{R}^{2n} \otimes \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. We shall extend this notion to the bundle $\mathbf{S}$ where we will replace the $\mathbb{R}^{2n}$ with the tangent bundle and obtain a bundle map

$$\mu : TM \otimes \mathbf{S} \to \mathbf{S}.$$

We have already remarked that the tangent bundle can be identified with the associated bundle to the metaplectic structure, i.e. $TM \cong \mathbf{P} \times_{\nu\circ\rho} \mathbb{R}^{2n}$ where ν is the defining representation and ρ is the double covering between metaplectic and symplectic groups. We use this fact to obtain the desired bundle map

$$\mu([p, v] \otimes [p, f]) := [p, \mu_0(v \otimes f)],$$

where $p \in \mathbf{P}, v \in \mathbb{R}^{2n}$ and $f \in \mathbf{S}$. This map is well-defined. Consider $g \in Mp(2n, \mathbb{R})$ then

$$\mu([pg, \rho(g^{-1})v], [pg, \mathbf{m}(g^{-1})f]) = [pg, \mu_0(\rho(g^{-1})v \otimes \mathbf{m}(g^{-1})f)] = [p, \mu_0(v \otimes f)],$$

where the last equality follows from the $Mp(2n, \mathbb{R})$-equivariance of the symplectic Clifford multiplication and the definition of equivalence relation on the associated bundle.

As in the vector space case we shall omit the μ and write the dot $\cdot$ instead, i.e. $\mu(X \otimes \varphi) = X \cdot \varphi$, where $X \in \Gamma(TM)$ and $\varphi \in \Gamma(\mathbf{S})$.

We shall also use the convention that $X \cdot Y \cdot \varphi = X \cdot (Y \cdot \varphi)$.

Remark. Since all smooth sections of $\mathbf{Q}$ are in fact sections (not necessarily smooth) of the bundle $\mathbf{S}$ (see the last paragraph before the section 2.2) we can extend the notion of symplecitc Clifford multiplication, thought of as a map $\Gamma(\mathbf{S}) \to \Gamma(\mathbf{S})$ defined by $\varphi \mapsto X \cdot \varphi$ for a fixed $X \in \Gamma(TM)$, to the map from $\Gamma(\mathbf{Q})$. However, since the result need not always be smooth we shall implicitly consider it as a partial function whose domain D is contained between the space of sections $\Gamma(\mathbf{S}) \subseteq D \subseteq \Gamma(\mathbf{Q})$.

Important special case of these constructions is the case where the original manifold M is actually a Kähler manifold. In this case we can make a bundle of unitary frames in the same way as we make a bundle of symplectic frames. This corresponds to the reduction of the structure group of the tangent bundle from $Gl(2n, \mathbb{R})$ to $U(n)$. In this scenario the symplectic frame bundle is actually associated to the unitary frame bundle. Also the metaplectic structure is associated to the principal $\hat{U}(n)$-bundle, obtained as the equivariant lift of the unitary frame bundle, here the $\hat{U}(n)$ denotes the preimage of the unitary group $U(n)$ under the double cover $\rho : Mp(2n, \mathbb{R}) \to Sp(2n, \mathbb{R})$.

We can then, of course, construct the symplectic spinor bundle as an associated vector bundle to the principal $\hat{U}(n)$-bundle. This allows the existence of some new operators. Consider the following observation, which will be of use later.

Observation 2.2.1. *Given a principal G-bundle $P \to M$ and an associated vector bundle $E = P \times_\mu V$, where $\mu : G \to Aut(V)$ is a representation of G, and we are given an operator T on V such that it commutes with the representation μ. Then the operator $\tilde{T} : \Gamma(E) \to \Gamma(E)$ defined by $\tilde{T}(\varphi)(p) = \tilde{T}(\varphi(p))$ where $\varphi \in \mathcal{C}(P, V)^G$ is a well-defined operator.*

Example 2.2.1. If we now consider the operator from the example 1.2.1 $H_0 : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ defined as

$$(H_0(f))(x) := \frac{1}{2}\sum_{j=1}^{n}\left(\frac{\partial^2 f}{\partial x_j^2}(x) - x_j^2 f(x)\right)$$

it can be shown [Habermann and Habermann, 2006, Proposition 1.5.1] that it commutes with the metaplectic representation restricted onto $\hat{U}(n)$. Thus, thanks to the observation above, it defines an operator on global sections of the symplectic spinor bundle.

We now bring our attention to constructing a connection on the symplectic spinor bundle, but first let us recall some basic properties of connections.

Remark. If we are given a fibre bundle $\pi : E \to M$ over a manifold and we talk about a connection we generally mean a horizontal distribution HE such that for all $p \in E$ it holds that $H_pE \oplus V_pE = T_pE$ where $V_pE = \mathrm{Ker}(d\pi_p)$. However, if the fibre bundle is a vector bundle there is a correspondence between the covariant derivative $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ and the horizontal distribution.

In short we can define the covariant derivative using the horizontal lift which is given by the distribution. On the other hand we can say that the differentials of sections with zero covariant derivatives are horizontal and thus they span the distribution. Therefore the covariant derivative is also referred to as a connection.

Let us also recall that on a principal bundle $\pi : P \to M$ there is the notion of a principal connection. It is such a horizontal distribution that is compatible with the action, i.e. for every $p \in P$ it holds that $H_{pg}P = (dR_g)(H_pP)$ where g is an element of the group G that has a free transitive action on the fibers of the bundle P and R_g denotes this action.

This principal connection can be equivalently described in terms of the so called connection 1-form, that is an element $\eta \in \Omega^1(P, \mathfrak{g})$ such that some compatible relations hold. Then the original distribution is the kernel of such 1-form, i.e. $H_pP := \mathrm{Ker}(\eta)$

If we are given a principal G-bundle $P \to M$ and its associated vector bundle $\hat{\pi} : P \times_\rho V = E \to M$. Then each principal connection on the principal bundle induces connection on the associated vector bundle. Let us briefly show this inducing.

First note that for every $v \in V$ there is a bundle map $F_v : P \to E$ defined by $F_v(p) := [p, v]$. Given a principal connection on P, i.e. a horizontal distribution HP we can use the differential of F_v to obtain a distribution on E. That is we define $H_{[p,v]}E := dF_v(H_pP)$. It can be shown that this correctly defines a horizontal distribution and moreover, it does not depend on the element $v \in V$.

This works in the opposite direction in some cases, in particular for the frame bundles of the vector bundles. We refer for details to [Čap and Slovák, 2009, in section 1.3.5]

In the case that we are interested in, this means that given a symplectic connection on the tangent bundle $\nabla : \Gamma(TM) \to \Gamma(T^*M \otimes TM)$ we obtain a principal connection of the symplectic frame bundle $\mathbf{R}$. That is we have the connection 1-form $\eta \in \Omega^1(\mathbf{R}, \mathfrak{sp}(2n, \mathbb{R}))$. We can take the lift of this form to obtain a 1-form $\bar{\eta} \in \Omega^1(\mathbf{P}, \mathfrak{mp}(2n, \mathbb{R}))$ via the pullback by $F_{\mathbf{P}}$ and the differential of ρ, i.e.

$$F_{\mathbf{P}}^* \eta = d\rho \circ \eta.$$

This lift is possible since $d\rho$ is an isomorphism. We now use this connection to induce a connection on the symplectic spinor bundle $\mathbf{Q}$. Which in turn translates to the covariant derivative $\nabla : \Gamma(\mathbf{Q}) \to \Gamma(T^*M \otimes \mathbf{Q})$. This covariant derivative is known as spinor derivative.

2.3 Symplectic Dirac operators

We shall now define the core operator of this thesis whose part of the point spectrum we are going to calculate.

Recall that from the last section we have the bundle map $\mu : TM \otimes \mathbf{S} \to \mathbf{S}$ which can also be viewed as a map between sections, i.e. $\mu : \Gamma(TM \otimes \mathbf{S}) \to \Gamma(\mathbf{S})$. Also given a symplectic connection we have constructed the spinor derivative $\nabla : \Gamma(\mathbf{Q}) \to \Gamma(T^*M \otimes \mathbf{Q})$. We can see that these two maps "can almost be composed", we are only missing the translation from T^*M to TM which can be provided using the isomorphism $\psi_\omega : TM \to T^*M$, where $\psi_\omega(X) := \omega(X, -)$. In this way we can define the symplectic Dirac operator.

Definition 2.3.1 (Symplectic Dirac operator)**.** *Given a symplectic manifold (M, ω) together with a symplectic connection and a metaplectic structure, we can define the symplectic Dirac operator*

$$D = \mu \circ \nabla : \Gamma(\boldsymbol{Q}) \xrightarrow{\nabla} \Gamma(T^*M \otimes \boldsymbol{Q}) \xrightarrow{\psi_\omega^{-1} \otimes Id} \Gamma(TM \otimes \boldsymbol{Q}) \xrightarrow{\mu} \Gamma(\boldsymbol{Q}).$$

If we are furthermore given a Riemannian metric $g \in S^2(T^*M)$ we can define the so called second symplectic Dirac operator in a similar way but using the isomorphism $\psi_g : TM \to T^*M$ defined as $\psi_g(X) := g(X, -)$.

Definition 2.3.2 (Second symplectic Dirac operator)**.** *Given a symplectic manifold (M, ω) together with a symplectic connection, a metaplectic structure and the Riemannian metric we can define the second symplectic Dirac operator*

$$\tilde{D} = \mu \circ \nabla : \Gamma(\boldsymbol{Q}) \xrightarrow{\nabla} \Gamma(T^*M \otimes \boldsymbol{Q}) \xrightarrow{\psi_g^{-1} \otimes Id} \Gamma(TM \otimes \boldsymbol{Q}) \xrightarrow{\mu} \Gamma(\boldsymbol{Q}).$$

Remark. It is always possible to choose a Riemannian metric that is compatible with the symplectic structure and thus defines the second symplectic Dirac operator. However this choice is usually not unique.

Note that these operator generally may depend (and do) on the symplectic connection and choice of a metaplectic structure. However in some situations these dependencies do not matter. For example on a simply connected Kähler manifold there is a canonical choice of the connection (the Levi-Civita one) which is also compatible with the symplectic structure and the simple connectedness guarantees the uniqueness of the metaplectic structure.

It is possible to give local descriptions of these operators using the local symplectic frames as it is given in [Habermann and Habermann, 2006, Lemma 4.1.2]

Lemma 2.3.1 (Local expression for symplectic Dirac operators)**.** *Let* $(e_1, \dots, e_n, f_1, \dots, f_n)$ *be a local symplectic frame on the symplectic manifold* (M, ω) *of dimension* $2n$ *with a Riemannian metric* g*. Then the symplectic Dirac operators can be written as*

$$D(\varphi) = \sum_{j=1}^{n} (e_j \cdot \nabla_{f_j} \varphi - f_j \cdot \nabla_{e_j} \varphi)$$

and

$$\tilde{D}(\varphi) = \sum_{j=1}^{n} (J e_j \cdot \nabla_{f_j} \varphi - J f_j \cdot \nabla_{e_j} \varphi),$$

where J *is the almost complex structure associated to the Riemannian and symplectic structure to be a compatible triple and* $\varphi \in \Gamma(\boldsymbol{Q})$.

Proof. First we write

$$\nabla \varphi = \sum_{j=1}^{n} (e_j^* \otimes \nabla_{e_j} \varphi + f_j^* \otimes \nabla_{f_j} \varphi).$$

Since the basis at each point is symplectic we can write now that $e_j^* = \omega(-, f_j) = -\omega(f_j, -)$ and so $\psi_\omega(f_j) = -e_j^*$.. After applying the isomorphism ψ_ω we get $\psi_\omega^{-1}(e_j^*) = -f_j$. Similarly with the other half of the coordinates we obtain $\psi_\omega^{-1}(f_j^*) = e_j$. Thus we have

$$(\psi_\omega \otimes Id) \circ \nabla \varphi = \sum_{j=1}^{n} (-f_j \otimes \nabla_{e_j} \varphi + e_j \otimes \nabla_{f_j} \varphi).$$

After applying μ we get the desired expression.

The case for the second symplectic Dirac $\tilde{D}$ is similar, only the isomorphism is different, i.e. $e_j^* = \omega(-, f_j) = -\omega(f_j, -) = -(J f_j, -)$ and thus $\psi_g^{-1}(e_j^*) = -J f_j$, similarly the other half of the coordinates.

□

The corollary, which will be useful later, is the case when we have local real unitary frame $(e_1, \dots, e_{2n})$. By the real unitary frame we mean symplectic frame which is also compatible with the almost complex structure J, i.e. $J e_i = e_{i+n}$ and $J e_{i+n} = -e_i$ for $i \in \{1, \dots, n\}$. This is equivalent to saying that we have a local symplectic frame $(e_1, \dots, e_n, J e_1, \dots, J e_n)$ or to saying that $(e_1, \dots, e_n)$ is a local unitary frame of the tangent bundle with the almost complex structure being J.

Corollary 2.3.2 (Dirac operator with respect to local unitary frame)**.** *Let* $((e_1, \ldots, e_{2n}))$ *be a local real unitary frame on a symplectic manifold* (M, ω) *with respect to a compatible triple* (ω, g, J)*. Then the symplectic Dirac operators can be expressed as follows*

$$D\varphi = -\sum_{j=1}^{2n} Je_j \cdot \nabla_{e_j}\varphi \qquad \tilde{D}\varphi = \sum_{j=1}^{2n} e_j \cdot \nabla_{e_j}\varphi$$

We now present the easiest non-trivial example, which can be also found as Example 4.1.4 in Habermann and Habermann [2006].

Example 2.3.1 (Dirac operators on $\mathbb{R}^2$)*.* Let $(\mathbb{R}^2, \omega_0)$ be the standard symplectic vector space thought of as a symplectic manifold. We denote the standard global coordinates as $x := (1, 0)$ and $y := (0, 1)$. The tangent bundle of this manifold is trivial, i.e. $T\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2$. Therefore it is easy to construct the symplectic frame bundle $\mathbf{R} = \mathbb{R}^2 \times Sp(2, \mathbb{R})$. If we define the bundle $\mathbf{P} := \mathbb{R}^2 \times Mp(2, \mathbb{R})$ we have the bundle map $F_{\mathbf{P}}(x, g) = (x, \rho(g))$ where ρ is the double covering. Therefore the bundle $\mathbf{P}$ is the metaplectic structure on $\mathbb{R}^2$. Since the manifold is simply connected (it is even contractible) we see that the metaplectic structure is unique.

We can easily construct the symplectic spinor bundle by just substituting the $L^2(\mathbb{R})$ to the fiber, i.e. $\mathbf{Q} := \mathbb{R}^2 \times L^2(\mathbb{R})$, since there are no 'twists'. We will use t as the coordinate along each fiber. Recall that sections of the symplectic spinor bundle $\Gamma(\mathbf{Q})$ are exactly maps $\mathbb{R}^2 \to \mathbb{R}^2 \times L^2(\mathbb{R})$ where on the first 2 coordinates it is just identity and in the last one we assign a function $\mathbb{R} \to \mathbb{C}$ which depends on (x, y). Therefore the section may be identified with function $\mathbb{R}^3 \to \mathbb{C}$.

$\mathbb{R}^2$ can be made into a Kähler manifold with the standard Euclidean inner product and the complex structure $J\frac{\partial}{\partial x} := \frac{\partial}{\partial y}$, i.e. at each point $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ is the real unitary frame. With this structure we have the canonical Levi-Civita connection, which is flat. This means that $\nabla_{\frac{\partial}{\partial x}} X = \frac{\partial}{\partial x} X$, where we identify the elements of $\Gamma(TM)$ with functions $\mathbb{R}^2 \to \mathbb{R}^2$ by assigning to each point coordinates of this vector field with respect to the basis $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$. This work similarly for the derivative along $\frac{\partial}{\partial y}$.

We can now use the corollary 2.3.2 to describe the symplectic Dirac operators on this manifold. Let $\varphi : \mathbb{R}^3 \to \mathbb{C}$ be a section of the symplectic spinor bundle, then we can compute

$$D\varphi = -\frac{\partial}{\partial y} \cdot \nabla_{\frac{\partial}{\partial x}}\varphi + \frac{\partial}{\partial x} \cdot \nabla_{\frac{\partial}{\partial y}}\varphi = -\frac{\partial}{\partial y} \cdot \frac{\partial \varphi}{\partial x} + \frac{\partial}{\partial x} \cdot \frac{\partial \varphi}{\partial y} = -\frac{\partial^2 \varphi}{\partial t \partial x} + it\frac{\partial \varphi}{\partial y}.$$

Almost identical is a computation for acquiring the expression for the second symplectic Dirac operator. It yields

$$\tilde{D}\varphi = it\frac{\partial \varphi}{\partial x} + \frac{\partial^2 \varphi}{\partial t \partial y}.$$

Note that since the unitary frame is global both of these expressions hold globally as well.

We are however mostly interested in the so-called associated second order operator. This operator is the analogue of the Laplace operator for the classical Riemannian Dirac operator.

Definition 2.3.3 (Associated second order operator)**.** *Given the same setting as we had when we defined the symplectic Dirac operators, we define the second order operator* $\mathcal{P} : \Gamma(\boldsymbol{Q}) \to \Gamma(\boldsymbol{Q})$ *as*

$$\mathcal{P} := i[\tilde{D}, D]$$

In the example above we would obtain an operator

$$\mathcal{P} = i[\tilde{D}, D] = i[(it\frac{\partial}{\partial x} + \frac{\partial^2}{\partial t \partial y}), (-\frac{\partial^2}{\partial t \partial x} + it\frac{\partial}{\partial y})] =$$
$$= i(i\frac{\partial^2}{\partial y^2} + i\frac{\partial^2}{\partial x^2}) = -\Delta$$

which is the standard Laplace operator on the space $\mathbb{R}^2$.

The last thing that we discuss is the formal self-adjointness of the operator, sometimes it is said that the operators are symmetric.

Recall that there is a Hermitian inner product on the space of $L^2(\mathbb{R}^n)$ given by the integration. From it we can construct a Hermitian structure on the symplectic spinor bundle $\mathbf{Q}$. We define

$$\langle [p, f_1], [p, f_2] \rangle := \langle f_1, f_2 \rangle,$$

where $f_1, f_2 \in L^2(\mathbb{R}^n)$ and $p \in \mathbf{P}$. This is again well-defined since the metaplectic representation is a unitary representation of the metaplectic group.

Moreover, we can further construct the inner product on the space of sections $\Gamma(\mathbf{Q})$ in the following way

Definition 2.3.4. *Let* (M, ω) *be a symplectic manifold which possesses a metaplectic structure, we then define*

$$(\varphi, \psi) := \int_M \langle \varphi, \psi \rangle dM,$$

where $\varphi \in \Gamma(\boldsymbol{Q})$ *and* $\psi \in \Gamma_0(\boldsymbol{Q})$*, where by* $\Gamma_0(\boldsymbol{Q})$ *we mean the space of compactly supported smooth sections of* $\boldsymbol{Q}$*.*

The condition of $\psi \in \Gamma_0(\mathbf{Q})$ is, of course, in order for the integral to converge.

We cite one more result, the [Habermann and Habermann, 2006, Theorem 4.5.3.].

Theorem 2.3.3. *If we are given a symplectic manifold* (M, ω) *with a Fedosov connection* ∇*. Then the symplectic Dirac operator* D *is symmetric with respect to the inner product* $(-, -)$*, that is*

$$(\varphi, D\psi) = (D\varphi, \psi),$$

where $\varphi \in \Gamma(\boldsymbol{Q})$ *and* $\psi \in \Gamma_0(\boldsymbol{Q})$*. If in addition the almost complex structure* J *is chosen to be compatible with the connection, i.e.* $\nabla J = 0$*, we also get the symmetry of the second symplectic Dirac operator, that is*

$$(\varphi, \tilde{D}\psi) = (\tilde{D}\varphi, \psi).$$

Note that a special consequence of this result is that in the second case ($\nabla J = 0$) we also get the symmetry of the associated second order operator $\mathcal{P}$. In particular, on a Kähler manifold, all of these operators are symmetric.

3. Grassmannian $Gr_2(\mathbb{C}^4)$ as a Kähler symmetric space with metaplectic structure

In this chapter we are going to present short introduction to homogeneous and symmetric spaces and then describe Grassmannians, in particular the Grassmannian $Gr_2(\mathbb{C}^4)$ as a symmetric space. We are going to show that this Grassmannian posses a unique metaplectic structure and therefore it is possible to define the symplectic Dirac operators on it.

We are also going to define local unitary frames, in which we can describe the symplectic Dirac operators as seen in 2.3.2, respectively the associated second order operator - $\mathcal{P}$. These frames will be essential in the main computation of the spectrum.

3.1 Symmetric spaces

We start by a definition of a homogeneous space, which formalises the intuition of a space which "locally looks everywhere the same".

Definition 3.1.1 (Homogeneous space). *Given a differentiable manifold M and a Lie group G with a smooth left action on M we say that M is a homogeneous space if G acts transitively (i.e. for each $p, q \in M$ there exists $g \in G$ such that $gp = q$).*

If we are given a homogeneous space M and a point $p \in M$ there is a notion of a isotropy subgroup H_p - a subgroup of all elements $g \in G$ such that it fixes the point $p \in M$, that is $H_p := \{g \in G \mid gp = p\}$. The Lie subgroup H is closed and this therefore produces a diffeomorphism of the manifold with the left coset space $M \cong G/H_p$ of M. This notion is independent of the chosen point p since if we are given some other point $q = g'p \in M$ there is a group morphism $\psi : H_q \to H_p$ given by $\psi(g) = g'^{-1}gg'$. This is well defined because $\psi(g)p = g'^{-1}gg'p = g'^{-1}gq = g'^{-1}q = p$ where we have used that $g \in H_q$. It has an obvious inverse and thus it is a group isomorphism.

Because of this we will omit the point and write the homogeneous spaces only as G/H.

Note that for any homogeneous space there is a canonical map $\pi : G \to G/H$ which is a surjective submersion and thus it can be made into a principal H-bundle. Also there is an inclusion of Lie algebrae $\mathfrak{h} \subseteq \mathfrak{g}$. Since the kernel of the differential of the projection π at the identity of the group G is exactly $\mathfrak{h}$, it is possible to identify the tangent space with the factor of the Lie algebrae (as vector spaces), i.e. $T_{eH}(G/H) \cong \mathfrak{g}/\mathfrak{h}$.

It can further be shown [see Čap and Slovák, 2009, Example 1.4.3] that the whole tangent bundle of the homogeneous spaces can be identified with the associated bundle to the bundle $\pi : G \to G/H$ via the adjoint representation. That is

$$T(G/H) \cong G \times_{Ad} \mathfrak{g}/\mathfrak{h}.$$

Note that $\mathfrak{h}$ is an invariant subspace for the adjoint representation restricted from G to H and thus the corresponding factor $\mathfrak{g}/\mathfrak{h}$ is also a representation, therefore the associated bundle on the right hand side is well defined.

If we put some extra constrains on the homogeneous space we will be able to create a canonical connection on it. With this notion there are two connected definitions of reductive homogeneous and symmetric spaces.

Definition 3.1.2 (reductive and symmetric space)**.** *Let G/H be a homogeneous space. If there exists vector space $\mathfrak{p}$ such that there is a vector space decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ where $\mathfrak{h}$ is the Lie algebra of the Lie subgroup H such that it holds that $Ad(H)(\mathfrak{p}) \subseteq \mathfrak{p}$ we say that G/H is a reductive homogeneous space.*

If it even holds that $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{h}$ we say that G/H is a symmetric space.

Remark. The condition on the reductivity of the homogeneous space implies that $dAd(\mathfrak{h})(\mathfrak{p}) = ad(\mathfrak{h})(\mathfrak{p}) = [\mathfrak{h}, \mathfrak{p}] \subseteq \mathfrak{p}$. In the case where H is a connected group it is actually equivalent condition.

Also note that if H is compact or its adjoint representation on $\mathfrak{g}$ is reductive we can always find such subspace.

There are a lot of equivalent definitions of symmetric spaces. One that is frequently seen in literature is a Riemannian manifold M with isometries $s_p : M \to M$ for each point $p \in M$ such that $ds_p = -Id$ and $s_p(p) = p$. An equivalence of different definitions and some basic properties of symmetric spaces may be find in Helgason [1962].

As we mentioned in front of the definition we can identify the tangent space $T(G/H)$ with the associated bundle via the adjoint representation. Note that in the reductive and symmetric cases we may go even further and restrict the adjoint representation only on the subspace $\mathfrak{p}$, thus giving us the isomorphism $T(G/H) \cong G \times_{Ad} \mathfrak{p}$.

Furthermore, we can extend by the left translation the vector subspace $\mathfrak{p}$ on the whole G and thus form a distribution $\mathcal{H}$ on the principal H-bundle $G \to G/H$ which is clearly complementary to the distribution obtained by left translation of the Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ which forms a vertical subbundle of the tangent bundle $TG \to G$. This therefore defines a so-called canonical connection on the reductive homogeneous space.

Given an associated bundle $E = G \times_\mu V$ where $\mu : H \to Aut(V)$ is a representation. We can induce a connection on E from the canonical one and thus define a covariant derivative on it. We now try to express this covariant derivative more explicitly.

Suppose we have a smooth section $\varphi \in \Gamma(E)$ which can be identified with a smooth H-equivariant map $\hat{\varphi} \in \mathcal{C}^\infty(G, V)^H$, i.e. $\varphi(gH) = [g, \hat{\varphi}(g)]$ and we are given a vector $[a, X] \in G \times_{Ad} \mathfrak{p} \cong T(G/H)$. There exists a curve $\gamma : t \mapsto a\exp(tX)H$ in G/H with tangent vector $[a, X]$ at a point aH, whose horizontal lift on the principal H-bundle is the curve $\hat{\gamma} : t \mapsto a\exp(tX)$. The parallel transport of the vector $\varphi(\gamma(t))$ from $\gamma(t)$ to $\gamma(0)$ is

$$P_0^t(\varphi(\gamma(t))) = [a, \hat{\varphi}(a\exp(tX))].$$

Therefore by the definition of covariant derivative we have found the expression

$$\nabla_{[a,X]}(\varphi) = [a, \frac{d}{dt}\Big|_0 \hat{\varphi}(a \exp(tX))] \tag{3.1}$$

From the Proposition 1.4.8 in [Čap and Slovák, 2009] we can see that the torsion of such connection is exactly

$$T(X,Y) = [X,Y]_{\mathfrak{h}} - [X,Y],$$

where $X, Y \in \mathfrak{p}$ and by the $Z_{\mathfrak{h}}$ we mean the component of the vector $Z \in \mathfrak{g}$ in the Lie algebra $\mathfrak{h}$ under the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$. Hence we can see that if we require the original homogeneous space to be a symmetric space the torsion vanishes.

If we are given an H-invariant inner product g on the vector space $\mathfrak{p}$. We can see that it induces a Riemannian metric on the underlying homogeneous space G/H since we may define

$$\tilde{g}([a,X],[a,Y]) := g(X,Y)$$

where $X, Y \in \mathfrak{p}$, $a \in G$ and we are using the identification of the tangent bundle mentioned above. The H-invariance is necessary for the expression to be well-defined. This procedure works similarly with the symplectic and complex structure.

Note that we may view $\tilde{g}$ as a section of the bundle $S^2(T^*(G/H))$ and thus, thanks to the identifications above, as an H-equivariant map $\hat{g} \in \mathcal{C}^\infty(G, S^2(\mathfrak{p}^*))^H$. Looking at the definition of $\tilde{g}$ we have actually defined $\hat{g}$ as a constant function, i.e. $\hat{g}(a) = g$.

From this it is easily seen that the covariant derivative of $\tilde{g}$ is 0, that is $\tilde{g}$ is parallel. This shows that the metric is then compatible with the connection. Again similar relations hold with the symplectic and complex structure.

Altogether and with the remark after definition 1.3.4 about integrability conditions with respect to the compatible connection, we have proved the following

Observation 3.1.1 (Structures on symmetric space)**.** *Let G/H be a symmetric space with decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$. An $Ad(H)$-invariant inner product on the vector space $\mathfrak{p}$ defines a Riemannian metric tensor on the manifold G/H and the canonical connection is the Levi-Civita connection for this metric.*

Moreover if we have an $Ad(H)$-invariant compatible triple on $\mathfrak{p}$ it defines a Kähler structure on the manifold G/H where again the canonical connection is the unique one compatible with all the structures.

These spaces are sometimes called Kähler symmetric spaces.

3.2 Grassmannian as symmetric space

In this section we are going to describe the Grassmannian $Gr_2(\mathbb{C}^4)$ as a Kähler symmetric space and show that it possesses a metaplectic structure (and thus symplectic Dirac operators can be defined). This description will be important for the computation of the spectrum later on.

For this we are mainly going to use Ballmann [2006].

Definition 3.2.1 (Grassmannians)**.** *We define a Grassmannian $Gr_k(\mathbb{C}^n)$ as all k-dimensional vector subspaces of the complex vector space $\mathbb{C}^n$. That is*

$$Gr_k(\mathbb{C}^n) := \{V \subseteq \mathbb{C}^n | dim_{\mathbb{C}}(V) = k\}.$$

We are mainly going to focus on the Grassmannian $Gr_2(\mathbb{C}^4)$, but a lot of these computations can be easily generalised.

Let us choose a plane $Gr_2(\mathbb{C}^4) \ni p = span\{\begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}^T, \begin{pmatrix}0 & 1 & 0 & 0\end{pmatrix}^T\}$. Now we define an action on p by the Lie group $SU(4)$ given by the multiplication from the left.

This action is transitive since for any 2-dimensional plane we can find two vectors inside that are orthonormal with respect to the standard complex inner product. And it is easy to see that we can find two more vectors in $\mathbb{C}^4$, so that these 4 vectors form a unitary basis. Thus a matrix with these vectors as columns is a unitary matrix and by multiplying one of the vectors by an appropriate complex unit we can take such matrix from $SU(4)$.

Let us compute the stabiliser of this action. Consider $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU(4)$ where $A, B, C, D \in M_2(\mathbb{C})$ that stabilises the point p. This happens if and only if $\begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} I_2 \\ 0 \end{pmatrix}$, thus forcing $C = 0$. But since the matrix is from $SU(4)$ its columns must be orthonormal with respect to the standard inner product, thus forcing also $B = 0$. It can be easily verified that each matrix of the form $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in SU(4)$ stabilises the point p. Also $A, D \in U(2)$, because the matrix is in $SU(4)$, therefore the stabiliser is the group $S(U(2) \times U(2))$. We have thus described the Grassmannian $Gr_2(\mathbb{C}^4)$ as homogeneous space $SU(4)/S(U(2) \times U(2))$.

Remark. Using essentially identical computations one may find that all complex Grassmannians are homogeneous spaces of the form $Gr_k(\mathbb{C}^n) = SU(n)/S(U(k) \times U(n-k))$.

Notation: We will write $G = SU(4)$ and $H = S(U(2) \times U(2))$.

Let us examine the Lie algebrae $\mathfrak{g}$ and $\mathfrak{h}$. It is well known That the Lie algebra of G is the space of all skew-Hermitian matricies of zero trace. Analogous arguments show that the Lie algebra of H is $\mathfrak{h} = \{\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} | Tr(X+Y) = 0, X^* = -X, Y^* = -Y\}$. Consider the space

$$\mathfrak{p} := \left\{ \begin{pmatrix} 0 & X \\ -X^* & 0 \end{pmatrix} \middle| X \in M_2(\mathbb{C}) \right\}.$$

These clearly serve for the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$. We are now going to show that the space $\mathfrak{p}$ is exactly the necessary subspace in order to make the Grasssmaniann a symmetric space.

Recall that if we have a matrix group, the adjoint representation is realised via conjugation. Consider the following computation

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 0 & X \\ -X^* & 0 \end{pmatrix} \begin{pmatrix} A^* & 0 \\ 0 & D^* \end{pmatrix} = \begin{pmatrix} 0 & AXD^* \\ -DX^*A^* & 0 \end{pmatrix}. \tag{3.2}$$

And since $(AXD^*)^* = DX^*A^*$, this shows that $Ad(H)(\mathfrak{p}) \subseteq \mathfrak{p}$. Thus the Grassmannian $Gr_2(\mathbb{C}^4)$ is a reductive homogeneous space. Furthermore we have that

$$\begin{pmatrix} 0 & X \\ -X^* & \end{pmatrix}\begin{pmatrix} 0 & Y \\ -Y^* & 0 \end{pmatrix} - \begin{pmatrix} 0 & Y \\ -Y^* & \end{pmatrix}\begin{pmatrix} 0 & X \\ -X^* & 0 \end{pmatrix} =$$
$$= \begin{pmatrix} -XY^* + YX^* & 0 \\ 0 & -X^*Y + Y^*X \end{pmatrix}.$$

Using the fact that $Tr(XY) = Tr(YX)$ we can easily see that this matrix is indeed in $\mathfrak{h}$. Thus we have also proved that $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{h}$, therefore showing that the considered Grassmannian is actually a symmetric space.

Remark. Again a similar choice of $\mathfrak{p}$ makes any Grassmannian into a symmetric space.

We will now introduce a Kähler structure on $\mathfrak{p}$. Consider the mapping

$$h\Big(\begin{pmatrix} 0 & X \\ -X^* & 0 \end{pmatrix}, \begin{pmatrix} 0 & Y \\ -Y^* & 0 \end{pmatrix}\Big) := Tr(XY^*).$$

this mapping defines a complex inner product on the space $\mathfrak{p}$. Using the equation 3.2 and the fact that $Tr(DXC^*CY^*D^*) = Tr(XY^*)$ we see that this mapping is $Ad(H)$-invariant. Furthermore since $Tr(XX^*) \geq 0$ and for each $X \in M_2(\mathbb{C})$ we may find $Y \in M_2(\mathbb{C})$ such that $Tr(XY^*) \neq 0$ the mapping h defines $Ad(H)$-invariant complex inner product on the space $\mathfrak{p}$. This defines an almost Hermitian structure on the Grassmannian.

Moreover, we may define $g := Re(h)$ and $\omega = Im(h)$, thus obtain the decomposition $h = g + i\omega$. It is possible to verify that g is $Ad(H)$-invariant inner product and ω is $Ad(H)$-invariant symplectic form. They are compatible and the complex structure on $\mathfrak{p}$ is then given by

$$J\Big(\begin{pmatrix} 0 & X \\ -X^* & 0 \end{pmatrix}\Big) = \begin{pmatrix} -iI_2 & 0 \\ 0 & iI_2 \end{pmatrix}\begin{pmatrix} 0 & X \\ -X^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & -iX \\ -iX^* & 0 \end{pmatrix}. \tag{3.3}$$

Using the observation 3.1.1 we have proved that Grassmannian $Gr_2(\mathbb{C}^4)$ has Kähler structure with the canonical connection being the Levi-Civita connection.

Note that with this Kähler structure the adjoint action of the Lie group H on the subspace $\mathfrak{p}$ is therefore unitary representation, that is $Ad : H \to U(\mathfrak{p})$. Thus the unitary frame bundle can also be written using this representation, i.e., $G \times_{Ad} U(\mathfrak{p})$.

Now we are going to show that the Grassmannian admits a metaplectic structure and therefore it is possible to define the symplectic Dirac operators on it.

If we choose some a unitary basis on $\mathfrak{p}$ we obtain an isomorphism $U(\mathfrak{p}) \cong U(4)$ with the unitary group realised as subgroup of complex matrices. Since $U(4) \subseteq Sp(8, \mathbb{R})$ consider the preimage of this group under the double cover ρ from the first chapter 1.2.2. We shall denote it by $\hat{U}(4) \subseteq Mp(8, \mathbb{R})$. Our strategy is to find a lift $\hat{Ad}$ of the adjoint representation such that the following diagram commutes

$$\begin{array}{ccc} & & \hat{U}(4) \\ & \overset{\hat{Ad}}{\nearrow} & \downarrow \rho \\ S(U(2) \times U(2)) & \xrightarrow{Ad} & U(4) \end{array}.$$

For this we are going to use the fact that $\hat{U}(4)$ is a covering space of $U(4)$ and there is a lifting criterion [see Hatcher, 2002, Proposition 1.3.3].

Proposition 3.2.1 (Lifting criterion)**.** *Suppose given a covering space $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ and a map $f : (Y, y_0) \to (X, x_0)$ with Y path-connected and locally path-connected. Then a lift $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ of f exists if and only if $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.*

Since we know that the group $H = S(U(2) \times U(2))$ is locally path-connected, because it is a manifold, we just need to prove that it is path connected and prove the property of the push-forward of the fundamental group.

For this we are going to use a topological result, whose proof can be found in Hatcher [2002] as Theorem 4.41 and Proposition 4.48.

Theorem 3.2.2 (Fibre bundles induce homotopy exact sequences)**.** *Given a fibre bundle $p : E \to B$ with B a path-connected space and base points $b_0 \in B$ and $x_0 \in p^{-1}(b_0) = F$. Then there is a long exact sequence of homotopy groups*

$$\ldots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \ldots \to \pi_0(E, x_0) \to 1$$

where 1 denotes the trivial group.

It is well known that the special unitary groups are connected and simply connected $\pi_0(SU(n)) = \pi_1(SU(n)) = 1$. Consider a short exact sequence

$$1 \to SU(2) \times SU(2) \to S(U(2) \times U(2)) \xrightarrow{\det^u} U(1) \to 1,$$

where the map $\det^u$ is the determinant of the upper-left 2×2 matrix. In other words we have the principal bundle $S(U(2) \times U(2)) \to S(U(2) \times U(2))/SU(2) \times SU(2) \cong U(1)$. By the above proposition this fiber bundle induces the following long exact sequence in homotopy

$$\begin{aligned} \ldots \to \pi_1(SU(2) \times SU(2)) \to \pi_1(S(U(2) \times U(2))) \xrightarrow{\det^u_*} \pi_1(U(1)) \to \\ \to \pi_0(SU(2) \times SU(2)) \to \pi_0(S(U(2) \times U(2))) \to 1. \end{aligned}$$

Using the fact that the homotopy groups are compatible with the product, that is $\pi_n(U \times V) = \pi_n(U) \times \pi_n(V)$ [see Hatcher, 2002, Proposition 4.2], the knowledge of the first two homotopy groups of $SU(n)$ and the fundamental group of a circle, i.e. $\pi_1(U(1)) = \pi_1(S^1) = \mathbb{Z}$, we obtain the following sequence

$$\ldots \to 1 \to \pi_1(S(U(2) \times U(2))) \xrightarrow{\det^u_*} \mathbb{Z} \to 1 \to \pi_0(S(U(2) \times U(2))) \to 1.$$

From this it is immediate that $\pi_0(S(U(2) \times U(2))) = 1$ and $\pi_1(S(U(2) \times U(2))) \cong \mathbb{Z}$. To compute the generator of the fundamental group it is enough to find a loop $\gamma : [0, 1] \to S(U(2) \times U(2))$ such that $\det^u \circ \gamma$ is the generator of the fundamental group of $U(1)$. One such a choice is

$$\gamma : t \mapsto \left(\begin{array}{cc|cc} e^{2\pi i t} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-2\pi i t} \end{array}\right).$$

Let us compute the adjoint representation of this curve on $\mathfrak{p}$.

$$Ad(\gamma(t))(\left(\begin{array}{cc|cc} 0 & 0 & z_1 & z_2 \\ 0 & 0 & z_3 & z_4 \\ \hline -\bar{z_1} & -\bar{z_3} & 0 & 0 \\ -\bar{z_2} & -\bar{z_4} & 0 & 0 \end{array}\right)) =$$

$$= \left(\begin{array}{cc|cc} e^{2\pi it} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-2\pi it} \end{array}\right) \left(\begin{array}{cc|cc} 0 & 0 & z_1 & z_2 \\ 0 & 0 & z_3 & z_4 \\ \hline -\bar{z_1} & -\bar{z_3} & 0 & 0 \\ -\bar{z_2} & -\bar{z_4} & 0 & 0 \end{array}\right) \left(\begin{array}{cc|cc} e^{-2\pi it} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{2\pi it} \end{array}\right) =$$

$$= \left(\begin{array}{cc|cc} 0 & 0 & e^{2\pi it}z_1 & e^{4\pi it}z_2 \\ 0 & 0 & z_3 & e^{2\pi it}z_4 \\ \hline -e^{-2\pi it}\bar{z_1} & -\bar{z_3} & 0 & 0 \\ -e^{-4\pi it}\bar{z_2} & -e^{-2\pi it}\bar{z_4} & 0 & 0 \end{array}\right),$$

where the coordinates $z_1, \ldots, z_4 \in \mathbb{C}$ come from the isomorphism $U(\mathfrak{p}) \cong U(4)$. Thus we have shown that

$$Ad(\gamma(t)) = \left(\begin{array}{cc|cc} e^{2\pi it} & 0 & 0 & 0 \\ 0 & e^{4\pi it} & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{2\pi it} \end{array}\right).$$

Now consider a short exact sequence

$$1 \to SU(4) \to U(4) \xrightarrow{\det} U(1) \to 1.$$

As before we get a long exact sequence of homotopy groups, but since $\pi_0(SU(4)) = \pi_1(SU(4)) = 1$ we obtain that $\det_*$ is an isomorphism of the fundamental groups of $U(4)$ and $U(1)$. Thus composing the morphism, one obtains $\det(Ad(\gamma(t))) = e^{8\pi it}$ which means that $Ad_* : \pi_1(S(U(2) \times U(2))) \to \pi_1(U(4))$ maps the chosen generator to the multiple of four, i.e., $Ad_*(\pi_1(S(U(2)\times U(2)))) = 4\pi_1(U(4)) \cong 4\mathbb{Z}$.

We know that the mapping $\rho : Mp(8, \mathbb{R}) \to Sp(8, \mathbb{R})$ is a double cover and so is its restriction $\rho|_{\hat{U}(4)} : \hat{U}(4) \to U(4)$ (recall that $U(n)$ is a deformation retract of $Sp(2n, \mathbb{R})$). Thus the fundamental group is mapped onto the multiple of 2, i.e. $\rho_*(\pi_1(\hat{U}(4))) = 2\pi_1(U(4)) \cong 2\mathbb{Z}$.

Putting this altogether we obtain $Ad_*(\pi_1(S(U(2) \times U(2)))) = 4\pi_1(U(4)) \subseteq 2\pi_1(U(4)) = \rho_*(\pi_1(\hat{U}(4)))$. Therefore the condition to the lifting criterion is fulfilled and there exists a lift $\hat{Ad}$ making the following diagram commutative

$$\begin{array}{ccc} & & \hat{U}(4) \\ & \overset{\hat{Ad}}{\nearrow} & \big\downarrow \rho \\ S(U(2) \times U(2)) & \xrightarrow{Ad} & U(4) \end{array} \quad . \tag{3.4}$$

We have thus proved the existence of a metaplectic structure on the Grassmannian $Gr_2(\mathbb{C}^4)$, since it is given by $SU(4) \times_{\hat{Ad}} Mp(8, \mathbb{R})$.

Consider the short exact sequence of the Grassmannian

$$1 \to S(U(2) \times U(2)) \to SU(4) \to Gr_2(\mathbb{C}^4) \to 1.$$

Let us look at the long exact sequence in a homotpy that it induces

$$\ldots \to \pi_1(SU(4)) \to \pi_1(Gr_2(\mathbb{C}^4)) \to \pi_0(S(U(2)\times U(2))) \to \pi_0(SU(4)) \to 1.$$

When we fill in the known facts we obtain a sequence

$$\ldots \to 1 \to \pi_1(Gr_2(\mathbb{C}^4)) \to 1 \to 1 \to 1.$$

So we can conclude that the Grassmannian is simply connected and thus by the corollary 2.1.2 we have shown that the metaplectic structure on the Grassmannian is unique.

Remark. We were (very specifically) talking about the Grassmannian $Gr_2(\mathbb{C}^4)$ in order not to introduce extra notation, but almost identical arguments work for a general Grassmannian $Gr_k(\mathbb{C}^{2l})$ where $k, l \in \mathbb{N}$.

We will make an (important) remark about working in local coordinates on symmetric spaces.

Remark. If we are given a local section of the principal H-bundle $G \to G/H$ denoted by $s : U \subseteq G/H \to G$ and an element $X \in \mathfrak{p}$ we automatically have a local section of the tangent bundle $T(G/H) \to G/H$ defined by $\tilde{s} : gH \mapsto [s(gH), X]$ where we are using the isomorphism $T(G/H) \cong G \times_{Ad} \mathbb{R}^8$.

Therefore if we are given a unitary basis of the vector space $\mathfrak{p}$ with respect to the Kähler structure mentioned above we can always (using a local section of the principal H-bundle $G \to G/H$) turn it into a local unitary frame. This justifies computing the Dirac operators only on $\mathfrak{p}$.

However, we are going to make one specific choice of such section coming from the exponential map on a Lie group G. Consider a map $\pi \circ \exp|_V : V \subseteq \mathfrak{p} \to G/H$. The exponential map is a local diffeomorphism and the projection restricted on the embedded submanifold $Im(\exp|_V)$ has full rank differential at an identity. This implies that there exists $V' \subseteq V \subseteq \mathfrak{p}$ such that we have an diffeomorphism $\pi : Im(\exp|_{V'}) \to G/H$. Taking an inverse of this map gives the required local section.

This section has an important property. Namely, the local vector fields obtained via the construction in the previous paragraph have actually zero covariant derivative at the point eH, where e is the identity of the group G.

Of course, the symplectic spinor bundle with the metaplectic structure characterised above is then $\mathbf{Q} := SU(4) \times_{\mathbf{m}\circ\hat{Ad}} L^2(\mathbb{R}^4)$. Given a local symplectic spinor field $\varphi : U \subseteq G/H \to \mathbf{Q}$ consider the associated H-equivariant function $\hat{\varphi} \in \mathcal{C}^\infty(\pi^{-1}(U), L^2(\mathbb{R}^4))^H$, where $\pi : G \to G/H$ is the projection, we define the application of the vector $X \in \mathfrak{p}$ as

$$X(\hat{\varphi})(a) := \left.\frac{d}{dt}\right|_0 \hat{\varphi}(a \exp(tX)), \tag{3.5}$$

where $a \in \pi^{-1}(U)$. Compare this definition with the covariant derivative3.1 together with the local frames in which we are working as introduced above.

Observe that with this definition, if we are given a local section $s : U \subseteq G/H \to G$ we may write

$$(\nabla_{[s(aH),X]}\hat{\varphi})(a) = X(\hat{\varphi})(a),$$

where $aH \in U$.

Analogous definition can be made for sections of other bundles, but we do not need those. Also let us write explicitly that if the section s is obtained from the exponential map as described in the remark above, we obtain

$$\nabla_{[s(eH),X]}[s(-),Y] = 0, \tag{3.6}$$

where $X, Y \in \mathfrak{p}$ are arbitrary.

Note also, that the definition of a field application may be used even for a global symplectic spinor fields.

Now we are going to introduce coordinates in which we will be computing the point spectrum in the next chapter. There are basically 4 independent entries in matricies in $\mathfrak{p}$. To have a more compact notation we are going to introduce a map $\tau : \{1,2,3,4\} \to \{1,2\} \times \{1,2\}$, defined by

$$\tau(1) := (1,1) \qquad \tau(2) := (1,2) \qquad \tau(3) := (2,1) \qquad \tau(4) := (2,2). \tag{3.7}$$

We are going to make the following choice of the real unitary frame of $\mathfrak{p}$.

$$\begin{aligned} F_j &:= \left(\begin{array}{c|c} 0 & -E^T_{\tau(j)} \\ \hline E_{\tau(j)} & 0 \end{array}\right) \\ JF_j &:= \left(\begin{array}{c|c} 0 & iE^T_{\tau(j)} \\ \hline iE_{\tau(j)} & 0 \end{array}\right) \end{aligned} \qquad \text{for } j \in \{1,2,3,4\}, \tag{3.8}$$

where by E_{ij} we mean a 2×2 matrix with one at a position (i,j) and with zeros elsewhere.

Using the corollary 2.3.2, we can write how do the symplectic Dirac operators look in this frame

$$\begin{aligned} D\varphi &= \sum_{j=1}^{4} F_j \cdot JF_j(\varphi) - \sum_{j=1}^{4} JF_j \cdot F_j(\varphi) \\ \tilde{D}\varphi &= \sum_{j=1}^{4} F_j \cdot F_j(\varphi) + \sum_{j=1}^{4} JF_j \cdot JF_j(\varphi). \end{aligned} \tag{3.9}$$

We recall that the dot $\cdot$ represents the symplectic Clifford multiplication.

4. Computation of the point spectrum of $\mathcal{P}$ on $Gr_2(\mathbb{C}^4)$

In this chapter we are going to perform the main computation of the point spectrum of the associated second order operator $\mathcal{P}$ to the symplectic Dirac operators on the Grassmannian $Gr_2(\mathbb{C}^4)$. We are going to us the notation introduced in the previous chapter.

Let us recall some notation introduced in the previous chapters. We have proven that $Gr_2(\mathbb{C}^4) \cong SU(4)/S(U(2) \times U(2))$ and we will continue with the shorter notation $G = SU(4)$ and $H = S(U(2) \times U(2))$. We have constructed the decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ together with an Ad-invariant Kähler structure and found a lift $\hat{Ad} : H \to \hat{U}(4)$.

Also recall the remark at the end of the previous chapter 3.2. Throughout the chapter we are going to work in the frame that is constructed using the section from the exponential map explained in that remark.

4.1 Split into 2 operators

Recall that we have a metaplectic representation $\mathbf{m} : Mp(8, \mathbb{R}) \to \mathrm{Aut}(L^2(\mathbb{R}^4))$. Note however, that the symplectic spinor bundle on the Grassmannian is given by $\mathbf{Q} = SU(4) \times_{\mathbf{m}\circ\hat{Ad}} L^2(\mathbb{R}^4)$. And since $\hat{Ad}(H) \subseteq \hat{U}(4)$, we can consider the restriction of the metaplectic representation onto the subgroup $\hat{U}(4) \subseteq Mp(8, \mathbb{R})$ and denote it by $\mathbf{u} := \mathbf{m}|_{\hat{U}(4)}$.

Consider the Hermite functions

$$h_\alpha(x) := h_{\alpha_1}(x_1)h_{\alpha_2}(x_2)h_{\alpha_3}(x_3)h_{\alpha_4}(x_4),$$

where

$$h_{\alpha_i}(x_i) := e^{\frac{x_i^2}{2}} \left(\frac{d}{dx_i}\right)^{\alpha_i} e^{-x_i^2}.$$

This convention is compatible with the one used by Habermann and differs only by a sign from the classical Hermite functions presented in Folland [1989] on p. 51. So we will refer there for the properties of these functions. Hermite functions gives the orthogonal decomposition

$$L^2(\mathbb{R}^4) = \widehat{\bigoplus}_{l=0}^{\infty} \mathcal{M}_l = \widehat{\bigoplus}_{l=0}^{\infty} \Big(\bigoplus_{\substack{\alpha \in \mathbb{N}_0^4 \\ |\alpha| = l}} \langle h_\alpha \rangle \Big).$$

Let us also mention that the space $\mathcal{M}_l$ of Hermite functions with the given order of the multiindex l is the eigenspace of the operator H_0 with eigenvalue $-(l + 2)$. Because of the observation 2.2.1 and the note below, we know that the operator H_0 commutes with $\mathbf{u}$. Thus we have that the subspace $\mathcal{M}_l$ is an invariant subspace for the representation $\mathbf{u}$.

Putting this together we obtain a splitting of the symplectic spinor bundle as follows

$$\mathbf{Q} = \widehat{\bigoplus}_{l=1}^{\infty} \mathbf{Q}_l = \widehat{\bigoplus}_{l=1}^{\infty} (SU(4) \times_{\mathbf{u} \circ \hat{Ad}} \mathcal{M}_l).$$

Furthermore, it can be shown [see Habermann and Habermann, 2006, section 5.3] that on a Kähler manifold the associated second order operator $\mathcal{P}$ leaves the subbundles $\mathbf{Q}_l$ invariant. Thus for the spectral problem we can restrict the computation only on these subbundles.

For a smooth section $\varphi \in \Gamma(\mathbf{Q}_l)$ considered as an element of $\mathcal{C}(G, \mathcal{M}_l)^H$, let us extend the definition of the application of an element of $\mathfrak{p}$ to any Lie algebra element $X \in \mathfrak{g} = \mathfrak{su}(4)$ as

$$X(\varphi)(a) := \frac{d}{dt}\Big|_0 \varphi(a \exp(tX)),$$

where $a \in G$. This is inspired by the covariant derivative obtained from the canonical connection (see 3.1). This extends the action from $\mathfrak{p}$ to $\mathfrak{g}$.

This application is linear. We distinguish 2 possibilities. If $X \in \mathfrak{p}$ we have already observed that $X(\varphi) = \nabla_X \varphi$, where we consider X and φ as appropriate local sections. If $X \in \mathfrak{h}$ we can compute

$$\begin{aligned} X(\varphi)(a) = \frac{d}{dt}\Big|_0 \varphi(a \exp(tX)) = \frac{d}{dt}\Big|_0 (\mathbf{m} \circ \hat{Ad})(\exp(-tX))\varphi(a) = \\ = -d(\mathbf{m} \circ \hat{Ad})(X)\varphi(a) = id\hat{Ad}(X) \cdot \varphi(a), \end{aligned} \tag{4.1}$$

where the second equality follows from the definition of an H-invariant function, the third is the definition of the Lie algebra representation and the fourth is the proposition 1.2.6. Note that this expression make sense since $\hat{Ad} : H \to \hat{U}(4)$ and thus $d\hat{Ad} : \mathfrak{h} \to \hat{\mathfrak{u}}(4) \subseteq \mathfrak{mp}(8, \mathbb{R}) \subseteq Cl(\mathbb{R}^8)$. We recall the identification of the Lie algebra of the metaplectic group as introduced in chapter 1 (see the paragraph above 1.2.

Let us observe, that this definition is actually a representation of a Lie algebra $\mathfrak{g}$ [see Wyss, 2003, p.33], i.e. it holds that

$$[X, Y](\varphi)(a) = X(Y(\varphi))(a) - Y(X(\varphi))(a) \tag{4.2}$$

Let us state without a proof a proposition 3.2.7 from Habermann and Habermann [2006], stating that for a vector fields $X, Y \in \Gamma(TM)$ and a symplectic spinor field $\varphi \in \Gamma(\mathbf{Q})$ it holds that

$$\nabla_X (Y \cdot \varphi) = (\nabla_X Y) \cdot \varphi + Y \cdot \nabla_X(\varphi).$$

The proposition applied to our homogeneous case (i.e. $X(-) = \nabla_X$) says that it even holds that

$$X(Y \cdot \varphi) = Y \cdot X(\varphi), \tag{4.3}$$

where $X, Y \in \mathfrak{p}$ are arbitrary, since we have noted that the local vector fields have zero covariant derivative at eH 3.6.

We will restrict ourselves to the computation on the sections of the subbundle $\mathbf{Q}_l$ whose fibers are finite-dimensional, and their smooth sections can be consider as maps $\varphi \in \mathcal{C}(G, \mathcal{M}_l)^H$.

We may now proceed to the main computation. We use the coordinates defined at the end of the previous chapter 3.8, F_j and JF_j. First step is to rewrite the associated second order operator into 2 operators, one of which only uses the applications of the vectors from the Lie algebra $\mathfrak{g}$ and the other one which can be completely described using just the symplectic Clifford multiplication. We will use the definition of the symplectic Dirac operators 3.9 and the fact 4.3. For $\varphi \in \Gamma(\mathbf{Q}_l)$, compute

$$\begin{aligned}
[\tilde{D}, D](\varphi) = \quad & \sum_{j,k=1}^{4} F_j \cdot F_j(F_k \cdot JF_k(\varphi)) + JF_j \cdot JF_j(F_k \cdot JF_k(\varphi)) - \\
& - F_j \cdot F_j(JF_k \cdot F_k(\varphi)) - JF_j \cdot JF_j(JF_k \cdot F_k(\varphi)) - \\
& - F_k \cdot JF_k(F_j \cdot F_j(\varphi)) + JF_k \cdot F_k(F_j \cdot F_j(\varphi)) - \\
& - F_k \cdot JF_k(JF_j \cdot JF_j(\varphi)) + JF_k \cdot F_k(JF_j \cdot JF_j(\varphi)) = \\
= \quad & \sum_{j,k=1}^{4} F_j \cdot F_k \cdot F_j(JF_k(\varphi)) + JF_j \cdot F_k \cdot JF_j(JF_k(\varphi)) - \\
& - F_j \cdot JF_k \cdot F_j(F_k(\varphi)) - JF_j \cdot JF_k \cdot JF_j(F_k(\varphi)) - \\
& - F_k \cdot F_j \cdot JF_k(F_j(\varphi)) + JF_k \cdot F_j \cdot F_k(F_j(\varphi)) - \\
& - F_k \cdot JF_j \cdot JF_k(JF_j(\varphi)) + JF_k \cdot JF_j \cdot F_k(JF_j(\varphi)).
\end{aligned}$$

We set $[X, Y](\varphi) := X(Y(\varphi)) - Y(X(\varphi))$ and $[X, Y]^{\cdot} \cdot \varphi := X \cdot Y \cdot \varphi - Y \cdot X \cdot \varphi$ for any X, Y, not only for $X, Y \in \mathfrak{p}$. Also we are going to use the equality

$$X \cdot Y \cdot W(Z(\varphi)) - Y \cdot X \cdot Z(W(\varphi)) = [X, Y]^{\cdot} \cdot W(Z(\varphi)) + Y \cdot X \cdot [W, Z](\varphi).$$

We use this equality to obtain the expression

$$\begin{aligned}
[\tilde{D}, D](\varphi) = \quad & \sum_{j,k=1}^{4} ([F_j, F_k]^{\cdot} \cdot F_j(JF_k(\varphi)) + F_k \cdot F_j \cdot [F_j, JF_k](\varphi) + \\
& + [JF_j, F_k]^{\cdot} \cdot JF_j(JF_k(\varphi)) + F_k \cdot JF_j \cdot [JF_j, JF_k](\varphi) + \\
& + [JF_k, F_j]^{\cdot} \cdot F_k(F_j(\varphi)) + F_j \cdot JF_k \cdot [F_k, F_j](\varphi) + \\
& + [JF_k, JF_j]^{\cdot} \cdot F_k(JF_j(\varphi)) + JF_j \cdot JF_k \cdot [F_k, JF_j](\varphi)).
\end{aligned}$$

Here we note that the equation 1.3 can be written in the following way

$$[X, Y]^{\cdot} \cdot \varphi = -i\omega(X, Y)\varphi,$$

where ω denotes the symplectic form on $\mathfrak{p}$. Recall however, that the basis (F_j, JF_j) is chosen in such a way, that it forms a real unitary basis. In particular it is a symplectic basis. Therefore we see that

$$[F_j, F_k]^{\cdot} \cdot \varphi = [JF_j, JF_k]^{\cdot} \cdot \varphi = 0 \qquad [F_j, JF_k]^{\cdot} \cdot \varphi = -i\delta_{jk}\varphi.$$

By the equation 3.3 we have that $JX = J_0 X = -X J_0$ for any $X \in \mathfrak{p}$, where $J_0 = \begin{pmatrix} iI_2 & 0 \\ 0 & iI_2 \end{pmatrix}$. This implies that $J_0 X J_0 = X$, since J is a complex structure.

Using this and the observation 4.2 we get the following equalities

$$\begin{aligned} [JF_j, JF_k] &= J_0F_jJ_0F_k - J_0F_kJ_0F_j = F_jF_k - F_kF_j = [F_j, F_k] \\ [F_j, JF_k] &= F_jJ_0F_k - J_0F_kF_j = -J_0F_jF_k + F_kJ_0F_j = [F_k, JF_j], \end{aligned} \tag{4.4}$$

where we have used the fact that the Lie bracket in this situation is just the matrix commutator.

Altogether we arrive at the expression

$$\begin{aligned} [\tilde{D}, D](\varphi) = &\left(\sum_{j=1}^{4}(iF_j^2(\varphi) + iJF_j^2(\varphi))\right) + \\ &+ \sum_{j,k=1}^{4}((F_j \cdot F_k + JF_j \cdot JFk) \cdot [F_j, JF_k](\varphi) + \\ &+ (F_j \cdot JF_k - F_k \cdot JF_j) \cdot [F_k, F_j](\varphi)). \end{aligned} \tag{4.5}$$

We have shown, that the associated second order operator splits into two operators. One (that is in the brackets) is described only using the application and the other one is not using only symplectic Clifford multiplication as we wanted. In order to achieve the goal, we need to replace the applications of $[F_i, F_j](\varphi)$ and $[F_i, JF_j](\varphi)$ by the symplectic Clifford multiplication which can be done.

We introduce the notation

$$\begin{aligned} A_{ij} &:= F_i \cdot F_j + JF_i \cdot JF_j \\ B_{ij} &:= F_i \cdot JF_j - F_j \cdot JF_i. \end{aligned}$$

It is easy to see [Habermann and Habermann, 2006, p.17] that when we consider the Lie algebra $\hat{\mathfrak{u}}(4)$ of the preimage of $\hat{U}(4)$ of the unitary group $U(4)$ under the double cover $\rho : Mp(8, \mathbb{R}) \to Sp(8, \mathbb{R})$. This algebra has a basis consisting of A_{ij} and B_{ij} where $1 \leq i, j \leq 4$ and where we do not consider B_{ii} since those are clearly 0.

Note the behaviour when swapping the indices, i.e. $A_{ij} = A_{ji}$, since the commutators $[JF_i, JF_j]^{\cdot} \cdot = [F_i, F_j]^{\cdot} \cdot$ are trivial, and $B_{ij} = -B_{ji}$, which holds trivially.

We will now show how it is possible to replace the applications of $[F_i, F_j](\varphi)$ and $[F_i, JF_j](\varphi)$ in 4.5 with the symplectic Clifford multiplication.

Recall that we know from chapter 3 that the Grassmannian is a symmetric space, so $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{h}$ and thus we consider the second case mentioned in 4.1. This means that it is actually a symplectic Clifford multiplication. We just need to compute the vectors $d\hat{A}d(X)$ for all $X \in \mathfrak{h}$.

Remark. Observe that up to this point, our computations are general and work for any suitable Grassmannian, but from this point onward the computations require specifically the $Gr_2(\mathbb{C}^4)$.

We are now going to sketch how to compute $d\hat{A}d([\mathfrak{p},\mathfrak{p}])$ by computing the element $d\hat{A}d(E_0)$ and then present a table with our result. Others elements are computed in completely analogous way.

First we need to compute the commutators $[F_j, F_k]$ and $[F_j, JF_k]$ in $\mathfrak{g}$. In order to do that we need to choose suitable basis of $\mathfrak{h}$ so that we have coordinates on the whole $\mathfrak{g}$. We are going to make the following choice.

$$
\begin{gathered}
E_0 := \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \begin{matrix} i & 0 \\ 0 & -i \end{matrix} \end{array}\right) \quad
E^0 := \left(\begin{array}{c|c} \begin{matrix} i & 0 \\ 0 & -i \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right) \quad
E_1 := \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{array}\right) \\
E^1 := \left(\begin{array}{c|c} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right) \quad
E_2 := \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \begin{matrix} 0 & i \\ i & 0 \end{matrix} \end{array}\right) \quad
E^2 := \left(\begin{array}{c|c} \begin{matrix} 0 & i \\ i & 0 \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right) \\
S := \frac{1}{\sqrt{2}} \left(\begin{array}{c|c} \begin{matrix} i & 0 \\ 0 & i \end{matrix} & 0 \\ \hline 0 & \begin{matrix} -i & 0 \\ 0 & -i \end{matrix} \end{array}\right)
\end{gathered} \tag{4.6}
$$

Remark. This is clearly a basis of the Lie subalgebra $\mathfrak{h}$. Note that the inner product on the subspace $\mathfrak{p}$ defined as the real part of the trace is actually a negative multiple of the Killing form on the Lie algebra $\mathfrak{su}(4)$ which is $(X,Y) = 8Tr(XY)$. It is immediate to see that we get $g(X,Y) = -\frac{1}{2}Tr(XY)$.

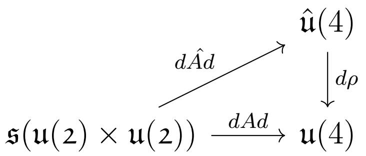
However, since the Lie algebra $\mathfrak{su}(4)$ comes from a compact group, its Killing form is negative definite, so the negative multiple (our g) is positive definite, and thus an inner product. This turns the whole Lie algebra $\mathfrak{g} = \mathfrak{su}(4)$ into a vector space with an inner product. With respect to this inner product the basis of $\mathfrak{h}$ is orthonormal and extends the already orthonormal basis (F_j, JF_j) into an orthonormal basis of $\mathfrak{g}$. This explains the choice of $\frac{1}{\sqrt{2}}$ in the element S.

Now we can compute the commutators which is just a straightforward computation of the regular matrix commutators

$$
\begin{array}{cc}
[F_1, JF_1] = -\sqrt{2}S - E^0 + E_0 & [F_2, JF_2] = -\sqrt{2}S + E^0 + E_0 \\
[F_3, JF_3] = -\sqrt{2}S - E^0 - E_0 & [F_4, JF_4] = -\sqrt{2}S + E^0 - E_0 \\
[F_1, JF_2] = [F_3, JF_4] = -E^2 & [F_1, JF_3] = [F_2, JF_4] = E_2 \\
[F_1, F_2] = [F_3, F_4] = -E^1 & [F_1, F_3] = [F_2, F_4] = -E_1.
\end{array} \tag{4.7}
$$

(All the other commutators we have not written are zero, except the ones which can be obtained from these ones by applying the identities mentioned above 4.4.)

Knowing the commutators, we can compute the explicit formula for the application using the symplectic Clifford multiplication. For this we need to calculate $d\hat{A}d$ applied on the elements of the basis 4.6. For this let us write the diagram



which is obtained from the diagram 3.4 by taking its differential. Map $d\rho$ can be find using the prescription given in 1.2 and the morphism $dAd = ad$ can be calculated by computing the commutators of appropriate elements from $\mathfrak{p}$ and $\mathfrak{h}$. Since the diagram above is commutative and the morphism $d\rho$ is an isomorphism (it comes from a differentiating the smooth double cover) we will be able to find the expression for the morphism $d\hat{Ad}$.

We need to compute the commutators $[E_0, F_j]$ for all $j \in \{1, \ldots, 4\}$ since $\mathfrak{p}$ is a complex vector space with a complex basis given by (F_j). Recall the notation introduced at the end of the previous chapter given by the map $\tau : \{1,2,3,4\} \to \{1,2\} \times \{1,2\}$.

$$dAd(E_0)(F_j) = [E_0, F_j] = \left(\begin{array}{c|cc} 0 & \multicolumn{2}{c}{0} \\ \hline 0 & i & 0 \\ & 0 & -i \end{array}\right) \left(\begin{array}{c|c} 0 & -E^T_{\tau(j)} \\ \hline E_{\tau(j)} & 0 \end{array}\right) -$$

$$- \left(\begin{array}{c|c} 0 & -E^T_{\tau(j)} \\ \hline E_{\tau(j)} & 0 \end{array}\right) \left(\begin{array}{c|cc} 0 & \multicolumn{2}{c}{0} \\ \hline 0 & i & 0 \\ & 0 & -i \end{array}\right) =$$

$$= \left(\begin{array}{c|c} 0 & (-1)^{\delta_{2\tau(j)_1}} iE^T_{\tau(j)} \\ \hline (-1)^{\delta_{2\tau(j)_1}} iE_{\tau(j)} & 0 \end{array}\right) = (-1)^{\delta_{2\tau(j)_1}} JF_j,$$

where by $\tau(j)_1$ we mean the first element from the pair $\tau(j)$, thus the expression $\delta_{2\tau(j)_1}$ is 1 only if the $\tau(j)$ is either $(2,1)$ or $(2,2)$.

Thus we can write $dAd(E_0)$ in the matrix form with respect to the complex basis (F_j) of the vector space $\mathfrak{p}$

$$dAd(E_0) = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix},$$

since multiplying by i is in the vector space $\mathfrak{p}$ just applying the map J.

On the other hand we wish to compute the image under the map $d\rho$ of the elements A_{ij} and B_{ij} introduced above, respectively their action on the basis vectors F_j. We have mentioned that the elements A_{ij} and B_{ij} form a basis of the Lie algebra $\hat{\mathfrak{u}}(4)$. Thus computing $d\rho$ of these describes the whole morphism $d\rho$. We are going to use the definition given in the chapter 1 (1.2).

$$d\rho(A_{kl})(F_j) = (F_k \odot F_l + JF_k \odot JF_l)(F_j) = \delta_{kj} JF_l + \delta_{lj} JF_k$$
$$d\rho(B_{kl})(F_j) = (F_k \odot JF_l - F_l \odot JF_k)(F_j) = \delta_{lj} F_k - \delta_{kj} F_l,$$

where in both equations we have first used the definition and then the isomorphism given in 1.1. In the matrix notation we get

$$d\rho(A_{kl}) = \begin{array}{c} \\ k \rightarrow \\ l \rightarrow \\ \\ \end{array} \overset{\begin{array}{cc} k & l \\ \downarrow & \downarrow \end{array}}{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}} \qquad d\rho(B_{kl}) = \begin{array}{c} \\ k \rightarrow \\ l \rightarrow \\ \\ \end{array} \overset{\begin{array}{cc} k & l \\ \downarrow & \downarrow \end{array}}{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}.$$

Where in the expression for $d\rho(A_{kk})$ we will have only one non-zero entry, namely $2i$ at the (k,k) position.

Putting these two computations together we get the following result

$$dAd(E_0) = \frac{1}{2} d\rho(A_{11} + A_{22} - A_{33} - A_{44}),$$

from which it immediately follows that

$$d\hat{A}d(E_0) = ((d\rho)^{-1} \circ dAd)(E_0) = \frac{1}{2}(A_{11} + A_{22} - A_{33} - A_{44}).$$

We will now write the table for all basis elements of the Lie subalgebra $\mathfrak{h}$.

$$\begin{aligned}
d\hat{A}d(E_0) &= \frac{1}{2}(A_{11} + A_{22} - A_{33} - A_{44}) \\
d\hat{A}d(E^0) &= \frac{1}{2}(-A_{11} + A_{22} - A_{33} + A_{44}) \\
d\hat{A}d(S) &= \frac{-1}{\sqrt{2}}(A_{11} + A_{22} + A_{33} + A_{44}) \\
d\hat{A}d(E_1) &= B_{13} + B_{24} \quad d\hat{A}d(E^1) = B_{12} + B_{34} \\
d\hat{A}d(E_2) &= A_{13} + A_{24} \quad d\hat{A}d(E^2) = -A_{12} - A_{34}.
\end{aligned} \tag{4.8}$$

We now proceed to a further computation. Using the commutators from 4.7 and the newly obtained table 4.8 for the morphism $d\hat{A}d$ together with the formula 4.2 we continue with the equation 4.5

$$\begin{aligned}
[\tilde{D}, D](\varphi) = &\left(\sum_{j=1}^{4} iF_j^2(\varphi) + iJF_j^2(\varphi))\right) + i(A_{11} \cdot (2A_{11} + A_{22} + A_{33}) \cdot + \\
&+ A_{22} \cdot (2A_{22} + A_{11} + A_{44}) \cdot + A_{33} \cdot (2A_{33} + A_{11} + A_{44}) \cdot + A_{44} \cdot (2A_{44} + \\
&+ A_{22} + A_{33}) \cdot + 2(A_{12} \cdot (A_{12} + A_{34}) + A_{13} \cdot (A_{13} + A_{24}) + A_{14} \cdot 0 + \\
&+ A_{23} \cdot 0 + A_{24} \cdot (A_{13} + A_{24}) + A_{34} \cdot (A_{12} + A_{34}) + B_{12} \cdot (B_{12} + B_{34}) + \\
&+ B_{13} \cdot (B_{13} + B_{24}) + B_{14} \cdot 0 + B_{23} \cdot 0 + B_{24} \cdot (B_{13} + B_{24}) + \\
&+ B_{34} \cdot (B_{12} + B_{34})) \cdot \varphi.
\end{aligned}$$

Here the zero terms comes from the vanishing of commutators $[F_1, F_4] = [F_2, F_3] = [F_1, JF_4] = [F_2, JF_3] = 0$, which vanish because they correspond to the diagonal entries in the matricies in $\mathfrak{p}$. Now we have achieved our initial goal, which is a splitting of the operator into 2 operators, one of which is described only by the application and other one only by the symplectic Clifford multiplication.

4.2 Simplification of the operators

Parallelly to the computations done by Habermann [Habermann and Habermann, 2006] on $\mathbb{C}P^1$ and by Wyss [Wyss, 2003] on $\mathbb{C}P^{2n+1}$ we define the Casimir operator of $SU(4)$ on the symplectic spinor fields by

$$\Omega(\varphi) := \sum_{j=1}^{15} b_i^2(\varphi), \tag{4.9}$$

where the terms b_i represent an arbitrary orthonormal basis of $SU(4)$ with respect to our choice of the inner product, which is the negative multiple of the Killing form. We will show later that this operator has a close relation to the Casimir element of $SU(4)$ (with respect to the same inner product) represented by the derivation of the induced representation, hence the name.

Because of this relation it may be a good idea to create the Casimir operator in the expression (by adding a 0 in a particular form) which would make the computation of the spectrum more manageable.

Remember that we actually have an orthonormal basis with respect to the inner product $g = -\frac{1}{2}Tr(-,-)$, i.e. the basis $(F_j, JF_j, E_0, E^0, E_1, E^1, E_2, E^2, S)$ where $j \in \{1,2,3,4\}$. Thus we almost have the Casimir operator in our formula. We are just missing the elements coming from the Lie subalgebra $\mathfrak{h}$. Because the elements of $\mathfrak{h}$ can be viewed as the application or as the symplectic Clifford multiplication (see the formula 4.1), we can add and subtract them and don't break our decomposition into two operators.

We apply the observation and also make a few adjustments to simplify the formula at the end of the previous section. Thus we obtain

$$\begin{aligned}
[\tilde{D}, D](\varphi) = i\Omega(\varphi) - i\Big(&(E_0)^2(\varphi) + (E^0)^2(\varphi) + (E_1)^2(\varphi) + (E^1)^2(\varphi) + \\
&+ (E_2)^2(\varphi) + (E^2)^2(\varphi) + S^2(\varphi)\Big) + 2i\Big(\sum_{j=1}^{4} A_{jj}^2 + A_{11}\cdot A_{22} + A_{11}\cdot A_{33} + \\
&+ A_{22}\cdot A_{44} + A_{33}\cdot A_{44} + (A_{12}+A_{34})^2 + (A_{13}+A_{24})^2 + (B_{12}+B_{34})^2 + \\
&+ (B_{13}+B_{24})^2\Big)\cdot\varphi =: i\Omega(\varphi) + i\Psi_0\cdot\varphi,
\end{aligned}$$

where we have used

$$A_{ij}A_{kl} = A_{kl}A_{ij} \qquad B_{ij}B_{kl} = B_{kl}B_{ij}, \qquad \text{if } \{i,j\}\cap\{k,l\} = \emptyset. \tag{4.10}$$

As can be seen directly from the definition. Also by the A_{kl}^2 we mean the iterative symplectic Clifford multiplication.

Now we bring our focus to the operator $\Psi_0\cdot$. As hinted earlier we are going to use again the same trick as in the previous section, that is use the equation 4.1 and the table 4.8 to rewrite the first part of the operator via the multiplication. Note that the elements $(E_1)^2, (E^1)^2, (E_2)^2, (E^2)^2$ under the morphism $d\hat{A}d$ are exactly the squares at the end of the operator (the signs cancel out). We arrive at

$$i\Psi_0 \cdot \varphi = \frac{i}{4}((A_{11} + A_{22} - A_{33} - A_{44})^2 \cdot \varphi + \frac{i}{4}(-A_{11} + A_{22} - A_{33} + A_{44})^2 \cdot \varphi +$$
$$+ \frac{i}{2}(A_{11} + A_{22} + A_{33} + A_{44})^2 \cdot \varphi + 2i\Big(\sum_{j=1}^{4} A_{jj}^2 + A_{11} \cdot A_{22} + A_{11} \cdot A_{33} +$$
$$+ A_{22} \cdot A_{44} + A_{33} \cdot A_{44}\Big) \cdot \varphi + 3i\Big((A_{12} + A_{34})^2 + (A_{13} + A_{24})^2 +$$
$$+ (B_{12} + B_{34})^2 + (B_{13} + B_{24})^2\Big) \cdot \varphi$$

We may further simplify the first part since

$$\frac{i}{4}((A_{11} + A_{22} - A_{33} - A_{44})^2 \cdot \varphi + \frac{i}{4}(-A_{11} + A_{22} - A_{33} + A_{44})^2 \cdot \varphi +$$
$$+ \frac{i}{2}(A_{11} + A_{22} + A_{33} + A_{44})^2 \cdot \varphi + 2i\Big(\sum_{j=1}^{4} A_{jj}^2 + A_{11} \cdot A_{22} + A_{11} \cdot A_{33} +$$
$$+ A_{22} \cdot A_{44} + A_{33} \cdot A_{44}\Big) \cdot \varphi =$$
$$= 3i\Big(\sum_{j=1}^{4} A_{jj}^2 + A_{11} \cdot A_{22} + A_{11} \cdot A_{33} + A_{22} \cdot A_{44} + A_{33} \cdot A_{44}\Big) \cdot \varphi.$$

Furthermore there is an observation.

Observation 4.2.1. *With the notation described above, consider $k, l \in \{1, 2, 3, 4\}$ such that $k \neq l$. Then it holds that*

$$A_{kl}^2 + B_{kl}^2 = A_{kk} \cdot A_{ll} - 1.$$

Proof. Note that we have proved earlier that $[F_k, JF_l]^{\cdot} = -i\delta_{kl}$ and we will use this identity in the next calculation without mentioning it. Also for this computation we will not write the symplectic Clifford multiplication, since there is no other one to be confused with. By expanding the definitions we have that

$$A_{kl}^2 + B_{kl}^2 = F_k^2 F_l^2 + JF_k^2 JF_l^2 + F_k F_l JF_k JF_l + JF_k JF_l F_k F_l + F_k^2 JF_l^2 +$$
$$+ F_l^2 JF_k^2 - F_k JF_l F_l JF_k - F_l JF_k F_k JF_l.$$

Since

$$JF_k JF_l F_k F_l = F_k JF_k JF_l F_l + iJF_l F_l = F_k F_l JF_k JF_l + iF_k JF_k + iF_l JF_l - 1$$
$$F_k JF_l F_l JF_k = F_k F_l JF_k JF_l + iF_k JF_k$$
$$F_l JF_k F_k JF_l = F_k F_l JF_k JF_l + iF_l JF_l,$$

we obtain that

$$A_{kl}^2 + B_{kl}^2 = F_k^2 F_l^2 + JF_k^2 JF_l^2 + F_k^2 JF_l^2 + F_l^2 JF_k^2 - 1 = A_{kk} A_{ll} - 1.$$

□

Using the previous observation, the computation above it, and the commutation rule 4.10 we have that

$$
\begin{aligned}
i\Psi_0 \cdot \varphi = 3i\Big(\sum_{j=1}^{4} A_{jj}^2 + A_{11}\cdot A_{22} + A_{11}\cdot A_{33} + A_{22}\cdot A_{44} + A_{33}\cdot A_{44}\Big)\cdot\varphi + \\
+ 3i\Big(A_{12}+A_{34})^2 + (A_{13}+A_{24})^2 + (B_{12}+B_{34})^2 + (B_{13}+B_{24})^2\Big)\cdot\varphi = \\
= 3i\Big(\sum_{j=1}^{4} A_{jj}^2 + A_{11}\cdot A_{22} + A_{11}\cdot A_{33} + A_{22}\cdot A_{44} + A_{33}\cdot A_{44}\Big)\cdot\varphi + \\
+ 3i\Big((A_{12}^2+B_{12}^2) + (A_{34}^2+B_{34}^2) + (A_{13}^2+B_{13}^2) + (A_{24}^2+B_{24}^2) + \\
+ 2(A_{12}\cdot A_{34} + A_{13}\cdot A_{24} + B_{12}\cdot B_{34} + B_{13}\cdot B_{24})\Big)\cdot\varphi = \\
= 3i\Big(\sum_{j=1}^{4} A_{jj}^2 + 2(A_{11}\cdot A_{22} + A_{11}\cdot A_{33} + A_{22}\cdot A_{44} + A_{33}\cdot A_{44})\Big)\cdot\varphi - \\
- 12i\varphi + 6i\Big(A_{12}\cdot A_{34} + A_{13}\cdot A_{24} + B_{12}\cdot B_{34} + B_{13}\cdot B_{24}\Big)\cdot\varphi.
\end{aligned}
$$

It is possible to simplify this expression even a bit further if we recall the definition of the Hamiltonian of the Harmonic oscillator 1.2.1 which in our current notation can be written as

$$
H_0(f) = \frac{1}{2}\sum_{j=1}^{4} A_{jj}\cdot f,
$$

where $f \in L^2(\mathbb{R}^4)$. Therefore we have that

$$
\begin{aligned}
i\Psi_0\cdot\varphi = i12H_0^2(\varphi) - 12i\varphi + \\
+ 6i\Big(A_{12}\cdot A_{34} + A_{13}\cdot A_{24} + B_{12}\cdot B_{34} + B_{13}\cdot B_{24} - A_{11}\cdot A_{44} - A_{22}\cdot A_{33}\Big).
\end{aligned}
$$

An interesting property of this operator is that although we have up until this point worked locally this definition is actually global. It can be shown via a tedious and lengthy computation that for each element $X \in \mathfrak{h}$, it holds that $d(\mathbf{m}\circ\hat{Ad})(X)\circ(\Psi_0\cdot) = (\Psi_0\cdot)\circ d(\mathbf{m}\circ\hat{Ad})(X)$. And since the Lie subgroup $H = S(U(2)\times U(2))$ is connected (as was shown earlier) its representation $\mathbf{m}\circ\hat{Ad}$ also commutes with the operator $\Psi_0\cdot$. We can hence make the global definition. Let $\varphi \in \mathcal{C}(G,\mathcal{M}_l)^H$ we define

$$
(\Psi\cdot\varphi)(a) := \Psi_0\cdot(\varphi(a)).
$$

Because the representation of the group H commutes with the map $\Psi_0\cdot$ we get a well defined global operator.

Remark. Note that the operator $\Psi_0\cdot$ is an operator on the vector space $\mathcal{M}_l$ (even on $L^2(\mathbb{R}^4)$), while the operator $\Psi\cdot$ is an operator on the vector space $\Gamma(\mathbf{Q}_l)$. It is defined using the operator $\Psi_0\cdot$ at each point of any fiber and since the operator $\Psi_0\cdot$ commutes with the representation $\mathbf{m}\circ\hat{Ad}$ it is well defined.

Also as mentioned before the operators defined via the application map are defined globally as well.

Furthermore recall that the Casimir operator Ω can be realised as the applications of the vectors of the chosen basis of $\mathfrak{g}$. We have seen that for vectors from $\mathfrak{p}$, the multiplication commutes with the application at a point (see 4.3). On the other hand we know that the operator $\Psi\cdot$ commutes with the representation of the Lie subalgebra $\mathfrak{h}$ given by $d\hat{Ad}$ which is basically the same as applications of the vectors from $\mathfrak{h}$ to the symplectic spinor field.

We have thus proved the following theorem.

Theorem 4.2.2. *Given the standard Kähler structure on the manifold $Gr_2(\mathbb{C}^4)$ described in the previous section, then the second order operator $\mathcal{P}$ associated to the symplectic Dirac operators on this manifold can be decomposed into two commuting operators as follows*

$$\mathcal{P} = i[\tilde{D}, D] = -\Omega(\varphi) - \Psi \cdot \varphi,$$

where the operators Ω and Ψ are defined above.

4.3 Computing the point spectrum of Ω and $\Psi\cdot$

Point spectrum of the operator $\Psi\cdot$

We will start with the operator $\Psi\cdot$. As noted at the end of the previous section we can calculate its spectrum considering it only as an operator on the spaces $\mathcal{M}_l$. Because clearly it cannot have other eigenvalues then the ones coming from operating on the space $\mathcal{M}_l$, but of course it can have (and it will have) bigger multiplicities when acting on the "whole" $\Gamma(\mathbf{Q}_l)$. On the other hand, given an eigenfunction of this operator on $\mathcal{M}_l$ we can multiply it by a bump function for some neighbourhood and create a section which will be an eigenvector for the associated global operator.

Consider a basis B of $\mathcal{M}_l$ given by the normalised Hermite functions $\tilde{h}_\alpha$ where $\alpha \in \mathbb{N}_0^4$ and $|\alpha| = l$. This is an orthonormal basis with respect to the L^2-product on $\mathcal{M}_l$. We will show that the operator $\Psi_0\cdot$ has a symmetric matrix with respect to this basis, and thus it is orthogonally diagonalizable with respect to the L^2-product.

This means that there is an orthogonal decomposition of each vector space $\mathcal{M}_l$ into subspaces of eigenvectors $\mathcal{M}_l = V^l_{\gamma_1} \oplus \ldots \oplus V^l_{\gamma_{k_l}}$, where γ_j are eigenvalues of the operator $\psi_0 \cdot |_{\mathcal{M}_l}$ with V_{γ_j} as the appropriate eigenspace. This induces the further splitting of the vector bundle $\mathbf{Q}_l$

$$\mathbf{Q}_l = \bigoplus_{j=1}^{k_l} \mathbf{Q}_l^j := \bigoplus_{j=1}^{k_l} (SU(4) \times_{\mathbf{m}\circ\hat{Ad}} V^l_{\gamma_j}).$$

We will now show that the matrix of Ψ_0 is symmetric with respect to the basis B.

We will start by noticing that Hermite functions are eigenfunctions of the operators A_{jj} since

$$A_{jj} \cdot h_\alpha(x) = A_{jj} \cdot (h_{\alpha_1}(x_1)h_{\alpha_2}(x_2)h_{\alpha_3}(x_3)h_{\alpha_4}(x_4)) = -(2\alpha_j + 1)h_\alpha(x)$$

where $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = |\alpha| = l$.

This implies that the operator $H_0 = \sum_{j=1}^4 A_{jj}$ maps the Hermite functions to their multiple, the same works for the operator $A_{11} \cdot A_{44} + A_{22} \cdot A_{33}$ and, of course, also for the multiplication by a constant -12. Thus this part of the operator $\Psi_0 \cdot$ makes up the diagonal and we can focus on the remaining part, i.e. on

$$6\Phi \cdot := 6(A_{12} \cdot A_{34} + A_{13} \cdot A_{24} + B_{12} \cdot B_{34} + B_{13} \cdot B_{24}) \cdot .$$

We will rewrite the map $\Phi \cdot$ using the definition of the symplectic Clifford multiplication. Also for a less overwhelming notation we will write ∂_i instead of $\frac{\partial}{\partial x_i}$ We have

$$\begin{aligned}\Phi \cdot =&(\partial_1\partial_2\partial_3\partial_4 - \partial_1\partial_2 x_3 x_4 - \partial_3\partial_4 x_1 x_2 + x_1 x_2 x_3 x_4)+\\ &+(\partial_1\partial_2\partial_3\partial_4 - \partial_1\partial_3 x_2 x_4 - \partial_2\partial_4 x_1 x_3 + x_1 x_2 x_3 x_4)+\\ &+(-\partial_2\partial_4 x_1 x_3 + \partial_1\partial_4 x_2 x_3 - \partial_1\partial_3 x_2 x_4 + \partial_2\partial_3 x_1 x_4)+\\ &+(-\partial_3\partial_4 x_1 x_2 + \partial_1\partial_4 x_2 x_3 - \partial_1\partial_2 x_3 x_4 + \partial_2\partial_3 x_1 x_4),\end{aligned}$$

where we have used the fact that $\partial_j x_k = x_k \partial_j$ if $j \neq k$. Via a simple calculation it can be shown that this operator equals

$$\begin{aligned}\Phi \cdot =&(\partial_1 - x_1)(\partial_4 - x_4)(\partial_3 + x_3)(\partial_2 + x_2)+\\ &+(\partial_1 + x_1)(\partial_4 + x_4)(\partial_3 - x_3)(\partial_2 - x_2).\end{aligned}$$

We can see, for example, that elements with an odd number of "x's" cancel out since they have the opposite sign, while the those with an even number of "x's" are exactly twice each pair from the set $\{1, 2, 3, 4\}$, twice the one with no "x's" and twice the one with only "x's".

It is possible to show that the operators $\partial_i - x_i$ may be called rising operators for the Hermite functions, i.e. $(\partial_1 - x_1)h_{\alpha_1}(x_1) = h_{\alpha_1+1}(x_1)$ and the operators $\partial_i + x_i$ are lowering operators up to a constant, i.e. $(\partial_1 + x_1)h_{\alpha_1}(x_1) = (-2\alpha_1)h_{\alpha_1-1}$.

This means that the operator $\Phi \cdot$ 'works' in the following way

$$\Phi \cdot h_\alpha = 4\alpha_2\alpha_3(h_{\alpha_1+1}h_{\alpha_2-1}h_{\alpha_3-1}h_{\alpha_4+1}) + 4\alpha_1\alpha_4(h_{\alpha_1-1}h_{\alpha_2+1}h_{\alpha_3+1}h_{\alpha_4-1}),$$

where we are using the convention that $h_{-1} = 0$. If we look only at the indicies and not at the coefficients, we can see that certain subsets of multiindicies are left invariant. To make it precise.

Proposition 4.3.1. *The operator $\Phi \cdot$ defined above leaves the subspaces $J^l_{k_1,k_2}$ invariant, where*

$$J^l_{k_1,k_2} := span\{h_{\alpha_1,\alpha_2,\alpha_3,\alpha_4} | \alpha_1 + \alpha_2 = k_1, \alpha_1 + \alpha_3 = k_2, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = l\},$$

where $k_1, k_2 \in \{0, 1, \ldots, l\}$ are arbitrary. Furthermore, these subspaces form an orthogonal decomposition of the space $\mathcal{M}_l$.

Proof. The proof of the first part is straightforward from the realisation of the operator $\Phi\cdot$ as seen above, i.e. $k_1 = \alpha_1 + \alpha_2 = (\alpha_1 + 1) + (\alpha_2 - 1)$, etc. It is also clear that these subspaces are orthogonal with respect to each other, since the Hermite functions are such. Also from the fact that each Hermite function belong exactly to one $J^l_{k_1,k_2}$, the last assertion follows.

□

Remark. It is even possible to show that if we require the subspaces of $\mathcal{M}_l$ to be generated by some Hermite functions, the subspaces $J^l_{k_1,k_2}$ are the smallest invariant subspaces meeting such requirements.

We can imagine the subspaces $J^l_{k_1,k_2}$ as a chains of Hermite functions, where the operator $\Phi\cdot$ maps each subspace generated by a Hermite function to the direct sum of the upper and lower segment. For example $J^4_{2,2}$ looks like as follows

$$0 \longleftrightarrow \langle h_{2,0,0,2}\rangle \longleftrightarrow \langle h_{1,1,1,1}\rangle \longleftrightarrow \langle h_{0,2,2,0}\rangle \longleftrightarrow 0$$

where by the angle-brackets we mean the complete linear span. The $\Phi\cdot$ then acts as follows

$$\begin{aligned}
\Phi(h_{2,0,0,2}) &= 16h_{1,1,1,1}\\
\Phi(h_{1,1,1,1}) &= 4h_{2,0,0,2} + 4h_{0,2,2,0}\\
\Phi(h_{0,2,2,0}) &= 16h_{1,1,1,1}.
\end{aligned}$$

Also note that we have thus achieved for each $l \in \mathbb{N}_0$ a decomposition of $\mathcal{M}_l$ into $(l+1)^2$ subspaces. The dimension of each subspace $J^l_{k_1,k_2}$ is $\min\{k_1, k_2, l - k_1, l - k_2\} + 1$ which can be observed by introspection of the chains. This leads to the fact that for an even l there is exactly one longest chain and for an odd l there are 4 of equal maximum length.

We are now going to prove that the operator $\Phi\cdot|_{J^l_{k_1,k_2}}$ has a symmetric matrix with respect to the basis B constituted of normalised Hermite functions $\tilde{h}_\alpha$ in the same order as they appear in the chain. That is, the inner indicies are rising and the outer indicies are lowering. We denote this matrix by P. From the way of how $\Phi\cdot$ acts we see that P must be a tri-diagonal matrix with zero diagonal. Therefore we just need to compute the elements $P_{i,i+1}$ and $P_{i+1,i}$ for all appropriate i's.

Let us assume we have $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{N}_0^4$ such that $l = \alpha_1+\alpha_2+\alpha_3+\alpha_4$ fixed. Further we assume that $\alpha_1 \neq 0 \neq \alpha_4$, i.e. the $\tilde{h}_\alpha$ is not the last element in the basis B. If the element $\tilde{h}_\alpha$ is on the i-th position in the basis B, we are going to show that $P_{i,i+1} = P_{i+1,i}$ and thus proving the symmetry of the matrix P.

We know from [Folland, 1989, p.51-52] that $||h_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}|| = \pi\sqrt{2^l\alpha_1!\alpha_2!\alpha_3!\alpha_4!}$, where as usual $l = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$. In other words $\tilde{h}_\alpha = \frac{h_\alpha}{\pi\sqrt{2^l\alpha_1!\alpha_2!\alpha_3!\alpha_4!}}$ Let us now compute the entries explicitly. We have that

$$\Phi\cdot(\tilde{h}_\alpha)=\Phi\cdot(\frac{h_\alpha}{\pi\sqrt{2^l\alpha_1!\alpha_2!\alpha_3!\alpha_4!}})=\frac{4\alpha_1\alpha_4 h_{\alpha_1-1,\alpha_2+1,\alpha_3+1,\alpha_4-1}}{\pi\sqrt{2^l\alpha_1!\alpha_2!\alpha_3!\alpha_4!}}+$$
$$+\frac{4\alpha_2\alpha_3 h_{\alpha_1+1,\alpha_2-1,\alpha_3-1,\alpha_4+1}}{\pi\sqrt{2^l\alpha_1!\alpha_2!\alpha_3!\alpha_4!}}=$$
$$=\frac{4\alpha_1\alpha_4\pi\sqrt{2^l(\alpha_1-1)!(\alpha_2+1)!(\alpha_3+1)!(\alpha_4-1)!}\tilde{h}_{\alpha_1-1,\alpha_2+1,\alpha_3+1,\alpha_4-1}}{\pi\sqrt{2^l\alpha_1!\alpha_2!\alpha_3!\alpha_4!}}+$$
$$+\frac{4\alpha_2\alpha_3\pi\sqrt{2^l(\alpha_1+1)!(\alpha_2-1)!(\alpha_3-1)!(\alpha_4+1)!}\tilde{h}_{\alpha_1+1,\alpha_2-1,\alpha_3-1,\alpha_4+1}}{\pi\sqrt{2^l\alpha_1!\alpha_2!\alpha_3!\alpha_4!}}=$$
$$=4\sqrt{\alpha_1\alpha_4(\alpha_2+1)(\alpha_3+1)}\tilde{h}_{\alpha_1-1,\alpha_2+1,\alpha_3+1,\alpha_4-1}+$$
$$+4\sqrt{\alpha_2\alpha_3(\alpha_1+1)(\alpha_4+1)}\tilde{h}_{\alpha_1+1,\alpha_2-1,\alpha_3-1,\alpha_4+1}.$$

Thus we see that $P_{i+1,i}=4\sqrt{\alpha_1\alpha_4(\alpha_2+1)(\alpha_3+1)}$. Similarly one finds out that $P_{i,i+1}=4\sqrt{(\alpha_2+1)(\alpha_3+1)((\alpha_1-1)+1)((\alpha_4-1)+1)}=$ $4\sqrt{(\alpha_2+1)(\alpha_3+1)\alpha_1\alpha_4}$, where the second equality can be seen by renaming the indicies appropriately in the above chain of equalities ($\alpha_1\leftrightarrow\alpha_1-1,\ldots$). Thus we have proved the following proposition.

Proposition 4.3.2 (Decomposition of $\mathcal{M}_l$)**.** *For each $l\in\mathbb{N}_0$ there is an orthogonal decomposition of $\mathcal{M}_l$ into spaces of eigenvectors $V^l_{\gamma_j}$, associated with the eigenvalues γ_j, for the operator $\Psi_0\cdot$*

$$\mathcal{M}_l=V^l_{\gamma_1}\oplus\ldots\oplus V^l_{\gamma_{k_l}}.$$

Furthermore there is a splitting of the bundle of symplectic spinors that is preserved by the operator $\mathcal{P}$

$$\boldsymbol{Q}_l=\bigoplus_{j=1}^{k_l}\boldsymbol{Q}^j_l:=\bigoplus_{j=1}^{k_l}(SU(4)\times_{\boldsymbol{m}\circ\hat{Ad}}V^l_{\gamma_j}).$$

Proof. The orthogonal decomposition follows from the computations above the proposition

The subbundles are well-defined because the transition function commutes with $\Psi\cdot$ and therefore they must keep the spaces of eigenvectors invariant.

The last assertion follows from the fact that the operator $\Psi\cdot$ commutes with the Casimir operator Ω.

□

Altogether this means that we can compute the spectrum only on the bundles $\boldsymbol{Q}^l_j$. We just need to describe the spaces $V^l_{k_j}$ using an orthonormal basis obtained from the decomposition of the $\mathcal{M}_l$'s.

Unfortunately by the time of writing the thesis we have not yet managed to find explicitly such an orthonormal basis. Nevertheless, we have found a neat little trick that inserts the smaller chains into the longer chains, while preserving the eigenvalues. This has 2 important consequences:

Number one, for computing the eigenvalues of $\Psi_0\cdot$ on the space $\mathcal{M}_l$ it is enough to diagonalise the operator on the longest chain, in case of the even l, or on any of the maximum length chains, in case of the odd l.

Number two, for computing the eigenvalues of such a chain we can work inductively. In more detail, we can compute the eigenvalue on the chain of length 1 (that is usually trivial) and insert it into a chain of a length 2. The image under the insertion is an eigenvector with the same eigenvalue, Therefore it is enough to find its orthogonal complement and it is guaranteed that it is also a space of eigenvectors. We just need to compute the eigenvalue by applying $\Psi_0\cdot$ on it. In this way we work through until we come to the desired length.

Remark. This, by the way, determines the dimensions of different eigenspaces. In a more detail, eigenspaces of the space $\mathcal{M}_l$ have the following dimensions: The first eigenspace has dimension same as is the number of all chains of length at least 1 - $(l+1)^2$, second has the dimension of all the chains of length at least 2 - $(l-1)^2$,... 1 or 4, where the last dimension (of the last eigenspace) is determined by the parity of l.

This is rather a tedious way of finding the eigenvectors and eigenvalues, but a possible one. We plan to find an easier, more compact, way and release an article later on with the whole spectrum. However for the thesis we are going to compute just a small part of the spectrum by taking only the eigenvalues and eigenvectors from $\mathcal{M}_0$, $\mathcal{M}_1$ and $\mathcal{M}_2$.

Explicitly these are the following:

1. $\mathcal{M}_0$ is one dimensional space with eigenvalue 24.
2. $\mathcal{M}_1$ is 4-dimensional and constitutes of 4 chains each of length 1 and thus with the same eigenvalues. The eigenvalue is 72.
3. $\mathcal{M}_2$ is 10-dimensional with one chain of length 2 and 8 chains of length 1. The eigenspace from the 'longer chain' is $\langle h_{0,1,1,0} - h_{1,0,0,1}\rangle$ with eigenvalue 96 and the orthogonal complement (of dimension 9: the 8 chains + the one dimensional subspace from the 2-chain) has eigenvalue 144.

Point spectrum of the Casimir operator Ω

As in the previous section we start with the overall principle and then apply it to the subspaces $\mathcal{M}_0, \mathcal{M}_1$ and $\mathcal{M}_2$, respectively to the subspaces of the decompositions obtained above, i.e. $\mathcal{M}_0 = V^0_{24}$, $\mathcal{M}_1 = V^1_{72}$ and $\mathcal{M}_2 = V^2_{96} \oplus V^2_{144}$.

First let us recall the definition of the induced representation (all of the representations considered below are over the field of complex numbers.).

Definition 4.3.1 (Induced representation)**.** *Let L be a Lie group with a closed Lie subgroup $K \subseteq L$, and consider a representation μ of the group K on the vector space V, $\mu : K \to Aut(V)$. Then we define the space of the induced representation as the space of K-equivariant maps from L to V*

$$Ind^L_K(V) := \{f : L \to V | f(gh) = \mu(h^{-1})f(g)\}$$

and the induced representation as

$$Ind_\mu : L \to \mathrm{Aut}(Ind_K^L(V)), \qquad (Ind_\mu(g)f)(g') := f(g^{-1}g'),$$

where $g, g' \in L$ are arbitrary.

It is easy to verify that this morphism is a well-defined left action. Furthermore if we recall the identification of the sections on a homogeneous bundle with the equivariant functions, we may see that this is indeed a representation on the sections of the associated bundle to $L \to L/K$, i.e. the bundle $L \times_\mu V$.

Important result that we are going to use is the so-called Frobenius reciprocity [see Sepanski, 2007, Theorem 7.47]

Theorem 4.3.3 (Frobenius reciprocity)**.** *Let L be a Lie group with a closed Lie subgroup $K \subseteq L$. If we are given a representations $\mu : K \to Aut(V)$ and $\nu : L \to Aut(W)$ then there is a natural isomorphism*

$$\mathrm{Hom}_L(W, Ind_K^L(V)) \cong \mathrm{Hom}_K(W|_L, V),$$

where by the $W|_K$ we mean the restriction of the representation $\nu|_K$ from L to K.

By $Hom_L(U_1, U_2)$ we denote the space of all intertwining operators between the representations U_1 and U_2.

Proof. The maps are defined as follows. Given $T \in \mathrm{Hom}_L(W, Ind_K^L(V))$ we assign $S_T \in \mathrm{Hom}_K(W|_L, V)$ to it defined by $S_T(w) := T(w)(e)$, where $w \in W$ and e is the neutral element in L.

On the other hand given $S \in \mathrm{Hom}_K(W|_L, V)$, we assign to it an operator $T_S \in \mathrm{Hom}_L(W, Ind_K^L(V))$ defined by $T_S(w)(g) := S(g^{-1}w)$, where $w \in W$ and $g \in L$.

Verifying that these maps are well defined and their compositions are identities is straightforward.

□

Remark. In the language of category theory this shows that the inducing functor $Ind_K^L : \mathrm{Rep}_K \to \mathrm{Rep}_L$ and the restricting functor $|_K : \mathrm{Rep}_L \to \mathrm{Rep}_K$ (also denoted as Res_K^L) form an adjoint pair.

Reminder on the notation: We used in the definition and statement of the theorem, the groups denoted by L and K, because we intend to still keep the notation from earlier that $G = SU(4)$ and $H = S(U(2) \times U(2))$.

We relate the Casimir operator to the representation of the Casimir element of the Lie algebra $\mathfrak{g}$. We follow the work of Wyss [Wyss, 2003] Given a representation $\mu : G \to \mathrm{Aut}(V)$ of the group G. We define a representation of the Casimir element by

$$\Omega_\mu(v) := \sum_{j=1}^{15} d\mu(b_j)(d\mu(b_j)(v)), \tag{4.11}$$

where $v \in V$ is arbitrary and the b_j's form an orthonormal basis with respect to our choice of the inner product, i.e. a negative multiple of a Killing form. Compare this definition to the definition of the Casimir operator given before in the chapter (see 4.9).

Remark. Given a representation of any Lie group, it induces a representation of its Lie algebra and thus it also induces the representation of the universal enveloping algebra of this Lie algebra (If the group is simply connected all of these are in one-to-one correspondence). Now given an *ad*-invariant bilinear form (for example Killing form) it is possible to define a Casimir element in the universal enveloping algebra [see Dixmier and Society, 1996]. The representation of the Casimir element given above is exactly the representation of the Casimir element (given by our choice of the inner product on $\mathfrak{g}$) under the appropriate representation of the universal enveloping algebra.

It is a well-known fact that Casimir elements lie in the center of the universal enveloping algebrae. Therefore they commute with the representation of each element from the Lie algebra. Using the Schur's lemma, we can conclude that given an irreducible representation $\mu : \mathfrak{g} \to \mathrm{End}(V)$ of a Lie algebra $\mathfrak{g}$, the representation of the Casimir element acts as a λ-multiple of the identity where $\lambda \in \mathbb{C}$.

The next proposition relates a representation of the Casimir element for some irreducible representation with eigenvalues of the Casimir operator, we have defined earlier in this chapter, via the intertwining map from the induced representation. This is the Lemma 5.4. in Wyss [2003].

Proposition 4.3.4 (Eigenvalues of Casimir operator)**.** *Given an irreducible representation $\mu : G \to Aut(W)$, where the representation of the Casimir element is given by $\Omega_\mu = \lambda_\mu Id$, and an intertwining map $A \in \mathrm{Hom}_G(W, \Gamma(\boldsymbol{Q}_l^j))$. Then*

$$\Omega(Aw) = \lambda_\mu Aw, \tag{4.12}$$

for any $w \in W$.

Proof. First recall that $\Gamma(\mathbf{Q}_l^j) = Ind_H^G(\mathcal{M}_l)$ where the representation is $\mathbf{m} \circ \hat{A}d : H \to \mathcal{M}_l$. Now the proof is just a straightforward computation. Take $g \in G$ then we have

$$Aw(g) = Ind_{\mathbf{m}\circ\hat{A}d}(g^{-1})Aw(e) = A(\mu(g^{-1})w),$$

where in the second equality we have used the defining property of the intertwining maps. Given an orthonormal basis (b_j) of $\mathfrak{g}$ with respect to our choice of an inner product, we can compute

$$\begin{aligned} b_j(b_j(Aw))(g) &= \frac{d}{dt}\Big|_0 b_j(Aw)(g\exp(tb_j)) = \\ &= \frac{d}{dt}\Big|_0 \frac{d}{ds}\Big|_0 Aw(g\exp(tb_j)\exp(sb_j)) = \\ &= \frac{d}{dt}\Big|_0 \frac{d}{ds}\Big|_0 A(\mu(\exp(-sb_j)) \circ \mu(\exp(-tb_j)) \circ \mu(g^{-1})w) = \\ &= (A \circ d\mu(b_j) \circ d\mu(b_j) \circ \mu(g^{-1}))(w). \end{aligned}$$

Thus summing over all j gives

$$\Omega(Aw) = A \circ \Omega_\mu(\mu(g^{-1})w) = \lambda_\mu Aw,$$

thus proving the assertion.

□

This explains the name Casimir operator.

Let us consider all of the unitary representations of the group G and denote it by $(\mu : G \to \mathrm{Aut}(W_\mu))_{\mu \in M}$. It is a fact from the theory of representations (that follows from the Schur's lemma) that we have the decomposition

$$\Gamma(\mathbf{Q}_l^j) = Ind_H^G(V_{\gamma_j}^l) = \widehat{\bigoplus}_{\mu \in M} W_\mu \otimes \mathrm{Hom}_G(W_\mu, Ind_H^G(V_{\gamma_j}^l)).$$

Where the inclusion of the subspaces $W_\mu \otimes \mathrm{Hom}_G(W_\mu, Ind_H^G(V_{\gamma_j}^l))$ is given by the obvious morphism

$$w \otimes A \mapsto Aw.$$

Together with the proposition 4.12, we see that the Casimir operator respects this decomposition and furthermore it operates as the λ_μ multiple of the identity on each, so-called, isotypic component.

Hence for the computing of the spectrum, we are now only interested when the Hom-space is nonzero. For this we are going to use the Frobenius reciprocity 4.3.3 to obtain that

$$\mathrm{Hom}_G(W_\mu, Ind_H^G(V_{\gamma_j}^l)) \cong \mathrm{Hom}_H((W_\mu)|_H, V_{\gamma_j}^l).$$

This leads to a problem of finding the representations of $H = S(U(2) \times U(2))$ that appears in the decomposition of the irreducible representations of $G = SU(4)$. Solution to the problem are the branching rules. However, to compute these Hom-sets we require more information about the spaces $V_{\gamma_j}^l$. So we are going to finish this subsection with the branching rules just for our special cases, i.e. $\mathcal{M}_0$, $\mathcal{M}_1$ and $\mathcal{M}_2$.

For this we are going to use the paper from Milhorat [Milhorat, 1998] about computing the spectrum of the classical Dirac operators on the Grassmannians $Gr_2(\mathbb{C}^{m+2})$ for $m = 2$. Namely we are going to use the Proposition 2 in section IV to determine the branching rules. Also, we are going to use the notation described in the beginning of section IV.

All weights of the representations $V_{\gamma_j}^l$ can be computed directly by applying the endomorphisms $d\hat{A}d(E^0), d\hat{A}d(E_0)$ and $d\hat{A}d(S)$ since they all preserve the Hermite function. All that is left is to find the dominant weight that characterises each of these representations. We do that using the Lemma 2 and 3 and the definition of simple roots in Milhorat [1998].

The dominant weight of the irreducible representations $V_{24}^0, V_{72}^1, V_{96}^2, V_{144}^2$ are $[-1,-1,1,1], [-1,-2,2,1], [-2,-2,2,2], [-1,-3,3,1]$, respectively.

Now we use the proposition 2 to obtain the dominant weights for the irreducible representations of $SU(4)$ that have the appropriate irreducible representation of H, characterised by the dominant weight above, as a direct summand. (We again skip some tedious computations). We denote by β^l_λ the obtained dominant weights.

For $[-1,-1,1,1]$ we get $\beta^0_{24} \in \{[k+l+1, l+1, -(l+1), -(k+l+1)] | k, l \in \mathbb{N}_0\}$.

For $[-1,-2,2,1]$ we get $\beta^1_{72}\{[k+l+2, l+2, -(l+2), -(k+l+2)],$
$[k+l+3, l+1, -(l+2), -(k+l+2)], [k+l+3, l+3, -(l+2), -(k+l+4)] | k, l \in \mathbb{N}_0\}$.

For $[-2,-2,2,2]$ we get $\beta^2_{96} \in \{[k+l+2, l+2, -(l+2), -(k+l+2)] | k, l \in \mathbb{N}_0$.

For $[-1,-3,3,1]$ we get $\beta^2_{144} \in \{[k+l+3, l+3, -(l+3), -(k+l+3)]$
$[k+l+2, l, -(l+1), -(k+l+1)]$, $[k+l+4, l, -(l+2), -(k+l+2)]$,
$[k+l+3, l+3, -(l+1), -(k+l+5)] | k, l \in \mathbb{N}_0\}$
or $\beta^2_{144} \in \{[k+l+2, l+2, -(l+1), -(k+l+3)] | k, l \in \mathbb{N}_0$ and $(k,l) \neq (0,2), (0,3)\}$.

All that is to compute the eigenvalues of the Casimir operator. For this we refer to the formula (42) in Milhorat [1998].

Proposition 4.3.5 (eigenvalues of Casimir operator)**.** *With the notation above, the eigenvalues of the Casimir operator on the sections of $\boldsymbol{Q}_l$ for $l \in \{0,1,2\}$ are given by*

$$c(\beta^l_\lambda) = \frac{1}{2}\langle \beta^l_\lambda, \beta^l_\lambda + 2\delta \rangle,$$

where $\langle -, - \rangle$ is the standard Euclidean inner product on $\mathbb{R}^4$,
$\delta = \frac{1}{2}[3,1,-1,-3]$ and by β^l_λ we mean any β^l_λ from the above list.

Note that we have made a correction by a multiple of four, since in the article the inner product on $\mathfrak{g}$ differs from our by a multiple of 2 and thus our Casimir element is 4-times bigger.

We conclude with

Theorem 4.3.6 (eigenvalues of second order operator $\mathcal{P}$)**.** *With the notation above, the eigenvalues of the second operator $\mathcal{P}$ associated to the symplectic Dirac operators on the sections of $\boldsymbol{Q}_l$, for $l \in \{0,1,2\}$ are given by*

$$c(\beta^l_\lambda) + \lambda.$$

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List of Abbreviations

$Gr_k(\mathbb{C}^n)$	Grassmannian of k-planes in $\mathbb{C}^n$
$O(n,\mathbb{R})$	orthogonal group
$U(n)$	unitary group
$H(2n)$	Heisenberg group
$Sp(2n,\mathbb{R})$	symplectic group
$Mp(2n,\mathbb{R})$	metaplectic group
$\hat{U}(n)$	double cover of the unitary group
$L^2(\mathbb{R}^n)$	space of square integrable functions on $\mathbb{R}^n$
$\mathcal{M}_l$	subspace of $L^2(\mathbb{R}^n)$ generated by Hermite functions
$Cl(V)$	symplectic Clifford algebra on V
$\mathbf{m}$	Segal-Shale-Weil representation
$\mathbf{R}$	symplectic frame bundle
$\mathbf{P}$	metaplectic structure
$\mathbf{Q}$	symplectic spinor bundle
H_0	Hamiltonian of the harmonic oscillator
$D, \tilde{D}$	symplectic Dirac operators
$\mathcal{P}$	associated second order operator
h_α	Hermite function
G_j, JF_j	local unitary frame of $Gr_2(\mathbb{C}^4)$
A_{ij}, B_{ij}	basis of $\hat{U}(4)$
Ω	Casimir operator
Ind_H^G	induced representation from H to G