Ellipticity of complexes of symplectic twistor operators

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Aim

Use of commutant algebra of a group action on a vector space for analysis of complexes of invariant operators defined on functions with values in that vector space. Inv. ops. are defined by projections.

The setting: 1) A associative algebra over the field $\mathbb{C}$ of complex numbers

2) $\rho : A \to \mathrm{End}(W)$ representation of A on vector space W

3) *Commutant algebra (centralizer)*
$B = \mathrm{Comm}_A(W) = \{T : W \to W | \, T \circ \rho(a) = \rho(a) \circ T \text{ for all } a \in A\}$ space of all A-equivariant maps/A-homomorphisms/intertwiners

4) If A is semi-simple, then W is multiplicity-free as $A \otimes B$-module. I.e., if $W' \neq W''$ are $A \otimes B$-submodules of W, then W' and W'' are not isomorphic as $A \otimes B$-modules.

Basic example: Schur duality (Commutant algebra for GL on k-tensors)

$G = GL(V)$ and $W = \bigotimes^k V$, $\rho(g)(v_1 \otimes \ldots \otimes v_k) = gv_1 \otimes \ldots \otimes gv_k$ $\Longrightarrow (c_1 g_1 + c_2 g_2) \cdot w = c_1 \rho(g_1)(w) + c_2 \rho(g_2)(w)$, $g_1, g_2 \in G$, $c_1, c_2 \in \mathbb{C}$ and $w \in W$ be the *extension of the action to the group algebra* $\mathbb{C}[\rho(G)]$; $\tau(\pi)(v_1 \otimes \ldots \otimes v_k) = v_{\pi^{-1}(1)} \otimes \ldots \otimes v_{\pi^{-1}(k)}$, $\pi \in S_k$.

Result: $\mathrm{Comm}_A(W) = \mathbb{C}[\tau(S_k)]$.

Example concerning **harmonic polynomials**: $O(n)$ on $P = P[x^1, \ldots, x^n]$ by regular representation. Commutant algebra of the action of $\mathbb{C}[O(n)]$ on P is generated by $\Delta = -\frac{1}{2}\sum_{i=1}^n \partial^2_{x^i}$, $E = -\sum_{i=1}^n x^i \partial_{x^i} - \frac{n}{2}$ and multiplication by $r^2 = \frac{1}{2}\sum_{i=1}^n (x^i)^2$. Forms a representation of $\mathfrak{sl}(2, \mathbb{C})$ by $X \mapsto \Delta$, $H \mapsto E$, $Y \mapsto r^2$.

Literature on - examples - of commutant algebras

Excerpt of further examples: Slupinski [Slup] - $Spin(n)$ acting on spinor valued anti-symmetric forms; Howe [Ho]; Goodman, Wallach [GN] (text-book); Leites, Shchepochkina [L]; Krýsl [KrLie]; Bracx, De Schepper, Ellbode, Lávička, Souček [Br]; De Bie, Souček, Somberg [Bie].

Symplectic spinors

(V,ω) real symplectic vector space of dimension $2n$

$\lambda : \widetilde{G} = Mp(V,\omega) \to Sp(V,\omega)$, connected double cover of $G = Sp(V,\omega)$, $\widetilde{G}$ - metaplectic group, non-compact Lie group - parallel to the covering $Spin(n) \to SO(n)$

$\mathbb{L} \subseteq V$ maximal isotropic vector subspace: $\omega(v,w) = 0$ for all $v, w \in \mathbb{L}$, $\mathbb{L} \simeq \mathbb{R}^n$

$L : Mp(V,\omega) \to U(L^2(\mathbb{L}))$ distinguished *Segal–Shale–Weil/ /symplectic spinor/metaplectic/oscillator representation* [Sh], [Weil], [Kos]

$S = L^2(\mathbb{L})$ - **symplectic spinors**, $E = \bigoplus_{i=0}^{2n} \bigwedge^i V \otimes S$ - **symplectic spinor valued anti-symmetric forms**

$$\rho(g)(\alpha \otimes s) = \lambda(g)^* \alpha \otimes L(g)s$$

Decomposition of $E = \bigoplus_{i=0}^{2n} \bigwedge^i V \otimes S$

The module E decomposes [KrLie] as $\widetilde{G}$-module into direct sum

$$\bigoplus_{(i,j)\in\Xi} E^{ij},$$

where Ξ is a finite set ($(n+1)(2n+1)$ elements),
$E^{ij} = E^+_{ij} \oplus E^-_{ij} \subseteq \bigwedge^i V \otimes S$ and $E^{\pm}_{ij}$ are irreducible $\widetilde{G}$-modules.

p^{ij} projection of $\bigwedge^i V \otimes S$ onto E^{ij}

Lie super algebras

$\mathfrak{f} = \mathfrak{f}_0 \oplus \mathfrak{f}_1$ is a $\mathbb{Z}_2$-graded/aka super vector space

$|z| = i$ if $0 \neq z \in \mathfrak{f}_i$, $i \in \mathbb{Z}_2 = \{0, 1\}$

$[\![,]\!] : \mathfrak{f} \times \mathfrak{f} \to \mathfrak{f}$ is bilinear

- $[\![,]\!] : \mathfrak{f}_i \times \mathfrak{f}_j \to \mathfrak{f}_{i+j}$
- super anti-symmetric: $[\![x, y]\!] = -(-)^{|x||y|}[\![y, x]\!]$
- super Jacobi rule
  $(-)^{|x||z|}[\![x, [\![y, z]\!]]\!] + (-)^{|z||y|}[\![z, [\![x, y]\!]]\!] + (-)^{|y||x|}[\![y, [\![z, x]\!]]\!] = 0$

where $x, y, z \in \mathfrak{f}_0 \cup \mathfrak{f}_1$, $i, j \in \mathbb{Z}_2$ and $i + j$ means $i + j \bmod 2$

Lie super algebra $\mathfrak{f} = \mathfrak{osp}(1|2)$

$\mathfrak{f} = \mathfrak{f}_0 \oplus \mathfrak{f}_1$ (bosonic and fermionic part)

$\mathfrak{f}_0 = \mathrm{Lin}_{\mathbb{C}}(e^+, h, e^-) \cong \mathfrak{sl}(2, \mathbb{C})$

$\mathfrak{f}_1 = \mathrm{Lin}_{\mathbb{C}}(f^+, f^-)$

$$\begin{aligned} [\![h, e^\pm]\!] &= \pm e^\pm & [\![e^+, e^-]\!] &= 2h \\ [\![h, f^\pm]\!] &= \pm\frac{1}{2} f^\pm & [\![f^+, f^-]\!] &= \frac{1}{2} h \\ [\![e^\pm, f^\mp]\!] &= -f^\pm & [\![f^\pm, f^\pm]\!] &= \pm\frac{1}{2} e^\pm \end{aligned}$$

Commutant for sympl. spinor valued anti-symmetric forms

Consider $E = E_0 \oplus E_1$ as super vector space ($\mathbb{Z}_2$-grading), where $E_0 = \bigoplus_{i=0}^{n} \bigwedge^{2i} V \otimes S$, $E_1 = \bigoplus_{i=1}^{n} \bigwedge^{2i-1} V \otimes S$.
$p_+(\alpha \otimes s) = \alpha \otimes s_+$, $p_-(\alpha \otimes s) = \alpha \otimes s_-$, where $s = (s_+, s_-) \in S_+ \oplus S_- = S = L^2(\mathbb{L})$ is the decomposition into even and odd part.

Definition:
$F^+(\alpha \otimes s) = \frac{\imath}{2} \sum_{i=1}^{2n} \epsilon^i \wedge \alpha \otimes e_i \cdot s$ (degree rising),
$F^-(\alpha \otimes s) = \frac{1}{2} \sum_{i,j=1}^{2n} \omega^{ij} \iota_{e_i} \alpha \otimes e_j \cdot s$ (degree lowering).

Theorem ([KrLie] 2012; ArXiv 2008): Setting $\tau(f^\pm) = F^\pm$ and extending it to a homomorphism of Lie super algebras $\mathfrak{osp}(1|2)$ and $\mathrm{End}(E)$, we get $\mathrm{Comm}_{\mathbb{C}[\widetilde{G}]}(E) = \langle \tau(\mathfrak{osp}(1|2)), p_\pm \rangle$.

Symplectic twistor operators

$(\mathbb{R}^{2n}, \omega_0)$ symplectic vector space

$(e_i)_{i=1}^{2n}$ symplectic basis, $(\epsilon^i)_{i=1}^{2n} \subseteq (\mathbb{R}^{2n})^*$ dual basis

$f : \mathbb{R}^{2n} \to E^{ij} \subseteq \bigwedge^i V \otimes S$, smooth (C^∞)

$(\nabla f)(y) = \sum_{k=1}^{2n} \epsilon^k \wedge (\frac{\partial f}{\partial x^k})(y) \in \bigwedge^{i+1} V \otimes S$, $y \in \mathbb{R}^{2n}$,

$(T_{\pm}^{ij} f)(y) = p^{i+1, j\pm 1}(\nabla f)(y)$ symplectic twistor operators

Parallel to Dolbeault operators in (almost)complex analysis.

Symplectic Dirac operators defined by K. Habermann [KH] in the nineties. ([Habs] monograph on sympl. Dirac.)

Complexes of symplectic twistor operators

Theorem [KrMon], [KrArch]: If (M, ω) is a smooth symplectic manifold ($d\omega = 0$), with vanishing second Stiefel–Whitney class, ∇ is a symplectic torsion-free connection ($\nabla\omega = 0$, torsion of $\nabla = 0$) and the symplectic Weyl curvature ([Vais]) of ∇ vanishes, then $(C^\infty(M, E^{i+k,j\pm k}), T_\pm^{i+k,j\pm k})_k$ is an elliptic complex, i.e.,

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$T_\pm^{i+k+1,j\pm k\pm 1} T_\pm^{i+k,j\pm k} = 0$ and
$\mathrm{Im}\,\sigma(T_\pm^{i+k,j\pm k},\xi) = \mathrm{Ker}\,\sigma(T_\pm^{i+k+1,j\pm k\pm 1},\xi)$, $0 \neq \xi \in T^*M$.

Use of commutant: Twistor ops. are given by $p^{i+1,j\pm 1}$ and the covariant derivative. $p^{i+1,j\pm 1}$ are projections onto $\widetilde{G}$-submodule, thus $\widetilde{G}$-homomorphisms. Thus they belong to the commutant algebra $\mathrm{Comm}_{\mathbb{C}[\widetilde{G}]}(E)$, which is generated by $F^\pm$ and the two projections $p_\pm$ onto the even and odd part.

Symbols of symplectic twistor complex

$$\sigma_i(\alpha \otimes f) = \sigma(T_+^{ii}, \xi)(\alpha \otimes f) = p^{ii}(\xi \wedge \alpha \otimes s) =$$

$$= \xi \wedge \alpha \otimes f + \frac{2}{i-n} F^+(\alpha \otimes \xi \cdot f) + \frac{\imath}{i-n} E^+(\iota_\xi \alpha \otimes f)$$

Ellipticity: $\operatorname{Im} \sigma_{i-1} = \operatorname{Ker} \sigma_i$ for $\xi \neq 0$

Assume $\alpha \otimes f \in \operatorname{Ker} \sigma_i$

Folded applying of operators F^-, E^- and using the relations defining $\mathfrak{osp}(1|2) \implies \xi \wedge \alpha \otimes f = 0$

$\implies$ trivial case of a version of Cartan lemma $\alpha = \xi \wedge \beta \implies$

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$\alpha \otimes f = \xi \wedge \beta \otimes f = p^{ii}(\xi \wedge \beta \otimes f) = \sigma_{i-1}(\beta \otimes f)$

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