Ann Glob Anal Geom (2014) 45:197–210
DOI 10.1007/s10455-013-9394-9

Hodge theory for elliptic complexes over unital C^*-algebras

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Received: 1 April 2013 / Accepted: 23 July 2013 / Published online: 3 August 2013

Abstract For a unital C^*-algebra A, we prove that the cohomology groups of A-elliptic complexes of pseudodifferential operators in finitely generated projective A-Hilbert bundles over compact manifolds are finitely generated A-modules and Banach spaces provided the images of certain extensions of the so-called associated Laplacians are closed. We also prove that under this condition, the cohomology groups are isomorphic to the kernels of the associated Laplacians. This establishes a Hodge theory for these structures.

Keywords Hodge theory · Elliptic complexes · A-Hilbert bundles

Mathematics Subject Classification (2000) 58J10 · 46L87 · 58A14 · 35J05

1 Introduction

In this paper, we deal with fields having values in infinite rank vector bundles and systems of pseudodifferential equations for these fields. To be more precise, we focus our attention at complexes of pseudodifferential operators acting between smooth sections of finitely generated projective A-Hilbert bundles over compact manifolds. We recall a definition of an A-elliptic complex and prove a Hodge theory for a certain class of them. By Hodge theory, we mean not only constructing an A-linear isomorphism between the cohomology groups and the spaces of harmonic elements of the complex, but also giving a description of the cohomology groups from a topological point of view.

For a unital C^*-algebra A, a finitely generated projective A-Hilbert bundle is, roughly speaking, a fiber bundle the total space of which is a Banach manifold and the fibers of which are finitely generated projective Hilbert A-modules. A Hilbert A-module is a module over the algebra A which is first equipped with a positive definite A-sesquilinear map with values

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in A and second, it is a Banach space with respect to the norm defined by the A-sesquilinear map and by the norm in the C^*-algebra A. Our reference for Hilbert A-modules is Lance [6], Manuilov and Troitsky [7] and Wegge-Olsen [18].

One of the basic steps in proving the so-called Fomenko–Mishchenko index theorem for *A-elliptic pseudodifferential operators* acting on sections of finitely generated projective A-Hilbert bundles over compact manifolds is a construction of pseudoinverses to extensions of these operators to Sobolev type completions of spaces of smooth sections. See Fomenko, Mishchenko [3] for this construction. In [3], not only a parametrix for an A-elliptic pseudodifferential operator in finitely generated projective A-Hilbert bundles over compact manifolds is constructed, but the authors prove also that such operators are A-Fredholm. In particular, kernel of these operators are Hilbert A-submodules of finitely generated projective Hilbert A-modules.

For a chain complex $(D_i, \Gamma^i)_{i\in\mathbb{N}_0}$ of pre-Hilbert A-modules and adjointable pre-Hilbert A-module homomorphisms, we can form a sequence of the so-called *associated Laplacians* $\triangle_i$, $i \in \mathbb{N}_0$, without supposing any topology on the modules. Namely, one sets $\triangle_i = D_i^* D_i + D_{i-1} D_{i-1}^*$ simply. Assume that each $\triangle_i$ possesses a parametrix, i.e., there exist pre-Hilbert A-module homomorphisms $g_i : \Gamma^i \to \Gamma^i$ and $p_i : \Gamma^i \to \Gamma^i$ such that parametrix equations $1_{|\Gamma^i} = \triangle_i g_i + p_i$, $1_{|\Gamma^i} = g_i \triangle_i + p_i$ are satisfied and p_i maps Γ^i into the kernel of $\triangle_i$. We call the complexes with such behaved Laplacians *parametrix possessing*. It is quite interesting and we prove that in this case, the pseudoinverses g_i are necessarily chain homomorphisms, i.e., $g_{i+1} D_i = D_i g_i$. This is at least implicitly known in the finite rank situation as well. We derive this property without assuming any topological structure on the modules. Knowing this, it is not difficult to show that the cohomology groups of a parametrix possessing complex are A-linearly isomorphic to the kernels of the appropriate Laplacians, i.e., to the spaces of *harmonic elements*.

Let $D^\bullet$ be a complex of pseudodifferential operators acting on smooth sections of A-Hilbert bundles. This complex is called A-elliptic if its symbol sequence is exact out of the zero section of the cotangent bundle. We prove that the symbols of the Laplacians associated with A-elliptic complexes are isomorphisms out of the zero section. In what follows, all mentioned bundles are supposed to be finitely generated projective A-Hilbert bundles and the base manifolds are supposed to be compact. Under this condition, we prove the smooth regularity for A-elliptic pseudodifferential operators using an analogue of the Sobolev embedding theorem for A-Hilbert bundles. If we assume that the images of extensions to Sobolev type completions (specified later in the text) of each of the associated Laplacians of an A-elliptic complex are closed, we are able to construct parametrix equations for the original Laplacians. We thus prove that such A-elliptic complexes are parametrix possessing. Especially, their cohomology groups are isomorphic to the spaces of harmonic elements as A-modules. This establishes the algebraic part of the Hodge theory (the isomorphism of the cohomology groups and the spaces of harmonic elements) for these structures. In the sequel, the mentioned A-Fredholm property of A-elliptic pseudodifferential operators is used to prove that the **cohomology groups** of this subclass of A-elliptic complexes are **finitely generated A-modules**. Due to some simple topological reasoning, we then prove that the cohomology groups of these complexes are **Banach spaces**.

The smooth regularity of an A-elliptic operator in finitely generated projective A-Hilbert bundles over compact manifolds was already proved. See, e.g., pp. 101 in [14, Theorem 2.1.145]. A formulation of the Sobolev type embedding can be also found there. See pp. 93 in [14, Theorem 2.1.80]. Let us remark that we do not give substantially new proofs of these assertions, but we try to write as much self-contained proofs of these facts as possible. Let us also notice that although we could have considered general C^*-algebras at least until Lemma 5 inclusively, we decided to suppose that all C^*-algebras are unital from the beginning.

We do not attempt to give a full reference to the topic of complexes of pseudodifferential operators in A-Hilbert bundles and refer the reader to the following more geometrically oriented K-theoretic works of Pavlov [10], Schick [11], Shubin [12], Troitsky [15,16], and Troitsky and Frank [17]. Let us remark that there are generalizations of the classical Hodge theory in directions different from that one described here. See, e.g., Bartholdi et al. [1] and Smale et al. [13] for a generalization to complete separable metric spaces and also for further references given there. We develop the presented theory mainly to enable a description of solutions to equations for operator-valued fields which are similar to equations appearing in Quantum field theory. However, the immediate motivation comes from geometric quantization via the so-called symplectic spinor fields. See Kostant [5], Fedosov [2], and Habermann [4] for this context.

In the second section, we prove a theorem on the chain homotopy properties of parametrix possessing complexes (Theorem 3) and in Theorem 4, we derive the algebraic form of the Hodge theory for them. In the third part of this paper, the definition of a finitely generated projective A-Hilbert bundle, Sobolev type completions of smooth sections of an A-Hilbert bundle and the definition of the Fourier transform of A-Hilbert bundle sections are recalled. The A-Hilbert bundle version of the Sobolev embedding theorem is stated as Lemma 5. The smooth regularity for A-elliptic operators is proved in Theorem 7, and the smooth pseudoinverses for the A-elliptic and self-adjoint ones are constructed in Theorem 8, building a core of the paper. In the fourth section, we mention a definition of an A-elliptic complex and prove that its cohomology groups are finitely generated A-modules and Banach spaces under the mentioned condition on the images of the extensions of the associated Laplacians (Theorem 11).

Preamble: In the whole text when not said otherwise, manifolds, fiber bundles (bundle projections, total and base spaces) and sections of fiber bundles are assumed to be smooth. Further, if an index exceeds its allowed range, the object labeled by this index is supposed to be zero.

2 Parametrix possessing complexes

To fix a terminology, we recall some notions from the theory of Hilbert A-modules. Let A be a unital C^*-algebra. For a pre-Hilbert A-module $(U, (,)_U)$, let $| \, |_U$ denote the associated norm on U defined by $|u| = \sqrt{|(u,u)_U|_A}$, $u \in U$, where $| \, |_A$ denotes the norm on A. If $(U, | \, |_U)$ is a complete normed space, $(U, (,)_U)$ is called a *Hilbert A-module*. If U is a pre-Hilbert A-module or a Hilbert A-module, the A-valued map $(,)_U : U \times U \to A$ is called an A-product or a Hilbert A-product, respectively. For definiteness, we consider left pre-Hilbert A-modules, and the A-products are supposed to be conjugate linear in the second variable. For pre-Hilbert A-modules U, V, the space of continuous A-module homomorphisms between U and V is denoted by $\mathrm{Hom}_A(U, V)$ and its elements are called pre-Hilbert A-module homomorphisms. If U, V are Hilbert A-modules, we omit the prefix pre. The notion of continuity is meant with respect to the norms $| \, |_U$ and $| \, |_V$. Further, we say that $u, v \in U$ are orthogonal if $(u, v)_U = 0$. When we write a finite direct sum, the summands are supposed to be mutually orthogonal pre-Hilbert A-modules. The *adjoints* of pre-Hilbert A-module homomorphisms are always thought with respect to the considered A-products. Let us remark that there exist Hilbert A-module homomorphisms of a Hilbert A-module $(U, (,)_U)$ which are adjointable with respect to a Hilbert space scalar product $U \times U \to \mathbb{C}$ on U, the norm induced by which is equivalent to the norm $|\,|_U$, but which do not posses an adjoint with respect to $(,)_U$. For this, see, e.g., Manuilov and Troitsky [7]. A Hilbert A-module U is called *finitely generated projective* if there exist $n \in \mathbb{N}_0$ and a Hilbert A-module V such $A^n \simeq U \oplus V$, where A^n denotes the direct sum of n copies of the standard Hilbert A-module A. Let us notice that instead of finitely generated the term algebraically finitely generated is also used. For more on these notions, we refer the reader to Manuilov and Troitsky [7] and to the pioneering work of Paschke [9].

Let us recall an A-theoretic generalization of the rank-nullity theorem from linear algebra.

Theorem 1 *Let U, V be Hilbert A-modules and $L \in Hom_A(U, V)$ be an adjointable Hilbert A-module homomorphism. If the image of L is closed, then also the image of L^* is closed and $Rng\, L = (Ker\, L^*)^\perp$, $Rng\, L^* = (Ker\, L)^\perp$ and $U = Ker\, L \oplus Rng\, L^*$.*

Proof See Lance [6] and a reference there for the original proof. □

Remark 1 Under the same assumptions as in Theorem 1, the equality $\mathrm{Ker}\, L^* = (\mathrm{Rng}\, L)^\perp$ holds.

Lemma 2 *Let U, V, W be pre-Hilbert A-modules and*

$$U \xrightarrow{D} V \xrightarrow{D'} W$$

be a sequence of adjointable pre-Hilbert A-module homomorphisms. Then for $\triangle = D'^* D' + DD^*$, we have

$$Ker\, \triangle = Ker\, D' \cap Ker\, D^*.$$

Proof From the definition of $\triangle$, we get the inclusion $\mathrm{Ker}\, D' \cap \mathrm{Ker}\, D^* \subseteq \mathrm{Ker}\, \triangle$. For each $v \in V$, we may write $(v, \triangle v)_V = (v, D'^* D' v + DD^* v)_V = (v, D'^* D' v)_V + (v, DD^* v)_V = (D'v, D'v)_W + (D^* v, D^* v)_U$. From this, the opposite inclusion follows using the positive definiteness of the A-products on W and U. □

To each complex $D^\bullet = (D_i, \Gamma^i)_{i\in\mathbb{N}_0}$ of pre-Hilbert A-modules and adjointable pre-Hilbert A-module homomorphisms, we attach the sequence

$$\triangle_i = D_{i-1} D_{i-1}^* + D_i^* D_i, \quad i \in \mathbb{N}_0$$

of *associated Laplacians*, where we assume $D_{-1} = 0$ (according to the preamble).

Theorem 3 *Let Γ^i, $i = 1, \ldots, 5$, be pre-Hilbert A-modules and*

$$\Gamma^1 \xrightarrow{D_1} \Gamma^2 \xrightarrow{D_2} \Gamma^3 \xrightarrow{D_3} \Gamma^4 \xrightarrow{D_4} \Gamma^5$$

be a complex of adjointable pre-Hilbert A-module homomorphisms. Suppose that for $i = 1, \ldots, 4$, there exist elements $g_i, p_i \in Hom_A(\Gamma^i, \Gamma^i)$ such that

$$1_{|\Gamma^i} = g_i \triangle_i + p_i, \tag{1}$$

$$1_{|\Gamma^i} = \triangle_i g_i + p_i \text{ and} \tag{2}$$

$$\triangle_i p_i = 0. \tag{3}$$

Then for $i = 1, 2, 3$,

$$p_{i+1} D_i = 0 \quad \text{and} \quad D_i g_i = g_{i+1} D_i. \tag{4}$$

Proof We use the relation $D_i p_i = 0,\ i = 1, \ldots, 4$, repeatedly, which follows from Lemma 2 and from Eq. (3). For $i = 1, 2, 3$ and $u \in \Gamma^i$, we may write

$$\begin{aligned} p_{i+1} D_i u &= p_{i+1} D_i (\triangle_i g_i u + p_i u) \\ &= p_{i+1} D_i \triangle_i g_i u + p_{i+1} (D_i p_i) u \\ &= p_{i+1} D_i (D_{i-1} D_{i-1}^* + D_i^* D_i) g_i u \\ &= p_{i+1} (D_i D_i^*) D_i g_i u \\ &= p_{i+1} \triangle_{i+1} D_i g_i u \\ &= (1 - \triangle_{i+1} g_{i+1}) \triangle_{i+1} D_i g_i u \\ &= \triangle_{i+1} D_i g_i u - \triangle_{i+1} g_{i+1} \triangle_{i+1} D_i g_i u \\ &= \triangle_{i+1} D_i g_i u - \triangle_{i+1} (1 - p_{i+1}) D_i g_i u \\ &= \triangle_{i+1} D_i g_i u - \triangle_{i+1} D_i g_i u + \triangle_{i+1} p_{i+1} D_i g_i u \\ &= 0 \end{aligned}$$

where we used $\triangle_{i+1} p_{i+1} = 0,\ i = 1, 2, 3$, in the last step. Notice that in rows 2 and 5 of the above computation, we derived the relation $D_i \triangle_i = \triangle_{i+1} D_i$ which we use in what follows. For $i = 1, 2, 3$ and $u \in \Gamma^i$, we have

$$\begin{aligned} g_{i+1} D_i u &= g_{i+1} D_i (\triangle_i g_i + p_i) u \\ &= g_{i+1} D_i \triangle_i g_i u + g_{i+1} (D_i p_i) u \\ &= g_{i+1} \triangle_{i+1} D_i g_i u \\ &= (1 - p_{i+1}) D_i g_i u = D_i g_i u \end{aligned}$$

where in the last step, we used the relation $p_{i+1} D_i = 0$ derived above. □

Remark 2 (1) We call Eqs. (1) and (2) the *parametrix equations* and any pre-Hilbert A-module homomorphism g_i for which the relations (1), (2) and (3) hold the *pseudoinverse* of the pre-Hilbert A-module homomorphism $\triangle_i$. In the special case $A = \mathbb{C}$, the name Green's function is also used.
(2) The second relation in row (4) implies that $g^\bullet = (g_i)_{i=1,2,3}$ is a *chain map*.

Remark 3 (1) Notice that from Eq. (1), we get $p_{i|\mathrm{Ker}\,\triangle_i} = 1_{|\mathrm{Ker}\,\triangle_i},\ i = 1, \ldots, 4$. Using this fact and relation (3), we see that p_i maps Γ^i not only into but also onto $\mathrm{Ker}\,\triangle_i$.
(2) Suppose now that Eqs. (1) and (3) are valid. Then applying p_i on Eq. (1) from the right, implies $p_i^2 = p_i$, i.e., p_i is an idempotent in $\mathrm{Hom}_A(\Gamma^i, \Gamma^i),\ i = 1, \ldots, 4$.

Definition 1 We call a complex $D^\bullet = (D_i, \Gamma^i)_{i \in \mathbb{N}_0}$ of pre-Hilbert A-modules and adjointable pre-Hilbert A-module homomorphisms *parametrix possessing* if for each $i \in \mathbb{N}_0$, there exist elements $g_i, p_i \in \mathrm{Hom}_A(\Gamma^i, \Gamma^i)$ such that Eqs. (1), (2) and (3) are satisfied.

Definition 2 For a complex $D^\bullet = (D_i, \Gamma^i)_{i \in \mathbb{N}_0}$ of pre-Hilbert A-modules and pre-Hilbert A-module homomorphisms, we define the cohomology groups

$$H^i(D, A) = \frac{\mathrm{Ker}(D_i : \Gamma^i \to \Gamma^{i+1})}{\mathrm{Rng}(D_{i-1} : \Gamma^{i-1} \to \Gamma^i)}, \quad i \in \mathbb{N}_0.$$

The nominator in the fraction, the cycles of the complex, is denoted by $Z^i(D, A)$, and the space of boundaries, $\mathrm{Rng}\,(D_{i-1} : \Gamma^{i-1} \to \Gamma^i)$, by $B^i(D, A)$.

Let us remark that we consider the cohomology groups with the canonical quotient A-module structure. We do not speak about a pre-Hilbert A-module structure on them because we even do not know whether $B^i(D, A)$ is a closed subspace of Γ^i.

Definition 3 For a complex $D^\bullet = (D_i, \Gamma^i)_{i \in \mathbb{N}_0}$ of pre-Hilbert A-modules and adjointable pre-Hilbert A-module homomorphisms, the elements of the kernel of the associated Laplacian $\triangle_i : \Gamma^i \to \Gamma^i$ are called harmonic elements of $D^\bullet$. We set $K^i(D, A) = \{a \in \Gamma^i \,|\, \triangle_i a = 0\}$.

Let us prove the algebraic part of the Hodge theory for parametrix possessing complexes—the isomorphism between the cohomology groups and the harmonic elements.

Theorem 4 *If $D^\bullet = (D_i, \Gamma^i)_{i \in \mathbb{N}_0}$ is a parametrix possessing complex, then for each $i \in \mathbb{N}_0$,*

$$H^i(D, A) \simeq K^i(D, A) \text{ as } A\text{-modules}.$$

Proof Because $D^\bullet$ is parametrix possessing, there exist maps $p_i, g_i \in \mathrm{Hom}_A(\Gamma^i, \Gamma^i)$ satisfying Eqs. (1) (2) and (3) for each $i \in \mathbb{N}_0$.

(1) Consider the map $\Phi_i : K^i(D, A) \to H^i(D, A)$ given by $\Phi_i a = [a]$, $a \in K^i(D, A)$. This map is well defined. Indeed, due to Lemma 2, $a \in \mathrm{Ker}\, \triangle_i$ implies $a \in \mathrm{Ker}\, D_i$. It is evident that Φ_i is an A-module homomorphism.

(2) Let us consider the map $\Psi_i : H^i(D, A) \to K^i(D, A)$ given by $\Psi_i[a] = p_i a$ for each $a \in Z^i(D, A)$. It follows from Theorem 3 that Ψ_i is well defined. Indeed, for $a = D_{i-1}b$ we have $p_i a = p_i D_{i-1} b = 0$ due to the first relation in row (4). It is evident that Ψ_i is an A-module homomorphism. Now, we prove that Ψ_i is inverse to Φ_i. For $a \in K^i(D, A)$, we may write $\Psi_i(\Phi_i a) = \Psi_i[a] = p_i a = a$. On the other hand, $\Phi_i(\Psi_i[b]) = \Phi_i(p_i b) = [p_i b]$ for any $b \in Z^i(D, A)$. The proof of the fact that b and $\tilde{b} = p_i b$ are cohomologous proceeds as follows: using the parametrix equation (1), we get $b - \tilde{b} = b - p_i b = (1 - p_i)b = g_i \triangle_i b = g_i D_{i-1} D^*_{i-1} b + g_i D^*_i D_i b = g_i D_{i-1} D^*_{i-1} b = D_{i-1}(g_{i-1} D^*_{i-1} b)$, where we used the second equation in row (4) from Theorem 3 and the fact that $b \in \mathrm{Ker}\, D_i$.

□

3 A-elliptic pseudodifferential operators

Let $p : \mathcal{E} \to M^n$ be a Banach bundle and A be a unital C^*-algebra. In particular, $\mathcal{E}$ is a Banach manifold. We call p an *A-Hilbert bundle* if

(1) there exists a Hilbert A-module $(U, (,)_U)$ (the typical fiber of p)
(2) for each $x \in M$, the fiber $\mathcal{E}_x = p^{-1}(x)$ is equipped with a Hilbert A-module structure and as such, it is isomorphic to $(U, (,)_U)$
(3) for each $x \in M$, the subset topology on $\mathcal{E}_x \subseteq \mathcal{E}$ is equivalent to the norm topology on $(U, |\,|_U)$ and
(4) the transition maps between the bundle charts of p are maps into the group $\mathrm{Aut}_A(U)$ of Hilbert A-module automorphisms of U.

Let us notice that all fiber bundles which appear in this text are supposed to be locally trivial. We call a smooth map $S : \mathcal{E}_1 \to \mathcal{E}_2$ acting between the total spaces of A-Hilbert bundles $p_1 : \mathcal{E}_1 \to M$ and $p_2 : \mathcal{E}_2 \to M$ with typical fibers U_1 and U_2, respectively, an A-Hilbert bundle *homomorphism* if it satisfies the equation $p_2 \circ S = p_1$ and if it is a Hilbert A-module homomorphism in each fiber. Let S be an

A-Hilbert bundle homomorphism. We call an A-Hilbert bundle homomorphism $T : \mathcal{E}_2 \to \mathcal{E}_1$ adjoint to S if $T_{|p_2^{-1}(x)} : p_2^{-1}(x) \to p_1^{-1}(x)$ is adjoint to the Hilbert A-module homomorphism $S_{|p_1^{-1}(x)} : p_1^{-1}(x) \to p_2^{-1}(x)$ for each $x \in M$. In this case, we write $T = S^*$. An A-Hilbert bundle is called *finitely generated projective* A-Hilbert bundle if its typical fiber is a finitely generated projective Hilbert A-module.

The vector space $\Gamma(\mathcal{E})$ of smooth sections of p carries a structure of an A-module given by $(as)(x) = a(s(x))$, where $a \in A$, $s \in \Gamma(\mathcal{E})$ and $x \in M$. For a compact manifold M and a Riemannian metric g on M, we fix a volume element on M and denote it by $|\mathrm{vol}_g|$. The volume element induces an A-product on $\Gamma(\mathcal{E})$ by the formula

$$(s, s')_{\Gamma(\mathcal{E})} = \int\limits_{x \in M} (s(x), s'(x))_U |\mathrm{vol}_g(x)| \in A$$

where $s, s' \in \Gamma(\mathcal{E})$. At the right-hand side, we consider the Bochner integral. In this way, $\Gamma(\mathcal{E})$ gains a structure of a pre-Hilbert A-module. We denote the induced norm by $|\ |_{\Gamma(\mathcal{E})}$. For each $t \in \mathbb{Z}$, one defines a further A-product $(,)_t$ on $\Gamma(\mathcal{E})$ by setting

$$(s, s')_t = \int\limits_{x \in M} ((1 + \triangle_g)^t s(x), s'(x))_U |\mathrm{vol}_g(x)| \in A$$

where $\triangle_g$ is the (positive definite) Laplace–Beltrami operator of (M, g) and $s, s' \in \Gamma(\mathcal{E})$. The induced norm will be denoted by $|\ |_t$, and the completion of $\Gamma(\mathcal{E})$ with respect to $|\ |_t$ by $W^t(\mathcal{E})$. In particular, $\Gamma(\mathcal{E})$ is dense in $W^t(\mathcal{E})$. Due to the construction, $W^t(\mathcal{E})$ together with the extended A-product $(,)_t$ form a Hilbert A-module. We call any $(W^t(\mathcal{E}), (,)_t)$ the *Sobolev type completion* of $\Gamma(\mathcal{E})$. See Fomenko, Mishchenko [3] for more information on the introduced A-products.

Because $|\ |_{0|\Gamma(\mathcal{E})} = |\ |_{\Gamma(\mathcal{E})}$, the Hilbert A-module $W^0(\mathcal{E})$ coincides with the completion of $\Gamma(\mathcal{E})$ with respect to $|\ |_{\Gamma(\mathcal{E})}$. For $t \in \mathbb{Z}$ and $k \geq t > 0$, it is not difficult to see that

$$\Gamma(\mathcal{E}) \subseteq \mathcal{C}^k(\mathcal{E}) \subseteq W^t(\mathcal{E}) \subseteq W^{t-1}(\mathcal{E}).$$

A section of $\mathcal{E}$ belongs to $\mathcal{C}^k(\mathcal{E})$ if and only if it is at least k times continuously differentiable. For accuracy reasons, let us notice that we consider the elements of $\Gamma(\mathcal{E})$, $\mathcal{C}^k(\mathcal{E})$ and $W^t(\mathcal{E})$ modulo the relation of being zero almost everywhere and with respect to it, also the inclusions above should be interpreted appropriately.

For any O open in M, we use the symbol $\mathcal{C}^k(O, \mathcal{E})$ to denote the space of the appropriate sections of the restricted bundle $p_O : p^{-1}(O) \to O$. The symbol $W^t(O, \mathcal{E})$, $t \in \mathbb{Z}$, has to be understood similarly. Let us recall a definition of the Fourier transform of local sections of A-Hilbert bundles over compact Riemannian manifolds. For a local chart $(\tilde{O} \subseteq \mathbb{R}^n, \phi : \tilde{O} \to \phi(\tilde{O}) \subseteq M^n)$, the Fourier transform of a section s of $p_{\phi(\tilde{O})}$ with support in a compact subset of $\phi(\tilde{O})$ is defined by

$$\hat{s}(q) = \int\limits_{x \in \phi(\tilde{O})} \mathrm{e}^{-2\pi\imath\langle q, x\rangle} s(x)\, \mathrm{d}x$$

where $q \in \mathbb{R}^n$ and $\langle , \rangle$ denotes the appropriate Euclidean product on $\mathbb{R}^n$. We use the same notation as for a section as for its coordinate expression and hope that this causes no confusion. Globally, one has to choose an atlas and a subordinate partition of unity, do the local Fourier transforms, and in the end, apply the appropriate gluing process. Let us notice that the Fourier transform depends on the choice of a particular partition, but it exists if the underlying

Riemannian manifold is compact independently of this choice. As in the classical case, one can show that the norm $|\ |_t$ is equivalent to the norm $||s||_t = [\int_{q\in\mathbb{R}^n} |\hat{s}(q)|^2_{\Gamma(\mathcal{E})}(1 + |q|^2)^t \mathrm{d}q]^{\frac{1}{2}} \in \mathbb{R}^+_0$, where $s \in \Gamma(\mathcal{E})$, $|q| = \langle q, q\rangle^{\frac{1}{2}}$ and $q \in \mathbb{R}^n$. To obtain a global version of this formula, one shall apply the same procedure as in the case of the Fourier transform.

In the next lemma, an A-Hilbert bundle analogue of the Sobolev embedding theorem is proved.

Lemma 5 *Let $p : \mathcal{E} \to M^n$ be an A-Hilbert bundle over a compact manifold M. Then for each $t > \lfloor\frac{n}{2}\rfloor + 1$ and $0 \le k < t - \lfloor\frac{n}{2}\rfloor - 1$, we have the inclusion $W^t(\mathcal{E}) \subseteq \mathcal{C}^k(\mathcal{E})$.*

Proof Let $t > \lfloor\frac{n}{2}\rfloor+1$ and $0 \le k < t-\lfloor\frac{n}{2}\rfloor-1$. If s is a section of p_O, $O \subseteq M$, we denote its coordinate expression by s as well. For $\alpha \in \mathbb{N}^n_0$ and $|\alpha| \le k$, we show that $\partial^\alpha s \in \mathcal{C}^0(O, \mathcal{E})$. It is sufficient to prove that $\int_{q\in\mathbb{R}^n} \hat{s}(q)q^\alpha e^{2\pi\imath\langle x,q\rangle} \mathrm{d}q$ converges for all $x \in O$. Because $s \in W^t(O, \mathcal{E})$, we know that $||\hat{s}||_t = [\int_{q\in\mathbb{R}^n} |\hat{s}(q)|^2_{\Gamma(\mathcal{E})} (1 + |q|^2)^t \mathrm{d}q]^{\frac{1}{2}} < \infty$. Let us compute

$$\begin{aligned}
\int_{q\in\mathbb{R}^n} |\hat{s}(q)|_{\Gamma(\mathcal{E})}|q^\alpha| \,\mathrm{d}q &\le \int_{q\in\mathbb{R}^n} |\hat{s}(q)|_{\Gamma(\mathcal{E})}|q|^{|\alpha|} \,\mathrm{d}q \\
&= \int_{q\in\mathbb{R}^n} |\hat{s}(q)|_{\Gamma(\mathcal{E})}(1 + |q|^2)^{t/2} \frac{|q|^{|\alpha|}}{(1 + |q|^2)^{t/2}} \,\mathrm{d}q \\
&\le \left(\int_{q\in\mathbb{R}^n} |\hat{s}(q)|^2_{\Gamma(\mathcal{E})}(1 + |q|^2)^t \,\mathrm{d}q \right)^{1/2} \left(\int_{q\in\mathbb{R}^n} \frac{|q|^{2|\alpha|}}{(1 + |q|^2)^t} \,\mathrm{d}q \right)^{1/2} \\
&= ||\hat{s}||_t \left(\int_{q\in\mathbb{R}^n} \frac{|q|^{2|\alpha|}}{(1 + |q|^2)^t} \,\mathrm{d}q \right)^{1/2}
\end{aligned}$$

where we used the Cauchy–Schwartz inequality in the Hilbert space $L^2(\mathbb{R}^n)$ in the second last step. Using polar coordinates in $\mathbb{R}^n$, we see that the finiteness of the last written integral is equivalent to the finiteness of $\int_0^{+\infty} r^{2|\alpha|+n-1}(1 + r^2)^{-t} \mathrm{d}r$. Near $r = 0$, the integrand is a bounded continuous function. At the infinity, the integrand behaves as $r^{2|\alpha|+n-2t}$ which is integrable over $(C, +\infty)$ for each $C > 0$ if and only if $2|\alpha|+n-2t < -1$. Thus, for each $|\alpha| < t - \frac{n}{2} - \frac{1}{2}$, the integral $\int_{q\in\mathbb{R}^n} \hat{s}(q)q^\alpha e^{2\pi\imath\langle x,q\rangle} \mathrm{d}q$ converges due to the absolute convergence of the Bochner integral. Therefore, $\partial^\alpha s \in \mathcal{C}^0(O, \mathcal{E})$ for each $|\alpha| \le k < t - \lfloor\frac{n}{2}\rfloor - 1$. □

A reference for statements in this paragraph is Solovyov, Troitsky [14]. For A-Hilbert bundles $p_1 : \mathcal{E}_1 \to M^n$ and $p_2 : \mathcal{E}_2 \to M^n$, one defines a symbol of a pseudodifferential operator of order $r \in \mathbb{Z}$. Besides other properties, the symbol is an adjointable A-Hilbert bundle homomorphism. Basically using the Fourier transform, to each symbol of a pseudodifferential operator of order r, one associates a pseudodifferential operator of order r and vice versa. See pp. 79 and 80 in [14]. Further, it is known that for an arbitrary $t \in \mathbb{Z}$, each pseudodifferential operator $D : \Gamma(\mathcal{E}_1) \to \Gamma(\mathcal{E}_2)$ of order r admits a unique Hilbert A-module homomorphism $D_t : W^t(\mathcal{E}_1) \to W^{t-r}(\mathcal{E}_2)$ extending D. See pp. 89 in [14, Theorem 2.1.60]. Moreover, the extensions are adjointable. See pp. in [14, Theorem 2.1.37]. Notice that any pseudodifferential operator $D : \Gamma(\mathcal{E}_1) \to \Gamma(\mathcal{E}_2)$ of order r is, in particular, an adjointable pre-Hilbert A-module homomorphism between the pre-Hilbert A-modules $\Gamma(\mathcal{E}_1)$ and $\Gamma(\mathcal{E}_2)$.

The order of D^* equals that of the operator D, and the symbol of D^* equals up to a non-zero constant complex multiple to the adjoint of the symbol of D. See pp. 83 in [14, Theorem 2.1.24]. One calls a pseudodifferential operator $K : \Gamma(\mathcal{E}_1) \to \Gamma(\mathcal{E}_2)$ *A-elliptic* if for each $x \in M$ and $\xi \in T_x^*M \setminus \{0\}$, the symbol

$$\sigma(x, \xi) : (\mathcal{E}_1)_x \to (\mathcal{E}_2)_x$$

of K at (x, ξ) is a Hilbert A-module isomorphism.

Definition 4 Let U, V be Hilbert A-modules and the decompositions $U = U_0 \oplus U_1$ and $V = V_0 \oplus V_1$ hold, where U_1 and V_1 are Hilbert A-modules and U_0 and V_0 are finitely generated projective Hilbert A-modules. Whenever $K_i \in \mathrm{Hom}_A(U_i, V_i)$ are adjointable $(i = 0, 1)$ and K_1 is an isomorphism, then $K = K_0 \oplus K_1$ is called an *A-Fredholm operator*.

Remark 4 It is immediately seen that $\mathrm{Ker}\, K = \mathrm{Ker}\, K_0$ and $\mathrm{Coker}\, K = V_0/\mathrm{Rng}\, K_0$. In particular, the Hilbert A-module Ker K and the A-module Coker K is a submodule and a quotient module of the finitely generated Hilbert A-modules U_0 and V_0, respectively. Let us notice that in general, the **image** of an A-Fredholm operator **is not closed**, contrary to the case of $A = \mathbb{C}$. Thus, in particular, Coker K need not be a Hilbert A-module. See Mingo [8] or Wegge-Olsen [18] for more information.

Now, we recall the theorem of Fomenko and Mishchenko on the existence of smoothing pseudoinverses mentioned in the Introduction.

Theorem 6 *Let $p : \mathcal{E} \to M$ be a finitely generated projective A-Hilbert bundle over a compact manifold M and $K : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ be an A-elliptic operator of order r. Then for each $t \in \mathbb{Z}$, the extension K_t of K is an A-Fredholm operator and there exists a Hilbert A-module homomorphism $g_{t-r} : W^{t-r}(\mathcal{E}) \to W^t(\mathcal{E})$ satisfying $g_{t-r}K_t - 1 : W^t(\mathcal{E}) \to W^{t+1}(\mathcal{E})$.*

Proof See pp. 101 in Solovyov and Troitsky [14, Theorems 2.1.142 and 2.1.146]. □

As a consequence of the preceding theorem, it is easy to prove the smooth regularity of A-elliptic operators. See also pp. 101 in Solovyov and Troitsky [14, Theorem 2.1.145 and also Theorem 2.1.143].

Theorem 7 *Let $p : \mathcal{E} \to M$ be a finitely generated projective A-Hilbert bundle over a compact manifold M and $K : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ be an A-elliptic operator of order r. Then for each $t \in \mathbb{Z}$, the equality Ker K_t = Ker $K \subseteq \Gamma(\mathcal{E})$ holds.*

Proof Let $t \in \mathbb{Z}$. The inclusion $\mathrm{Ker}\, K \subseteq \mathrm{Ker}\, K_t$ is obvious. Let us prove that $\mathrm{Ker}\, K_t \subseteq \mathrm{Ker}\, K$. Due to Theorem 6, there exists a map $g_{t-r} : W^{t-r}(\mathcal{E}) \to W^t(\mathcal{E})$ such that $g_{t-r}K_t - 1 : W^t(\mathcal{E}) \to W^{t+1}(\mathcal{E})$. For $s \in W^t(\mathcal{E})$, we may write

$$\begin{aligned} s &= (g_{t-r}K_t)s - (g_{t-r}K_t)s + s \\ &= g_{t-r}(K_t s) - (g_{t-r}K_t - 1)s. \end{aligned}$$

Assuming $K_t s = 0$ for $s \in W^t(\mathcal{E})$, we get $s = g_{t-r}(K_t s) - (g_{t-r}K_t - 1)s = -(g_{t-r}K_t - 1)s \in W^{t+1}(\mathcal{E})$ (Theorem 6). By induction, $s \in W^t(\mathcal{E})$ for each $t \in \mathbb{Z}$. Using Lemma 5, we obtain $s \in \bigcap_{k \in \mathbb{N}_0} \mathcal{C}^k(\mathcal{E}) = \Gamma(\mathcal{E})$. □

Using the previous theorem, we prove a smooth version of Theorem 6.

Theorem 8 *Let $p : \mathcal{E} \to M$ be a finitely generated projective A-Hilbert bundle over a compact manifold M and $K : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ be a self-adjoint A-elliptic operator of order r. If Rng K_r is closed in $W^0(\mathcal{E})$, then there exist pre-Hilbert A-module homomorphisms $P : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ and $G : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ such that*

$$1_{|\Gamma(\mathcal{E})} = GK + P,\ 1_{|\Gamma(\mathcal{E})} = KG + P \text{ and } KP = 0.$$

Proof Notice that $K_r : W^r(\mathcal{E}) \to W^0(\mathcal{E})$ and $K_r{}^* : W^0(\mathcal{E}) \to W^r(\mathcal{E})$.

(1) For each $a, b \in \Gamma(\mathcal{E})$, we have $(K_r{}^*a, b)_r = (a, K_r b)_0 = (a, Kb)_{\Gamma(\mathcal{E})} = (a, K^*b)_{\Gamma(\mathcal{E})} = (Ka, b)_{\Gamma(\mathcal{E})}$. Summing up, $(K_r{}^*a, b)_r = (Ka, b)_{\Gamma(\mathcal{E})}$ for any $a, b \in \Gamma(\mathcal{E})$. For each $a \in \operatorname{Ker} K \subseteq \Gamma(\mathcal{E})$, we thus get $(K_r{}^*a, b)_r = 0$ for any $b \in \Gamma(\mathcal{E})$. Because $(,)_r$ is continuous and $\Gamma(\mathcal{E})$ is dense in $W^r(\mathcal{E})$, we obtain $(K_r^*a, b)_r = 0$ for each $b \in W^r(\mathcal{E})$. Thus from the non-degeneracy of $(,)_r$, $K_r{}^*a = 0$ and consequently $\operatorname{Ker} K \subseteq \operatorname{Ker} K_r{}^*$. If $a \in \operatorname{Ker} K_r{}^* \cap \Gamma(\mathcal{E})$, then from the previous computation and the non-degeneracy of $(,)_{\Gamma(\mathcal{E})}$, we get $a \in \operatorname{Ker} K$. Therefore $\operatorname{Ker} K = \operatorname{Ker} K_r{}^* \cap \Gamma(\mathcal{E})$. Because $K_r{}^*$ is also an A-elliptic operator, Theorem 7 applies and we may omit the intersection with $\Gamma(\mathcal{E})$ in the previous expression obtaining

$$\operatorname{Ker} K = \operatorname{Ker} K_r{}^*. \tag{5}$$

(2) Because the image of $K_r : W^r(\mathcal{E}) \to W^0(\mathcal{E})$ is closed and K_r is adjointable, we have $W^r(\mathcal{E}) = \operatorname{Ker} K_r \oplus \operatorname{Rng} K_r{}^*$ due to Theorem 1. Therefore, the projection $p_{\operatorname{Ker} K_r} : W^r(\mathcal{E}) \to \operatorname{Ker} K_r$ from $W^r(\mathcal{E})$ onto $\operatorname{Ker} K_r$ is a well-defined Hilbert A-module homomorphism. Because $\operatorname{Rng} K_r = (\operatorname{Ker} K_r{}^*)^\perp \subseteq W^0(\mathcal{E})$ (Theorem 1), there exists a bijective Hilbert A-module homomorphism

$$\delta = K_{r|(\operatorname{Ker} K_r)^\perp} : (\operatorname{Ker} K_r)^\perp \to (\operatorname{Ker} K_r{}^*)^\perp.$$

Due to Banach's open mapping theorem, its inverse

$$\gamma : (\operatorname{Ker} K_r{}^*)^\perp \to (\operatorname{Ker} K_r)^\perp$$

is continuous. As any inverse of an A-module homomorphism, the map γ is an A-module homomorphism as well. Extending γ by zero on $\operatorname{Ker} K_r{}^* = (\operatorname{Rng} K_r)^\perp$ (Remark 1), we get a Hilbert A-module homomorphism

$$\tilde{\gamma} : W^0(\mathcal{E}) \to (\operatorname{Ker} K_r)^\perp \subseteq W^r(\mathcal{E}).$$

(3) Due to the construction, we obtain a parametrix equation $1_{|W^r(\mathcal{E})} = \tilde{\gamma} K_r + p_{\operatorname{Ker} K_r}$ for K_r, where $p_{\operatorname{Ker} K_r}$ denotes the projections onto $\operatorname{Ker} K_r$. Indeed, for $a \in \operatorname{Ker} K_r$, we have $\tilde{\gamma} K_r a + p_{\operatorname{Ker} K_r} a = 0 + a = a$. For $a \in (\operatorname{Ker} K_r)^\perp$, we get $\tilde{\gamma} K_r a + p_{\operatorname{Ker} K_r} a = \gamma K_r a + 0 = a$.

Let us denote the restriction to $\Gamma(\mathcal{E})$ of $p_{\operatorname{Ker} K_r}$ by P. Thus $P : \Gamma(\mathcal{E}) \to \operatorname{Ker} K_r$. Because $\operatorname{Ker} K_r = \operatorname{Ker} K \subseteq \Gamma(\mathcal{E})$ (Theorem 7), we have the pre-Hilbert A-module homomorphism $P : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ at our disposal.

Further, let us set $G = \tilde{\gamma}_{|\Gamma(\mathcal{E})}$. For $a \in \operatorname{Ker} K_r{}^* \subseteq \Gamma(\mathcal{E})$, we have $Ga = \tilde{\gamma} a = 0$. For $a \in (\operatorname{Ker} K_r{}^*)^\perp \cap \Gamma(\mathcal{E})$, there exists $b \in (\operatorname{Ker} K_r)^\perp$ (item 2 of this proof) such that $a = \delta b$. Since $a \in \Gamma(\mathcal{E})$ and δ is a restriction of an extension of a pseudodifferential operator of finite order, $b \in \Gamma(\mathcal{E})$. We have $Ga = \gamma a = \gamma \delta b = b$. Especially, $Ga \in \Gamma(\mathcal{E})$. Thus, we get a pre-Hilbert A-module homomorphism $G : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$. Now, we may restrict the parametrix type equation for K_r to the space $\Gamma(\mathcal{E})$ to obtain $1_{|\Gamma(\mathcal{E})} = GK + P$.

(4) Similarly as above, we prove that the parametrix type equation $1_{|W^0(\mathcal{E})} = K_r\tilde{\gamma} + p_{\mathrm{Ker}\,K_r{}^*}$ holds, where $p_{\mathrm{Ker}\,K_r{}^*}$ denotes the projection from $W^0(\mathcal{E})$ onto $\mathrm{Ker}\,K_r{}^*$. Indeed, for $a \in \mathrm{Ker}\,K_r{}^*$, we obtain $K_r\tilde{\gamma}a + p_{\mathrm{Ker}\,K_r{}^*}a = 0 + a = a$. For $a \in (\mathrm{Ker}\,K_r{}^*)^{\perp}$, we have $K_r\tilde{\gamma}a + p_{\mathrm{Ker}\,K_r{}^*}a = a + 0 = a$. Using formula (5) from item 1, we see that $(p_{\mathrm{Ker} K_r{}^*})_{|\Gamma(\mathcal{E})} = (p_{\mathrm{Ker}\,K_r})_{|\Gamma(\mathcal{E})}$. Thus restricting $1_{|W^0(\mathcal{E})} = K_r\tilde{\gamma} + p_{\mathrm{Ker}\,K_r{}^*}$ to $\Gamma(\mathcal{E})$, we get $1_{|\Gamma(\mathcal{E})} = KG + P$.

(5) The equation $KP = 0$ follows from the definition of P and the inclusion $\mathrm{Ker}\,K_r \subseteq \Gamma(\mathcal{E})$ (Theorem 7). □

Remark 5 Under the conditions of Theorem 8 and from the constructions in its proof, we get that $GP = 0$ and $P^2 = P$. See also Remark 3 item 2.

4 Complexes of A-elliptic pseudodifferential operators

In this section, we focus our attention at complexes of pseudodifferential operators. Let M^n be a manifold, $\mathcal{E}^\bullet = (p_i : \mathcal{E}^i \to M)_{i\in\mathbb{N}_0}$ be a sequence of A-Hilbert bundles over M and $D^\bullet = (D_i, \Gamma(\mathcal{E}^i))_{i\in\mathbb{N}_0}$ be a complex of pseudodifferential operators acting between sections of the appropriate bundles, A being a unital C^*-algebra. Let σ_i denote the symbol of the pseudodifferential operator $D_i : \Gamma(\mathcal{E}^i) \to \Gamma(\mathcal{E}^{i+1})$, $i \in \mathbb{N}_0$. For each $i \in \mathbb{N}_0$, we define σ_i' as the restriction of σ_i to $\pi'^*(\mathcal{E}^i)$, where $\pi' : T^*M \setminus 0 \to M$ denotes the foot-point projection from the cotangent bundle with the image of the zero section removed onto the manifold M. Let us denote the resulting complex consisting of A-Hilbert bundle homomorphisms σ_i', $i \in \mathbb{N}_0$, by $\sigma'^\bullet$.

Definition 5 A complex $D^\bullet = (D_i, \Gamma(\mathcal{E}^i))_{i\in\mathbb{N}_0}$ of pseudodifferential operators is called *A-elliptic* if $\sigma'^\bullet$ is an exact sequence in the category of A-Hilbert bundles.

The following lemma is a generalization to the case of Hilbert A-modules of a reasoning used in the classical Hodge theory. See, e.g., Wells [19] for the classical case, i.e., for $A = \mathbb{C}$.

Lemma 9 *Let U, V, W be Hilbert A-modules and σ and $\tilde{\sigma}$ be adjointable Hilbert A-module homomorphisms. If*

$$U \xrightarrow{\sigma} V \xrightarrow{\widetilde{\sigma}} W$$

is an exact sequence and the image of $\widetilde{\sigma}$ is closed, then $\Sigma = \sigma\sigma^* + \widetilde{\sigma}^*\widetilde{\sigma}$ is a Hilbert A-module automorphism of V.

Proof Due to the assumption, the Hilbert A-module homomorphisms σ^* and $\widetilde{\sigma}^*$ exist. Therefore, the map $\Sigma = \sigma\sigma^* + \widetilde{\sigma}^*\widetilde{\sigma}$ is a well-defined Hilbert A-module endomorphism. It is immediately seen that $\mathrm{Rng}\,\widetilde{\sigma}^* \subseteq \mathrm{Ker}\,\sigma^*$. Indeed, for any $v \in \mathrm{Rng}\,\widetilde{\sigma}^*$ there exists an element $w \in W$ such that $\widetilde{\sigma}^*w = v$. Thus, $\sigma^*v = (\sigma^*\widetilde{\sigma}^*)w = (\widetilde{\sigma}\sigma)^*w = 0$.

Now, we prove the injectivity of $\sigma\sigma^*$ restricted to $\mathrm{Rng}\,\sigma$. Suppose there exists an element $u \in U$ such that $v = \sigma u$ satisfies $\sigma\sigma^*v = 0$. We may write $0 = (\sigma\sigma^*v, v)_V = (\sigma^*v, \sigma^*v)_U$ which implies $\sigma^*v = 0$. Thus, $0 = (\sigma^*v, u)_U = (v, \sigma u)_V = (v, v)_V$ which implies $v = 0$. Now, we prove that $\tilde{\sigma}^*\widetilde{\sigma}$ is injective on $\mathrm{Rng}\,\widetilde{\sigma}^*$. Let $v = \widetilde{\sigma}^*w$ for an element $w \in W$ and $\widetilde{\sigma}^*\widetilde{\sigma}v = 0$. Then the equation $\widetilde{\sigma}^*\widetilde{\sigma}v = 0$ implies that $(\widetilde{\sigma}v, \widetilde{\sigma}v)_W = (v, \widetilde{\sigma}^*\widetilde{\sigma}v)_V = 0$, what in turn implies $\widetilde{\sigma}v = 0$. We may compute $0 = (\widetilde{\sigma}v, w)_W = (v, \widetilde{\sigma}^*w)_V = (v, v)_V$ from which $v = 0$ follows. Therefore, $\widetilde{\sigma}^*\widetilde{\sigma}$ is injective on $\mathrm{Rng}\,\widetilde{\sigma}^*$.

Because $\operatorname{Rng}\widetilde{\sigma}$ is closed, we may use Theorem 1 which implies $V = \operatorname{Ker}\widetilde{\sigma} \oplus \operatorname{Rng}\widetilde{\sigma}^*$. Because $\operatorname{Rng}\sigma = \operatorname{Ker}\widetilde{\sigma}$, we may write $V = \operatorname{Rng}\sigma \oplus \operatorname{Rng}\widetilde{\sigma}^*$. Since $\operatorname{Rng}\sigma \subseteq \operatorname{Ker}\widetilde{\sigma}$ and $\operatorname{Rng}\widetilde{\sigma}^* \subseteq \operatorname{Ker}\sigma^*$, we get $\Sigma_{|\operatorname{Rng}_\sigma} = \sigma\sigma^*_{|\operatorname{Rng}_\sigma}$ and $\Sigma_{|\operatorname{Rng}_{\widetilde{\sigma}^*}} = \widetilde{\sigma}^*\widetilde{\sigma}_{|\operatorname{Rng}_{\widetilde{\sigma}^*}}$. Suppose there exists $v = (v_1, v_2) \in \operatorname{Rng}\sigma \oplus \operatorname{Rng}\widetilde{\sigma}^*$ such that $\Sigma v = 0$. Then $0 = \Sigma v = \sigma\sigma^* v_1 + \widetilde{\sigma}^*\widetilde{\sigma} v_2$. Because $\operatorname{Rng}\sigma \cap \operatorname{Rng}\widetilde{\sigma}^* = 0$, we have $\sigma\sigma^* v_1 = 0$ and $\widetilde{\sigma}^*\widetilde{\sigma} v_2 = 0$. Due to the injectivity of the appropriate restrictions of the maps $\sigma\sigma^*$ and $\widetilde{\sigma}^*\widetilde{\sigma}$, the last two equations imply $v_1 = v_2 = 0$. Thus Σ is injective on V.

Since the images of σ ($\operatorname{Rng}\sigma = \operatorname{Ker}\widetilde{\sigma}$) and $\widetilde{\sigma}$ are closed, the images of $\sigma\sigma^*$ and $\widetilde{\sigma}^*\widetilde{\sigma}$ are closed as well. Because $\sigma\sigma^*$ and $\widetilde{\sigma}^*\sigma'$ have closed images, their sum Σ has closed image as well. Obviously, Σ is adjointable. Due to Theorem 1, Σ is surjective since $V = \operatorname{Rng}(\Sigma^*) \oplus \operatorname{Ker}(\Sigma) = \operatorname{Rng}(\Sigma^*) = \operatorname{Rng}(\Sigma)$. □

Remark 6 Going through the proof of the previous theorem, we see that under the same conditions on σ and σ', we obtain that the map $\lambda\sigma^*\sigma + \mu\widetilde{\sigma}\widetilde{\sigma}^*$ is a Hilbert A-module automorphism for all $\lambda, \mu \in \mathbb{C} \setminus \{0\}$ as well.

Corollary 10 *Let M be a manifold and $D^\bullet = (D_i, \Gamma(\mathcal{E}^i))_{i\in\mathbb{N}_0}$ be an A-elliptic complex of pseudodifferential operators in A-Hilbert bundles over M. Then for each $i \in \mathbb{N}_0$, the associated Laplacian $\triangle_i$ is A-elliptic.*

Proof Because $D^\bullet$ is an A-elliptic complex, the restricted symbol complex $\sigma'^\bullet$ is exact. From the exactness of $\sigma'^\bullet$, one concludes that the images of the Hilbert A-module homomorphisms σ_i' evaluated at any $(x, \xi) \in T^*M \setminus 0$ are closed. Further, they are adjointable, being restrictions of symbols of pseudodifferential operators. Using Remark 6 for $\lambda, \mu \in \mathbb{C} \setminus \{0\}$, the maps $\Sigma_i = \lambda\sigma_i^*\sigma_i + \mu\sigma_{i-1}\sigma_{i-1}^*$, $i \in \mathbb{N}_0$, are Hilbert A-module automorphisms when evaluated at $(x, \xi) \in T^*M \setminus 0$. Consequently, the associated Laplacians $\triangle_i$, $i \in \mathbb{N}_0$, of $D^\bullet$ are A-elliptic operators. □

For each $i \in \mathbb{N}_0$, let us denote the order of the associated Laplacian $\triangle_i$ by r_i. Further, we fix the topologies on the spaces which appear in Definition 2 of the cohomology groups of a complex $D^\bullet = (D_i, \Gamma(\mathcal{E}^i))_{i\in\mathbb{N}_0}$ of pseudodifferential operators. We consider the spaces $\Gamma(\mathcal{E}^i)$ with the topology given by the norm $|\ |_{\Gamma(\mathcal{E}^i)}$ and the spaces $Z^i(D, A)$, $B^i(D, A) \subseteq \Gamma(\mathcal{E}^i)$ with the subset topologies. Notice that $(\Gamma(\mathcal{E}^i), |\ |_{\Gamma(\mathcal{E}^i)})$ is not a Banach space in general. We assume each cohomology group $H^i(D, A)$ to be equipped with the quotient topology.

Theorem 11 *Let $D^\bullet = (D_i, \Gamma(\mathcal{E}^i))_{i\in\mathbb{N}_0}$ be an A-elliptic complex in finitely generated projective A-Hilbert bundles over a compact manifold M. Suppose that for each $i \in \mathbb{N}_0$, the image of $(\triangle_i)_{r_i}$ is closed in $W^0(\mathcal{E}^i)$. Then for any $i \in \mathbb{N}_0$,*

(1) *the cohomology group $H^i(D, A)$ of $D^\bullet$ is a finitely generated A-module*
(2) *the image of D_{i-1} is closed in $\Gamma(\mathcal{E}^i)$ and*
(3) *$H^i(D, A)$ is a Banach space.*

Proof Let us fix a non-negative integer $i \in \mathbb{N}_0$. According to Corollary 10, the associated Laplacian $\triangle_i$ is an A-elliptic operator. Because $\triangle_i$ is also self-adjoint and $\operatorname{Rng}(\triangle_i)_{r_i} \subseteq W^0(\mathcal{E}^i)$ is closed, we may use Theorem 8, from which the existence of pre-Hilbert A-module homomorphisms $P_i : \Gamma(\mathcal{E}^i) \to \Gamma(\mathcal{E}^i)$ and $G_i : \Gamma(\mathcal{E}^i) \to \Gamma(\mathcal{E}^i)$ follows such that $P_i + G_i\triangle_i = P_i + \triangle_i G_i = 1_{|\Gamma(\mathcal{E}^i)}$ and $\triangle_i P_i = 0$. Thus $D^\bullet$ is a parametrix possessing complex.

(1) Because $(\triangle_i)_{r_i}$ is A-Fredholm (Theorem 6), Ker $(\triangle_i)_{r_i}$ is a Hilbert A-submodule of a finitely generated Hilbert A-module (see Remark 4). Let us denote the latter module by U_0. Since $(\triangle_i)_{r_i}$ is adjointable and has a closed image, we get using Theorem 1, Ker $(\triangle_i)_{r_i} \oplus$ Rng $(\triangle_i)^*_{r_i} = U_0$. In particular, there exists a well-defined projection q from U_0 onto Ker $(\triangle_i)_{r_i}$. If $\{u_i\}_{i=1}^m$ is a set of generators of U_0, taking $\{q(u_i)\}_{i=1}^m$ implies that Ker $(\triangle_i)_{r_i}$ is a finitely generated Hilbert A-module. Because $D^\bullet$ is a parametrix possessing complex, $K^i(D, A)$ is A-linearly isomorphic to $H^i(D, A)$ (Theorem 4). Since $\triangle_i$ is A-elliptic, Ker $(\triangle_i)_{r_i} = K^i(D, A)$ due to the regularity (Theorem 7). Summing up, $H^i(D, A)$ is finitely generated over A.

(2) Now, we prove that $H^i(D, A)$ is a Hausdorff space. It is obvious that the maps

$$\Phi_i : K^i(D, A) \to H^i(D, A) \text{ and } \Psi_i : H^i(D, A) \to K^i(D, A)$$

defined by $\Phi_i a = [a]$ and $\Psi_i[a] = P_i a$, $a \in Z^i(D, A)$ are continuous. For the fact that they are mutually inverse, see item 2 in the proof of Theorem 4. Thus, $K^i(D, A)$ is homeomorphic to $H^i(D, A)$. Because $K^i(D, A)$ is a Hausdorff space, being a subspace of the Hausdorff space $\Gamma(\mathcal{E}^i)$, $H^i(D, A)$ is a Hausdorff space as well. Since $H^i(D, A) = Z^i(D, A)/B^i(D, A)$ and $Z^i(D, A)$ is closed in $\Gamma(\mathcal{E}^i)$, being the kernel of a continuous map, the space $B^i(D, A)$ of boundaries in $\Gamma(\mathcal{E}^i)$ is closed in $\Gamma(\mathcal{E}^i)$ as well.

(3) Proving that $K^i(D, A)$ is a Banach space with respect to the norm topology on $\Gamma(\mathcal{E}^i)$, the proof is done due to item 2. Let us consider the subset topology on Ker $(\triangle_i)_0 \subseteq (W^0(\mathcal{E}^i), |\ |_0)$. We have $|\ |_{\Gamma(\mathcal{E}^i)} = |\ |_{0|\Gamma(\mathcal{E}^i)}$. Because $K^i(D, A)$ equals Ker $((\triangle_i)_0 : W^0(\mathcal{E}^i) \to W^{-r_i}(\mathcal{E}^i))$ (Theorem 7), which is closed in $W^0(\mathcal{E}^i)$ being the kernel of a continuous map, $K^i(D, A)$ is also closed in the Banach space $W^0(\mathcal{E}^i)$. Now, the assertion follows because the norms $|\ |_0$ and $|\ |_{\Gamma(\mathcal{E}^i)}$ coincide on the space $\Gamma(\mathcal{E}^i)$. $\square$

Remark 7 Let us remark that any A-Fredholm operator can be perturbed by a so called A-compact operator such that the image of the resulting operator is closed. See, e.g., Fomenko and Mishchenko [3] or Mingo [8].

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