Dirac operators in symplectic and contact geometry - a problem of infinite dimension

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Statements of the Hodge theory

- $(E^i \to M)_{i\in\mathbb{Z}}$ sequence of finite rank vector bundles over compact manifold
- $D_i : \Gamma(E^i) \to \Gamma(E^{i+1})$ (pseudo-)differential operators forming a complex $D^\bullet = (\Gamma(E^i), D_i)_{i\in\mathbb{Z}}$, $D_{i+1}D_i = 0$
- principal symbols $\sigma_i(\xi, -) : E^i \to E^{i+1}$ form a complex
- If $\sigma_i(\xi, -)$ form an exact sequence for any $0 \neq \xi \in T^*M \implies$ elliptic complexes, then
- $H^i(E^\bullet)$ is finite dimensional and
- $H^i(E^\bullet) \simeq \mathrm{Ker}\,\triangle_i$ where $\triangle_i = D_i^* D_i + D_{i-1}D_{i-1}^*$. The adjoint is with respect to the inner product induced by a metric on E^i

Examples of complexes satifying the results of Hodge theory

- deRham complex over a compact manifold, $\sigma_i(\xi, \alpha) = \xi \wedge \alpha$
- Dolbeault complex on a compact complex manifold
- Not an example: deRham complex over $\mathbb{R}^n$, $H^0(\mathbb{R}^n) = \mathbb{R}$ and $\operatorname{Ker} \triangle_0$ is infinite dimensional (already when restricted to polynomials and $n = 2 : 1; x, y; x^2 - y^2, 2xy$ - Weyl duality)
- "Brilinsky complex over a compact symplectic manifold and Laplace defined via adjoints with respect to the symplectic form - not an example from a simple reason $\triangle_i = 0$"
- $\mathbb{E}$ infinite dimensional Hilbert space, M compact manifold $E = \mathbb{E} \times M \to M$, $\nabla : \Gamma(E) \to \Gamma(E \otimes T^*M)$ trivial connection, $\nabla s = 0$; kernel are constant functions with values in $\mathbb{E}$. Kernel is $\{s \in \Gamma(E) | \, \exists e \in \mathbb{E} \, \forall m \in M \; s(m) = (e, m)\} \simeq \mathbb{E}$ - thus infinite dimensional.

Key steps in the proof of Hodge theory

- Construction of the Green operators for the extensions $\widetilde{\triangle_i}$ of $\triangle_i$ to the Sobolev spaces
  - orthogonal projections
  - proof that the extensions are Fredholm
- Completion of the pre-Hilbert $\Gamma(E^i) \Longrightarrow$ Sobolev or Hardy spaces with values in vector spaces (they are Hilbert spaces)
- Continuous extensions of D_i and $\triangle_i$ to the completions (diff ops are of finite integer order)
- Elliptic implies regular (Ker$(\triangle_i) =$ Ker$(\widetilde{\triangle_i})$)
- Elliptic implies extension is Fredholm

Metaplectic or Segal-Shale-Weil representation

- (V, ω) symplectic vector space of diemsnion $2n$
- $Sp(V, \omega)$, connected smooth double covering $G = Mp(V, \omega)$ unique up to a covering isotopy
- non-universal since $U(n)$ is a maximal compact subgroup of $Sp(V, \omega)$ and $\pi_1(U(n)) = \mathbb{Z}$
- G can be given a Lie group structure (unique, making the covering a group homomorphism)
- Parallel to $SO(p, q)$ and $Spin(p, q)$
- $Mp(V, \omega)$ is not a matrix group, no faithful representation in finite dimensional vector space

Segal-Shale-Weil representation

- L Lagrangian subspace in V, J a complex structure on (V, ω), metric $g(-,-) = \omega(J-,-)$
- $\mathbb{E} = L^2(L)$ for the Lebesgue measure induced by $g_{|L}$
- $\sigma : G \to \mathrm{Aut}(\mathbb{E})$ Segal-Shale-Weil representation
- Construction: Schrödinger representation of the Heisenberg group
- Stone-Neumann theorem on uniqueness of representation of the Heisenberg group up to multiple
- "co-cycle counting"of Weil
- $\sigma(J) = \pm\mathcal{F} : L^2(L) \to L^2(L)$

Properties of the Segal-Shale-Weil representation

- It is faithful, unitary, reducible
  $\mathbb{E} = L^2(L) = L^2(L)_{even} \oplus L^2(L)_{odd}$
- Lie alg. rep. is highest weight, parallel to spin representations via realizing it in the symplectic Clifford algebra
  $sCliff(V,\omega) = T(V)/ < x \otimes y - y \otimes x - \omega(x,y)1, x, y \in V >$
- V. Berezin, V. Bargmann (Fock space), I. Segal, Shale (quantizing of KG-fields), Weil (number theory for locally compact fields)

Metaplectic structures

Metaplectic structures

- (M, ω) symplectic manifold
- $P = \{f | \, f$ is a symplectic basis of $(T^*M, \omega_m), m \in M\}$
- $Q \to M$ any $Mp(2n, R)$-bundle compatible with the projection structures is called metaplectic structure

Basic exmples: even dimensional tori, S^2, $\mathbb{C}P^{2n+1}$, T^*M cotangent bundle of orientable manifold M.
Theorem (Kostant): (M, ω) admits a metaplectic structure iff the Chern class of (TM, J) is even for any J almost complex compatible.
There is a notion of an Mp^c structure which exists on any symplectic manifold (Rawnsley, Gutt, Cahen)

Some a priori constructions

- (M, ω) sympletic manifold admitting a metaplectic structure $\mathcal{P} \to M$
- $E = \mathcal{P} \times_\sigma \mathbb{E}$ Segal-Shale-Weil bundle (Kostant)
- sections $\Gamma(\mathbb{E})$ Kostants spinors
- ∇ symplectic connection $\Longrightarrow$ Z principal connection on $\mathcal{Q} \Longrightarrow$ lift on $\mathcal{P} \Longrightarrow \nabla^E$ associated connection on $E \Longrightarrow$ exterior covariant derivative d_k^∇ on $\Omega^k(E)$ -forms with values in the SSW-bundle
- If ∇ is flat, then $(d_k^\nabla, \Omega^k(E))$ forms a complex - SSW twisted de Rham complex

Symplectic Dirac operators of Habermann

$p : L^2(L) \otimes \mathbb{R}^{2n} \simeq L^2(L) \oplus T \to L^2(L)$, T is a G-module
$L^2(L) = \mathbb{E}$ the Segal-Shale-Weil representation

- (M^{2n}, ω) symplectic manifold admitting a metaplectic structure
- $Ds = p \circ \nabla^E s$, $s \in \Gamma(E)$
- $Ds = \sum_{i=1}^{2n} e_i \cdot \nabla^E_{e_i} s$
- $e_i \cdot s = \imath x^i s$, $e_{i+n} \cdot s = \frac{\partial s}{\partial x^i}$ (defined on dense subset, smooth vectors of $\mathbb{E}$)

Contact projective Dirac operator

$G = Mp(2n, \mathbb{R})$, P parabolic subgroup of contact grading, i.e.,
$\mathfrak{p} = \mathfrak{sp}(2n-2, \mathbb{R}) \oplus \mathbb{R}^{2n-2} \oplus \mathbb{R}$
$(\mathcal{G} \to M, \omega)$ any Cartan geometry of type (G, P)
$G_0 = Mp(2n-2, \mathbb{R}) \times \mathbb{R}^\times$ (reductive part)
$\mathbb{E}' = L^2(\mathbb{R}^{2n-2})$
Extension of the SSW-representation $\sigma : Mp(2n-2, \mathbb{R}) \to \mathrm{Aut}(\mathbb{E}')$ to P; $\mathbb{R}^{2n-2}$ acts by identity, on $\mathbb{R} \setminus \{0\}$ by a character - σ'
$E' = \mathcal{G} \times_{\sigma'} \mathbb{E}'$
$Ds = p(\nabla^{\omega, E'} s)$ contact projective Dirac operator, $\nabla^{\omega, E'}$ connection associated to E' via σ'

Generalization Hodge thy to infinite dimension

- Our aim: generalize the Hodge theory to infinite dimension
- $\mathbb{E}$ infinite dimensional Hilbert space, M compact manifold $E = \mathbb{E} \times M \to M$, $\nabla : \Gamma(E) \to \Gamma(E \times T^*M)$ trivial connection, $\nabla s = 0$; kernel are constant functions with values in $\mathbb{E}$. Kernel is $\{s | \exists e \forall m\, s(m) = (e, m)\} \simeq E$ - thus infinite dimensional.
  - Do the fibers cause it?
  - Solutions are somehow "finite over E".
- What if $\mathbb{E}$ is a Banach space only, no inner product structures, cannot produce the Green operators via self adjoint projections
- $\mathbb{E}$ a Hilbert module over the C^*-algebra $B(\mathbb{E}) \implies$ Hilbert modules over C^*-algebras

$\mathbb{E}^*$ as a $B(H)$-module

- Multiplication
  $\cdot : \mathbb{E}^* \times B(\mathbb{E}) \to \mathbb{E}^*$, $l \cdot a = l \circ a$, $l \in \mathbb{E}^*$ and $a \in B(\mathbb{E})$
- "Inner product"
  $(,) : \mathbb{E}^* \times \mathbb{E}^* \to B(\mathbb{E})$, $(k, l) \in B(\mathbb{E})$ defined by
  $(k, l)(v) = l(v)k^*$ for any $v \in \mathbb{E}$ where $k^* \in \mathbb{E}$ such that
  $(k^*, v)_{\mathbb{E}} = k(v)$ (unique)
- Note: $(,)$ maps into $F(\mathbb{E}) \subseteq K(\mathbb{E}) \subset B(\mathbb{E})$

C^*-algebra (J. von Neumann, I. M. Gelfand, Naimark, I.E. Segal)

- A is a complex associative algebra ($za, a+b, ab$, distributive)
- $|\,| : A \to [0, \infty)$ is a norm on A making A a Banach space
- $* : A \to A$ is an involutive antihomomorphism of A
- $|a^*a|^2 = |a|^2$ for each $a \in A$

- If $1 \in A$, $\lambda \in \mathbb{C}$ is a spectral element of $a \in A :=$ if $(a - \lambda 1)$ is not invertible in A
- $1 \notin A$, make extension $A' = A \oplus \mathbb{C}1$, the spectrum of a is defined by non-invertibility in A'
- We say that $a = a^*$ is positive if any spectral element λ of a is non-negative

Examples of C^* algebras

- $\mathbb{C}$ with $z^* = \bar{z}$ and $|a+ib| = (a^2+b^2)^{1/2}$
- $(\mathbb{E}, (,)_{\mathbb{E}})$ Hilbert space

$$B(\mathbb{E}) = \{a : \mathbb{E} \to \mathbb{E} |\, a \text{ linear and bounded}\}$$

$(a^* v, w) = (v, aw)$, $|a| = \sup_{|v|=1} |a(v)|_H$

- $M_n(\mathbb{C})$ is $B(\mathbb{E})$ for $\mathbb{E} = \mathbb{C}^n$ with the Hermitian norm on $\mathbb{C}^n$
- $K(\mathbb{E}) \subseteq B(\mathbb{E})$ algebra of compact operators on $\mathbb{E}$
- X topological space, continuous functions with compact support $\mathcal{C}_c(X)$ with $f^*(x) = \overline{f(x)}$ and the supremum norm (non-unital)
- X locally compact topological space, continuous functions vanishing at infinity; operations as above

Not C^* algebras

Which algebras are not C^*?

- $\mathbb{E}$ infinite dimensional Hilbert space $F(\mathbb{E})$ linear operators $F(\mathbb{E}) \subseteq B(\mathbb{E})$ operators with finite rank (star not well defined)
- $S(\mathbb{E}) \subseteq B(\mathbb{E})$ self-adjoint operators; $(ab)^* = b^* a^* = ba \neq ab$ if $\dim \mathbb{E} > 1$ - Jordan algebras
- Several $L^p(\mathbb{R}^n)$, $f^*(x) = \overline{f(x)}$, $|f|_p = \int_{\mathbb{R}^n} |f|^p d\lambda$ with multiplication convolution (point-wise multiplication, not even an algebra)

Definition of Hilbert A-modules

Definition

Let A be a C^*-algebra and $\mathbb{E}$ be a vector space over the complex numbers. We call $(\mathbb{E}, (,))$ a Hilbert A-module if

- $\mathbb{E}$ is a right A-module – operation $\cdot : \mathbb{E} \times A \to \mathbb{E}$
- $(,) : \mathbb{E} \times \mathbb{E} \to A$ is a $\mathbb{C}$-bilinear mapping
- $(u, v + \lambda w) = (u, v) + \lambda(u, w)$
- $(u, v \cdot a) = (u, v)a$ - at right is the the product in A
- $(u, v)^* = (v, u)^*$
- $(u, u) \geq 0$ and $(u, u) = 0$ implies $u = 0$

Definition of Hilbert and pre-Hilbert A-modules

Definition

If $(\mathbb{E}, (\,,\,))$ is a pre-Hilbert A-module we call it Hilbert A-module if it is complete with respect to the norm $|\,| : \mathbb{E} \to [0, \infty)$ defined by $u \ni E \mapsto |u| = \sqrt{|(u,u)|_A}$ where $|\,|_A$ is the norm in A.

- Closed submodules need to have neither orthogonal nor only topological complements: $\mathcal{C}^0((0,1)) \subseteq \mathcal{C}^0([0,1])$
- Continuous linear maps need not be adjointable $(=(T^*u, v) = (u, Tv))$
- $\mathrm{Aut}_A(\mathbb{E}) = \{T : \mathbb{E} \to \mathbb{E} | T(u \cdot a) = T(u) \cdot a,\, T$ is continuous and bijective$\}$

Examples of Hilbert A-modules

- For $A = \mathbb{C}$, a Hilbert $\mathbb{C}$-module is = Hilbert space
- For A a C^*-algebra, $\mathbb{E} = A$, $a \cdot b = ab$ and $(a, b) = a^* b$.
- For $A = K(\mathbb{E})$, the C^*-algebra of bounded operators on a separable Hilbert space $\mathbb{E}$, $\mathbb{E}^*$ is a Hilbert A-module with respect to $(,) : \mathbb{E}^* \times \mathbb{E}^* \to K(\mathbb{E})$ as above.
- $A = \mathbb{E} = C^0([0,1])$, $(f \cdot g)(x) = f(x)g(x)$, $(f, g) = fg \in \mathcal{C}^0([0,1])$
- $\ell^2(A) = \{(a_i)_{i=1}^{\infty} \subseteq A | \sum_i |a_i|^2 < \infty\}$, $(a_i)_i \cdot b = (a_i \cdot b)_i$, $((a_i)_i, (b_i)_i) = \sum_i a_i^* b_i$.
- Sections of A-Hilbert bundles over compact manifolds form pre-Hilbert modules. (A. S. Mishchenko, A. T. Fomenko, champs contunud des algebres C^*; construction of Sobolev type completions of the section spaces)

Generalization of the Fredholm property

- $F_{u,v}(w) = u \cdot (v, w)$, $F : \mathbb{E} \to \mathbb{E}$
- A-finite rank any of form $\mathbb{E} \ni v \mapsto \sum_i^n \lambda_i F_{u_i,v_i}(v)$ for some $u_i, v_i \in \mathbb{E}$ and $\lambda_i \in \mathbb{C}$
- A-compact operators = closure of A-finite rank, ($\mathbb{E}$ is Banach, $B(\mathbb{E})$ is a normed space with the operator norm)
- A-Fredholm = invertible modulo A-compact
- The image of A-Fredholm neeed not be closed (F $\mathbb{C}$-Fredholm implies $M/\mathrm{Im}\, F < \infty$ implies $\mathrm{Im}\, F$ is closed)!
  $Ff = xf$, $A = \mathbb{E} = \mathcal{C}^0([0,1])$ - counterexample: A-Fredholm but not closed range

Bundles of Hilbert modules

Definition

$p : E \to M$, $\mathcal{A}$ is called an A-Hilbert bundle if

- there exists a Hilbert A-module $(\mathbb{E}, (,))$
- p is a smooth Banach bundle with typical fiber $\mathbb{E}$ and $\mathcal{A}$ is (a maximal) smooth bundle atlas of p
- the transition maps of the atlas are maps into $\mathrm{Aut}_A(\mathbb{E})$

Example: the Segal-Shale-Weil bundle $E = \mathcal{P} \times_\sigma \mathbb{E}$ over a symplectic manifold admitting a metaplectic structure.
Atlas - any atlas containing the global trivialization map (Kuiper thm. $\Longrightarrow$ exists) Take any maximal atlas from the (nonempty) set of atlae containing this trivialization.

Key technical tool

Theorem [Fomenko, Mishchenko, 1979]: Let A be a C^*-algebra, M a compact manifold and $E \to M$ be finitely generated projective A-Hilbert bundle over M. If D is an elliptic operator, its extension is an A-Fredholm operator. In particular, $\operatorname{Ker} D$ is a finitely generated projective Hilbert A-module.

Generalization of the procedures from parametrix construction for elliptic operators on compact manifolds. However, no results connected to cohomologies, their topology (closedness of operator images) and their projective/finite properties.

Complexes and their cohomology groups

Theorem [Krysl, J. Geom. Phys. (accepted)]: Let $A = K(\mathbb{E})$ be the algebra of compact operators and M be a compact manifold and $D^\bullet = (\Gamma(E^i), D_i)_i$ be an elliptic complex on finitely generated projective $K(\mathbb{E})$-Hilbert bundles over M. Then

- $H^i(D^\bullet) \simeq \operatorname{Ker} \triangle_i$ and it is finitely generated projective Hilbert $K(\mathbb{E})$-module
- $\Gamma(E^i) = \operatorname{Ker} \triangle_i \oplus \operatorname{Im} d_{i-1} \oplus \operatorname{Im} d_i^*$
- $\operatorname{Im} \triangle_i = \operatorname{Im} d_{i-1} \oplus \operatorname{Im}\, d_i^*$. In particular images of all operators involved are closed.

Complexes and their cohomology properties

Theorem [Krysl, J. Global. Analysis Geom.]: Any elliptic complex of operators $D_i, i \in \mathbb{Z}$ in sections of finitely generated projective Hilbert bundles E^i over a compact manifold whose Laplace operators have a closed image satisfies

- $H^i(D^\bullet) \simeq \operatorname{Ker} \triangle_i$ and it is finitely generated projective Hilbert modules
- $\Gamma(E^i) = \operatorname{Ker} \triangle_i \oplus \operatorname{Im} D_{i-1} \oplus \operatorname{Im} D_i^*$
- $\operatorname{Im} \triangle_i = \operatorname{Im} d_{i-1} \oplus \operatorname{Im}\ d_i^*$

Application

(in progress)

Theorem: Let M be a symplectic manifold equipped with a metaplectic structure and ∇^E be a connection for which the associated Segal-Shale-Weil bundle $E \to M$ is trivial. The cohomology groups of the SSW-twisted deRham complex $(\Omega^k(E), d_k^\nabla)_k$ are isomorphic to the kernels of the associated Laplace operators and they are finitely generated projective $K(\mathbb{E})$-Hilbert modules.

Paralelly for the contact projective case.

Theorem: Let (M, ω) admitting a metaplectic structure be symplectic manifold and ∇ be a symplectic connection. Then the kernel of the Habrmann Dirac operator is a finitely generated projective $K(\mathbb{E})$-module.

Question: Do we have $H^k(\Omega^k(E)) \simeq H^k_{deRham}(M) \otimes K(\mathbb{E})$.

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