Complexes of symplectic twistor operators

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- Known: Riemannian or pseudoriemannian spin geometry. Twistor operators in *classical spin geometry* for specific manifolds with $Spin(p, q)$-structure - **[Penrose]** for signature (1,3). Spinor bundles are *associated bundles* to the spinor reps of spin group; model for the bundle's fibres

- Known: **Dolbeault operators:** (M^n, J) almost complex manifold; $(\Gamma(\mathcal{E}^{i,j+k}), \overline{\partial}^{i,j+k})_{k\in\mathbb{Z}}$ holomorphic–antiholomorphic differential forms

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- If the Nijenhuis tensor $N_J(X,Y) = [X,Y] + J[JX,Y] + [X,JY] - [JX,JY] = 0$ for all smooth vector fields X, Y, the Dolbeault operators form (families of) complexes $\overline{\partial}^{i,j+k+1}\overline{\partial}^{i,j+k} = 0$ for each (i,j)
- Moreover $N_J = 0 \Longrightarrow \partial^{i+k+1,j}\partial^{i+k,j} = 0$ and $\overline{\partial}\partial + \partial\overline{\partial} = 0$
- Newlander–Nirenberg: $N_J = 0 \Leftrightarrow J$ induces a holomorphic subatlas on M (complex structure)
- holomorphic–antiholomorphic forms are *associated bundles* to representations of the unitary group $U(n)$. Defining rep. of $U(n)$ on $\mathbb{C}^n$ is the model for the bundle's fibre
- Use Lie groups representation theory (of so-called $(\mathfrak{g}, K)$-modules)

Plan

- Present the "not too known" structure by introducing the "symplectic spin group", the representation model for the complex, and the "symplectic spin structure"
- Define sequence of symplectic twistor operators on induced bundles' sections
- Analyse curvature of the induced connection
- Connect the curvature to the complex condition "$\partial\partial = 0$" or "$TT = 0$"

Symplectic Vector Spaces

- (V, ω) real symplectic vector space of dimension $2n$; model of the tangent space
- $Sp(2n, \mathbb{R})$ symplectic group (the non-compact one), $\pi_1(Sp(2n, \mathbb{R})) = \pi_1(U(n)) = \mathbb{Z}$; symmetry group
- There exists connected Lie group that covers $Sp(2n, \mathbb{R})$ twice
- unique as Lie group up to choice of neutral element and deck-transformation
- the metaplectic group, denoted by $\widetilde{G}$ or $Mp(2n, \mathbb{R})$
- Choose a maximal ω-isotropic subspace $L \subseteq V$ and a complex structure J on V such that $g(u, v) = \omega(Ju, v)$ is positive definite - sometimes called *adapted cplx str.*; (J is then g-orthogonal, ω-symplectic)

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  called the symplectic spinor representation (oscillator, metaplectic, Segal–Shale–Weil, Shale–Weil). Denote it by $\mathfrak{m} : \widetilde{G} \to U(E)$, where $E = L^2(L)$ square integrable on the Euclidean vector space $(L, g_{|L \times L})$, ([Shale] ('60), I. E. Segal, [Weil] ('60), Berezin). U denotes unitary operators. Realizable by minimal left ideals in the infinite dimensional $sCliff(V, \omega) = T(V)/\langle v \otimes w - w \otimes v - \omega(v, w)1 | \, v, w \in V \rangle$

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- There is no faithful representation of $\widetilde{G}$ on a finite dimensional vector space, **non-matrix group**
- $E = E_+ \oplus E_-$, even and odd square integrable functions on Lagrangian space L, it is a decomposition into irreducibles

Model for the Complex - Symplectic Spinor Valued Exterior Forms

Notation:

- The double cover $\lambda : \widetilde{G} \to Sp(2n, \mathbb{R}) \simeq \lambda^* : \widetilde{G} \to \mathrm{Aut}(V^*)$ is a representation; wedge-powers $\lambda^i : \widetilde{G} \to \mathrm{Aut}(\bigwedge^i V^*)$ exterior forms

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- tensor product with symplectic spinors: $\mathfrak{m}^i_\pm : \widetilde{G} \to \mathrm{Aut}(\bigwedge^i V^* \otimes E_\pm)$ sympl. spinor valued ext. forms; often considered by Penrose in the pseudoriemannian case of sign. $(1, 3)$ $\mathfrak{m}^i = \mathfrak{m}^i_+ \oplus \mathfrak{m}^i_-$

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- **Theorem** [Krysl, Lie Theory]: For each i, there are irreducible modules $E^{ij}_\pm$, $j = 0, \ldots, k_i = n - |n - i|$, such that

$$\bigwedge^i V^* \otimes E_\pm = E^i_\pm = \oplus_{j=0}^{k_i} E^{ij}_\pm.$$

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- We set $E^{ij} = E^{ij}_+ \oplus E^{ij}_-$.

Decomposition Diagram (in dimension six)

Representations $E^{ij}_{\pm}$ described by highest weight (of the "infinitesimal" $\mathfrak{g}$-structure) with respect to a Cartan subalgebra and a choice of positive roots, $2n = 6$.

$$
\begin{array}{ccccccc}
E^0 & E^1 & E^2 & E^3 & E^4 & E^5 & E^6 \\
E^{00} & E^{10} & E^{20} & E^{30} & E^{40} & E^{50} & E^{60} \\
 & E^{11} & E^{21} & E^{31} & E^{41} & E^{51} & \\
 & & E^{22} & E^{32} & E^{42} & & \\
 & & & E^{33} & & &
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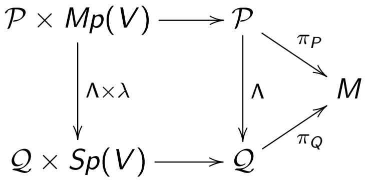

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\Big\downarrow{\scriptstyle \Lambda \times \lambda} & & \Big\downarrow{\scriptstyle \Lambda} & & M \\
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- Existence [Forger, Hess] - obstacle: second Stiefel-Whitney class non-zero; T^*N for N orientable, tori, $\mathbb{CP}^{2n+1}$

Symplectic Connections

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- **Theorem** [Tondeur ('60)]: The affine space of Fedosov connections is in an affine bijection with the affine space $(\Gamma(S^3 M), 0)$.

Symplectic Curvature Tensors

- Curvature definition and symmetries:
  $R_{ijkl} = \omega(R(X_k, X_l)X_j, X_i)$, $R_{ijkl} = R_{jikl}$ and
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- **Definition:** A Fedosov connection is called *Weyl-flat* (or of Ricci-type) if $W = 0$.

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- i-th curvature $R^i = \nabla^{i+1}\nabla^i$. Total curvature $R = \sum_{i=0}^{2n-2} R^i$

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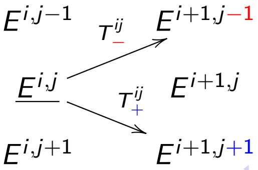
- In the decomposition diagram of $\bigwedge^i V^* \otimes E$ into irreducible representations $\Longrightarrow$



Condition for Forming a Complex

- We would like to investigate the chain-complex condition

$$T_{\pm}^{i+k+1,j+k\pm1} \circ T_{\pm}^{i+k,j+k} = 0, \text{ i.e.,}$$

$$p^{i+k+2,j+k\pm2} \circ \nabla^{i+k+1,j+k\pm1} \circ p^{i+k+1,j+k\pm1} \circ \nabla^{i+k,j+k} = 0$$

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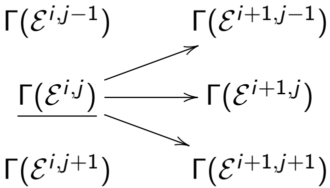

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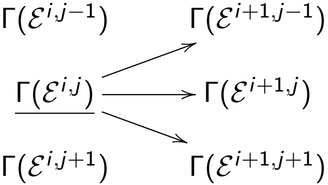

- $\Longrightarrow T_{\pm}^{i+1,j\pm1} T_{\pm}^{i,j} = p^{i+2,j\pm2} \nabla^{i+1} \nabla^{ij} = p^{i+2,j\pm2} R^{\nabla}_{|\Gamma(\mathcal{E}^{ij})}$. Thus analyze the curvature.

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- Symplectic spinor multiplication $\cdot : V \times E \to E$

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Curvature Structure of Weyl-Flat Connection - Representation Approach

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- Equivariant properties of this multiplication with respect to $\mathfrak{m}$ $\implies$ it can be defined on symplectic spinor bundle

Representation of $\mathfrak{osp}(1|2)$ on Symplectic Spinor Forms

- $F^+(\alpha \otimes f) := \frac{\imath}{2}\sum_{i=1}^{n} \epsilon^i \wedge \alpha \otimes e_i \cdot f$, $\alpha \otimes f \in \bigwedge^i V^* \otimes \mathcal{S}(L)$

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- **Lemma** (curvature). If ∇ is a symplectic Weyl-flat connection, then

$$R = \frac{1}{n+2}(E^+\Theta^\sigma + 2F^+\Sigma^\sigma).$$

Curvature in Diagram Decomposition

- F^+:

$$\underline{E^{i,j}} \xrightarrow{F^+} E^{i+1,j}$$

- Σ^σ:

$$\begin{array}{ccc} E^{i,j-1} & & E^{i+1,j-1} \\ & \nearrow & \\ \underline{E^{i,j}} & \longrightarrow & E^{i+1,j} \\ & \searrow & \\ E^{i,j+1} & & E^{i+1,j+1} \end{array}$$

- E^+:

$$E^{i,j} \xrightarrow{\quad F^+ \quad} E^{i+2,j}$$

- Θ^σ:

$$\begin{array}{c} E^{i,j-1} \\ \uparrow \\ \underline{E^{i,j}} \\ \downarrow \\ E^{i,j+1} \end{array}$$

Curvature and Connection

- Curvature $R = \frac{1}{n+2}(E^{+}\Theta^{\sigma} + 2F^{+}\Sigma^{\sigma})$:

$$\begin{array}{ccccc} E^{i,j-1} & & E^{i+1,j-1} & & E^{i+2,j-1} \\ & & & \nearrow & \\ \underline{E^{i,j}} & \longrightarrow & E^{i+1,j} & \longrightarrow & E^{i+2,j} \\ & & & \searrow & \\ E^{i,j+1} & & E^{i+1,j+1} & & E^{i+2,j+1} \end{array}$$

- Cov. derivative ∇^{ij} :

$$\begin{array}{ccc} \Gamma(\mathcal{E}^{i,j-1}) & & \Gamma(\mathcal{E}^{i+1,j-1}) \\ & \nearrow & \\ \underline{\Gamma(\mathcal{E}^{i,j})} & \longrightarrow & \Gamma(\mathcal{E}^{i+1,j}) \\ & \searrow & \\ \Gamma(\mathcal{E}^{i,j+1}) & & \Gamma(\mathcal{E}^{i+1,j+1}) \end{array}$$

Cov. derivative's target are right also if the connection has torsion and ω is pre-symplectic only ($d\omega \neq 0$).

Theorem and proof

Theorem: Let (M, ω) be symplectic manifold admitting a metaplectic structure and let ∇ a Weyl-flat Fedosov connection on (M, ω). Then for all pairs of integers (i, j), the sequences $(\Gamma(\mathcal{E}^{i+k,j\pm k}), T_{\pm}^{i+k,j\pm k})_{k\in\mathbb{Z}}$ form complexes.
Proof. Basic steps, ideas

Composition of the $(+)$-twistor operators (upwards going)

$$p^{i+2,j+2} \circ \nabla^{i+1} \circ \nabla^{i}_{|\Gamma(\mathcal{E}^{ij})}$$

$$p^{i+2,j+2} \circ R^{\nabla}_{|\Gamma(\mathcal{E}^{ij})} = p^{i+2,j+2} \circ \nabla^{i+1}\nabla^{i}_{|\Gamma(\mathcal{E}^{ij})} = 0$$

From the structure of R^{∇} for Weyl-flat Fedosov connection, we see that $p^{i+2,j+2} \circ R^{\nabla}_{|\Gamma(\mathcal{E}^{ij})} = 0$

Topology on cohomology of complexes

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Topology on cohomology of complexes

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- *Related questions*: Complexes of infinite rank bundles, topological cohomology questions: Images with or without completion $\mathrm{Ker} D^i/\mathrm{Im} D^{i-1}$ or $\mathrm{Ker} D^i/\overline{\mathrm{Im} D^{i-1}}$? (Important in Analysis and Quantum Physics of constraint systems - Becchi–Rouet–Stora–Tyutin)

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