Ellipticity of complexes of symplectic twistor operators

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Geometry and Physics, Srní, Jan. 13 - 20, 2024

Motivation

1) Symplectic twistor operators are parallel to Penrose's twistor operators ([Penrose-Rindler]) used for studying spin particles in general rel. of Gravitation; also in gauge complexes for charges of massless 3/2-spin fields.
2) We study them in **symplectic** and infinite rank bundle setting.
3) The symplectic operators (called symplectic twistor operators) are known to form *complexes* under an integrability condition expressed by the Weyl curvature ([KryslMonat, KryslACCA]).
4)Two of them already known to be *elliptic* ([KryslArch]).
5) We show how to prove the ellipticity of the remaining ones.

Background from algebra

Use of centralizers of a *group action on a (complex) vector space* W in analysis of operators on functions with values in W

Setting:
1) A associative algebra over $\mathbb{C}$, ie., A is a ring and and $\mathbb{C}$-module (the ring plus-operation is considered to coincide with the module plus-operation) with compatibility $r \cdot (s \cdot t) = (rs) \cdot t$, $r, s \in \mathbb{C}$ and $t \in A$.

2) $\rho : A \to \mathrm{Aut}_{\mathbb{C}}(W)$ representation of A on W. Thus W is A-mod.

3) *Centralizer (commutant algebra)*
$B = \mathrm{Comm}_A(W) = \{T : W \to W | \, T \circ \rho(a) = \rho(a) \circ T$ for all $a \in A\}$ space of all A-equivariant maps/A-homomorphisms/intertwiners

4) If A is semi-simple (sum of algs w. no proper $\neq 0$ ideals), W is multiplicity-free as $A \otimes B$-module, i.e., if $W' \neq W''$ are $A \otimes B$ -submodules of W, then $W' \not\cong W''$ as $A \otimes B$-modules.

i) Basic example: Schur duality

$G = GL(V)$ and $W = \bigotimes^k V$, $\rho(g)(v_1 \otimes \ldots \otimes v_k) = gv_1 \otimes \ldots gv_k$
$\implies (c_1 g_1 + c_2 g_2) \cdot w = c_1 \rho(g_1)(w) + c_2 \rho(g_2)(w)$, $g_1, g_2 \in G$,
$c_1, c_2 \in \mathbb{C}$ be the *extension of the action to the group algebra* $A = \mathbb{C}[G]$; $\tau(\pi)(v_1 \otimes \ldots \otimes v_k) = v_{\pi^{-1}(1)} \otimes \ldots \otimes v_{\pi^{-1}(k)}$, where $\pi \in S_k$ (permutation group on k letters), $v_1 \otimes \ldots \otimes v_k \in \bigotimes^k V$

Result: $\mathrm{Comm}_A(W) = \mathbb{C}[\tau(S_k)]$.

ii) Example of **harmonic polynomials**: $O(n, \mathbb{R})$ on
$P = P[x^1, \ldots, x^n]$ by the regular representation: $p \in P$,
$g \in O(n, \mathbb{R})$, $x = (x^1, \ldots, x^n)$, $[\rho(g)(p)](x) = p(g^{-1}(x))$.
Centralizer algebra generated by $\Delta = -\frac{1}{2} \sum_{i=1}^n \partial^2_{x^i}$,
$E = -\sum_{i=1}^n x^i \partial_{x^i} - \frac{n}{2}$ and multiplication by $r^2 = \frac{1}{2} \sum_{i=1}^n (x^i)^2$.
Forms a representation of Lie algebra $\mathfrak{sl}(2, \mathbb{C}) = \langle e^+, h, e^- \rangle$ by
$e^+ \mapsto \Delta$, $h \mapsto E$, $e^- \mapsto r^2$.

Literature on examples of commutant algebras

iii) **Further examples**: H. Weyl (Theory of Groups and Quantum mechanics); R. Howe [Ho] (systematic unifying approach for so called classical groups); Goodman, Wallach [GN] (text-book); Slupinski [Slup] - $Spin(n)$ acting on spinor valued anti-symmetric forms; Leites, Shchepochkina ('super cases') [L], Krýsl [KrLie] - $Mp(2n)$ acting on symplectic spinor valued wedge forms; Bracx, De Schepper, Eelbode, Lávička, Souček [Br]; De Bie, Souček, Somberg [Bie] (in Clifford algebras).

Symplectic spinors

(V, ω) real symplectic vector space of dimension $2n$

$\lambda : \widetilde{G} = Mp(V, \omega) \to Sp(V, \omega)$, connected double cover of $G = Sp(V, \omega)$, $\widetilde{G}$ - so called **metaplectic group**, non-compact Lie group - parallel to the covering $Spin(n) \to SO(n)$

$\mathbb{L} \subseteq V$ maximal isotropic vector subspace: $\omega(v, w) = 0$ for all $v, w \in \mathbb{L}$, $\mathbb{L} \simeq \mathbb{R}^n$

$L : Mp(V, \omega) \to U(L^2(\mathbb{L}))$ distinguished *Segal–Shale–Weil/ /symplectic spinor/metaplectic/oscillator representation* [Shale], [Weil], [Kostant]

$S = L^2(\mathbb{L})$ - **symplectic spinors**, $E = \bigoplus_{i=0}^{2n} \bigwedge^i V \otimes S$ - **symplectic spinor valued wedge forms**

$$\rho(g)(\alpha \otimes s) = \lambda(g)^* \alpha \otimes L(g)s$$

Decomposition of $E = \bigoplus_{i=0}^{2n} \bigwedge^i V \otimes S$

The module E decomposes [KrLie] as $\widetilde{G}$-module into direct sum

$$\bigoplus_{(i,j)\in\Xi} E^{ij},$$

where Ξ is a finite set ($(n+1)(2n+1)$ elements),
$E^{ij} = E_{ij}^{+} \oplus E_{ij}^{-} \subseteq \bigwedge^i V \otimes S$ and $E_{ij}^{\pm}$ are irreducible $\widetilde{G}$-modules.

p^{ij} projection of $\bigwedge^i V \otimes S$ onto E^{ij}

E^0	E^1	E^2	E^3	E^4	E^5	E^6
E^{00}	E^{10}	E^{20}	E^{30}	E^{40}	E^{50}	E^{60}
	E^{11}	E^{21}	E^{31}	E^{41}	E^{51}	
		E^{22}	E^{32}	E^{42}		
			E^{33}			

Lie super algebras

$\mathfrak{f}$ is a super-graded vector space, i.e., $\mathfrak{f} = \mathfrak{f}_0 \oplus \mathfrak{f}_1$ is a direct sum of vector spaces

$|z| = i$ if $0 \neq z \in \mathfrak{f}_i$, $i \in \mathbb{Z}_2 = \{0, 1\}$

$[\![,]\!] : \mathfrak{f} \times \mathfrak{f} \to \mathfrak{f}$ is 1) complex bilinear

2) $[\![,]\!] : \mathfrak{f}_i \times \mathfrak{f}_j \to \mathfrak{f}_{i+j}$, $i + j$ is considered $mod\, 2$

3) super anti-symmetric: $[\![x, y]\!] = -(-)^{|x||y|}[\![y, x]\!]$

4) super-Jacobi rule
$$(-)^{|x||z|}[\![x, [\![y, z]\!]]\!] + (-)^{|z||y|}[\![z, [\![x, y]\!]]\!] + (-)^{|y||x|}[\![y, [\![z, x]\!]]\!] = 0$$

where for each $x, y, z \in \mathfrak{f}$ satisfy $x \in \mathfrak{f}_{|x|}, y \in \mathfrak{f}_{|y|}, z \in \mathfrak{f}_{|z|}$, i.e., they are homogeneous wr. to $\mathfrak{f}_0 \oplus \mathfrak{f}_1$

Lie super algebra $\mathfrak{f} = \mathfrak{osp}(1|2)$

$\mathfrak{f} = \mathfrak{f}_0 \oplus \mathfrak{f}_1$ (bosonic and fermionic part)

$\mathfrak{f}_0 = \mathrm{Lin}_{\mathbb{C}}(e^+, h, e^-) \cong \mathfrak{sl}(2, \mathbb{C})$

$\mathfrak{f}_1 = \mathrm{Lin}_{\mathbb{C}}(f^+, f^-)$

$$
\begin{aligned}
[\![h, e^\pm]\!] &= \pm e^\pm & \qquad [\![e^+, e^-]\!] &= 2h \\
[\![h, f^\pm]\!] &= \pm\frac{1}{2} f^\pm & \qquad [\![f^+, f^-]\!] &= \frac{1}{2} h \\
[\![e^\pm, f^\mp]\!] &= -f^\pm & \qquad [\![f^\pm, f^\pm]\!] &= \pm\frac{1}{2} e^\pm
\end{aligned}
$$

Commutant for sympl. spinor valued anti-symmetric forms

Consider $E = E_0 \oplus E_1$ as super vector space ($\mathbb{Z}_2$-grading), where $E_0 = \bigoplus_{i=0}^{n} \bigwedge^{2i} V \otimes S$, $E_1 = \bigoplus_{i=1}^{n} \bigwedge^{2i-1} V \otimes S$.
$p_+(\alpha \otimes s) = \alpha \otimes s_+$, $p_-(\alpha \otimes s) = \alpha \otimes s_-$, where $s = (s_+, s_-) \in S_+ \oplus S_- = S = L^2(\mathbb{L})$ is the decomposition into even and odd part.

Definition: $(e_i)_{i=1,\dots,2n}$ symplectic basis of (V,ω), $(\epsilon^i)_{i=1}^{2n} \subseteq V^*$ dual basis
$F^+(\alpha \otimes s) = \frac{\imath}{2} \sum_{i=1}^{2n} \epsilon^i \wedge \alpha \otimes e_i \cdot s$ (degree rising),
$F^-(\alpha \otimes s) = \frac{1}{2} \sum_{i,j=1}^{2n} \omega^{ij} \iota_{e_i} \alpha \otimes e_j \cdot s$ (degree lowering).

Theorem ([KrLie] 2012; ArXiv 2008): Setting $\tau(f^{\pm}) = F^{\pm}$ and extending it to a homomorphism of Lie super-algebras $\mathfrak{osp}(1|2)$ and $\mathrm{End}(E)$, we get $\mathrm{Comm}_{\mathbb{C}[\widetilde{G}]}(E) = \langle \tau(\mathfrak{osp}(1|2)), p_\pm \rangle$.

Symplectic twistor operators

$(\mathbb{R}^{2n}, \omega_0)$ symplectic vector space

For $f : \mathbb{R}^{2n} \to E^{ij} \subseteq \bigwedge^i V \otimes S$

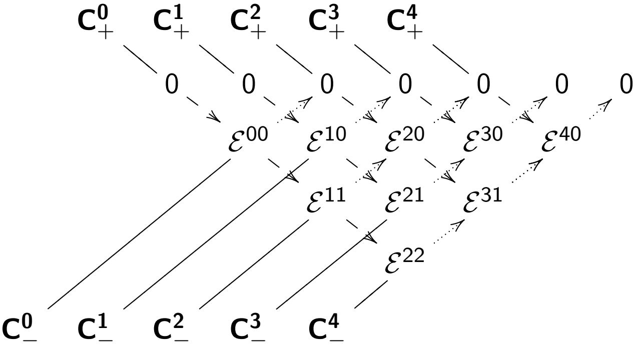
$(\nabla f)(y) := \sum_{k=1}^{2n} \epsilon^k \wedge (\frac{\partial f}{\partial x^k})(y) \in \bigwedge^{i+1} V \otimes S,\ y \in \mathbb{R}^{2n},$

$(T_{\pm}^{ij} f)(y) = p^{i+1,j\pm 1}(\nabla f)(y)$ *symplectic twistor operators*

Symplectic Dirac operators defined by K. Habermann [KH] in the nineties. ([Habs] monograph on sympl. Dirac.)

Structure of complexes twistor operators

Dim $M = 4$



Complexes of symplectic twistor operators

Theorem [KrMon], [KrArch]: If (M, ω) is a smooth symplectic manifold ($d\omega = 0$), with vanishing second Stiefel–Whitney class, ∇ is a symplectic torsion-free connection ($\nabla\omega = 0$, torsion of $\nabla = 0$) and the **symplectic Weyl curvature** of ∇ vanishes, then $(C^\infty(M, E^{i+k,j\pm k}), T_{\pm}^{i+k,j\pm k})_k$ is a complex, i.e.,

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Theorem [subm.]: Under the assumptions of preceding theorem, the symplectic twistor complexes are elliptic.
Proof. C_+^0 and C_-^{2n} proved elliptic in [KryslArch].

Symbols of symplectic twistor complex

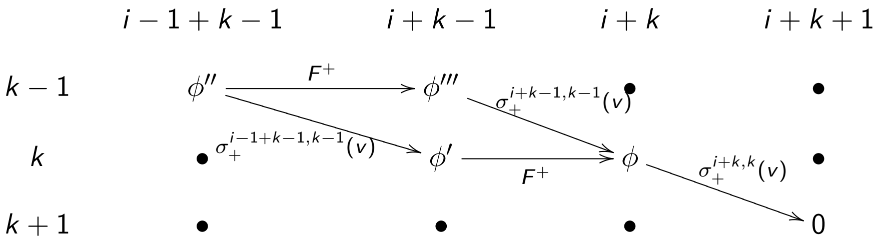
Induction (the $+$-case): Suppose the complex C_+^{i-1} is elliptic.
$\phi \in \operatorname{Ker} \sigma_+^{i+k,k}(v) \subseteq E^{i+k,k}$, $\phi' = (F^+)^{-1}\phi \in E^{i+k-1,k}$,

$$\sigma_+^{i+k,k}(v \wedge F^+\phi') = 0.$$

Schur on intertwiners: $\exists$ complex numbers $\lambda \neq 0$ such that

$$\lambda(F^+ \circ \sigma_+^{i+k-1,k})(v \wedge \phi') = \sigma_+^{i+k,k}(v \wedge F^+\phi') = 0$$

(commutativity up to multiple).



$\Rightarrow \sigma_+^{i+k-1,k}(v \wedge \phi') = 0 \Rightarrow \sigma_+^{(i-1)+k),k}(v \wedge \phi') = 0.$
Because C_+^{i-1} is elliptic by the induction hypothesis, there exists an element ϕ'' such that ϕ' is the image of ϕ'' by the map $\sigma_+^{(i-1)+(k-1),k-1}(v)$.
Setting $\phi''' = \lambda^{-1}F^+\phi'' \in E^{i-1,k-1}$, we get the desired element in the preimage of ϕ by $\sigma_+^{i+k-1,k-1}(v)$.

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