Riemann surfaces

1. Riemann → harmonicity & topology
Intro Klein → algebr. fctions, multivalued maps (pictures of)
Weyl → definition (quite up to date : Über die Idee der Riemannschen Fläche)

Analysis, Topology („Analysis situ" - Poincaré), Geometry (holonomy, alg. geom), Physics (CFT), inspiring for Number theory

Literature : Forster, Lectures on Riemann Surfaces
∄ Černý, Analýza v kplx. oboru
∄ Černý, Foundations of Analysis in Cplx Dom.
Narasimhan, Comp. Riem. Surfaces

Content :
1. Introduction
2. Definition & Examples of Riem. surf. Basic properties of holom. maps.
3. Covering spaces and homotopy
4. Sheaves and merom. continuation
5. Dolbeault's differential and cohomology
6. Divisors & Riemann-Roch thm.

2. Definition and Ex. of Riemann surfaces. Basic prop.

Def: Biholomorphism (in $\mathbb{C}$): $f : U \twoheadrightarrow V$ bijective ~~and~~, holomorphic, inverse f^{-1} is holomorphic

Def: Topological manifold of dimension n $\equiv$ Hausdorff top. space

with countable basis (of open neighb.), locally homeomorphic
to $\mathbb{R}^n$, i.e., $\forall p \in M$ $\exists U \underset{\text{open}}{\subseteq} M$, $p \in U$ s.t. $\exists \varphi : U \to \mathbb{R}^n$
φ is homeo (onto an open set).

Definitions of $\mathcal{C}^k$-atlas, C^k-structure; notes on maximal atlas
↑ Recall of manifolds

Definition: Holomorphic atlas ≡ atlas, the transition functions of whic are biholomorphisms of open sets in $\mathbb{C}$

! Remark ("identification"): $f : \mathbb{R}^2 \to \mathbb{R}^2$ associate $\tilde{f} : \mathbb{C} \to \mathbb{C}$ by $\tilde{f}(x+iy) = f^1(x,y) + i f^2(x,y)$, where $f^i = pr^i \circ f$, $i = 1,2$ and $pr^i : \mathbb{R}^2 \to \mathbb{R}$ projects onto the i-th component in the (Cartesian) product.

Thus, we may speak about holomorphic maps $\mathbb{R}^2 \to \mathbb{R}^2$.

Typical vocabulary:

Def: Holomorphically compatible atlases (as classical $\mathcal{C}^k$-compatibility): $\mathcal{U}_1, \mathcal{U}_2$ atlases on M are holomorphically compatible ≡ $\mathcal{U}_1 \cup \mathcal{U}_2$ is a holomorphic atlas

Def: Holomorphic structure ≡ a class of holomorphically compatible holomorphic atlases.

Remark: Holomorphic structures $\overset{\dots}{\longleftrightarrow}$ maximal holomorphic atlases

Definition: $(M, \mathcal{U})$ is called a Riemann surfaces if M is a connected topologic manifold of dim 2 and $\mathcal{U}$ is a holomorphic structure.

Remark: Riemann surfaces is a connected complex mnfld of (real) dimension 2 or a cplx mnfld of cplx mnfld 1. ~~[or just a cplx curve [in alg. geom.]~~ ~~smooth if one knows Kodaira emb.]~~

Examples: 1) Domains in $\mathbb{C}$

+2 $(M, \mathcal{U})$, M a domain $\mathcal{U} = \langle\{(M, Id_M)\}\rangle$, where $\langle B \rangle$ for an atlas B is the holo-structure containing B. Esp. $(\mathbb{C}, \langle(\mathbb{C}, Id_{\mathbb{C}})\rangle)$ is Riem surface.

3) $\mathbb{C}P^1 := \mathbb{C} \cup \{\infty\}$, where $\infty \notin \mathbb{C}$

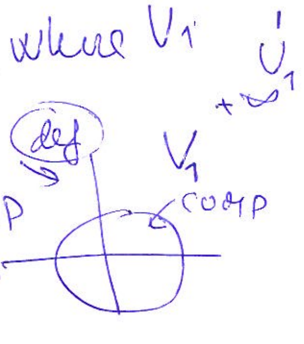
Topology: $U' \subseteq \mathbb{C}P^1$ is open iff

a) $\infty \notin U'$ and U' is open in $\mathbb{C}$

b) $\infty \in U'$ and $U' = V \cup \{\infty\}$, where V_1 a complement of cpt in $\mathbb{C}$

$V_1 = \mathbb{C} \setminus comp$

$U_1' = V_1 \cup \{\infty\}$



$U_2' = U_2'$

$\longleftarrow$ picture

$\longleftrightarrow$ under so-called stereogr. proj

[The def. is an example of the so-called Alexandrov's compactification]

Atlas: $U_1 = \mathbb{C}P^1 \setminus \{\infty\} = \mathbb{C}$ $\varphi_1(z) = z$

$U_2 = \mathbb{C}P^1 \setminus \{0\} = \mathbb{C}^\times \cup \{\infty\}$, where $\mathbb{C}^\times = \mathbb{C}\setminus\{0\}$,

$$\varphi_2(z) = \begin{cases} \frac{1}{z}, & z \neq \infty \\ 0, & z = \infty \end{cases}$$

[Easy φ_2 is homeo]

$$\varphi_2 \circ \varphi_1^{-1} : \underbrace{\varphi_1(U_1 \cap U_2)}_{\overset{\parallel}{\mathbb{C}^\times}} \to \underbrace{\varphi_2(U_1 \cap U_2)}_{\mathbb{C}^\times}$$

$\varphi_2 \circ \varphi_1^{-1}(z) = \frac{1}{z}$ holo on $\mathbb{C}^\times$. $\frac{1}{z}$ is holo!

(Notice that,
$\varphi_1 \circ \varphi_2^{-1}(z) = z$ holo as well.)

Exercise/HW: Compute the real diff $d(\varphi_2 \circ \varphi_1^{-1})$ in $z_0 = x_0 + iy_0$ in direction $h = (h_1, h_2)$. (Ident. $\mathbb{C} \leftrightarrow \mathbb{R}^2$.) Prove that it is $\mathbb{C}$-linear.

4) $\omega_1, \omega_2 \in \mathbb{C}$ over $\mathbb{R}$ $\{\omega_1, \omega_2\}$ linearly indep. in this vect. sp.
$\Gamma := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is a lattice (abelian discrete group)
$\mathbb{C}/\Gamma$, $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$ ← final topol (= finest s.t. π is cont.)

① Holostructure (inducing of):

a) π is open: V open $\Rightarrow \pi(V)$ open? ~~$\pi(V) = \pi^{-1}(\pi(V))$~~

$\pi(V)$ open in $\mathbb{C}/\Gamma$ iff (def. of top in $\mathbb{C}/\Gamma$) $\pi^{-1}(\pi(V)) =: V'$ is open. But $V' = \bigcup_{\omega \in \Gamma} \underbrace{(\omega + \overset{\uparrow \text{open}}{V})}_{\text{open}}$ (union of open sets), is open.

b) "inverse of suitable restr. of π"

We can do it more definitely as in the lecture!

→ Let U be open in $\mathbb{C}$ s.t. no points in it are Γ-equivalent ($\overset{*}{=}$ differ by Γ)*)? Then form $\pi(U)$. easy $\twoheadrightarrow U$

These sets cover $\mathbb{C}/\Gamma$. Moreover $\varphi = (\pi_{|\pi(U)})^{-1} : \pi(U) \to \mathbb{C} \simeq \mathbb{R}^2$ ②

exists (as a map) and it is bijective onto U. Since π is ~~cont.~~ open $\Rightarrow \pi_{|\pi(U)}$ is open $\Rightarrow (\pi_{|\pi(U)})^{-1}$ is cont.

Since π is cont $\Rightarrow \pi_{|\pi(U)}$ is cont $\Rightarrow (\pi_{|\pi(U)})^{-1}$ is open and $((\pi_{|\pi(U)})^{-1})^{-1} = \pi_{|\pi(U)}$ is cont. Thus $\pi_{|\pi(U)}^{-1}$ is not only bijective, but also a homeo.

→ Compatibility: φ_1, φ_2 be inversions of restrictions to sets as above. Thus elements of the atlas

$\psi := \varphi_2 \circ \varphi_1^{-1}$ is holo?

$$\pi(\psi(w)) = (\pi \circ \varphi_2 \circ \varphi_1^{-1})(w) \overset{\pi \,\&\, \varphi_2 \text{ annihilate}}{=} \varphi_1^{-1}(w) = \pi(w) \underset{\text{def of } \pi}{\Longrightarrow}$$

$\psi(w) - w \in \Gamma$. But ψ is cont. + Γ discr. $\Rightarrow$

$\Rightarrow \exists\, \omega_0 \in \Gamma$ such that $\psi(w) = w + \omega_0$ on $\varphi_1(U_1 \cap U_2)$, where $U_i = \mathrm{Dom}(\varphi_i)$, $i = 1, 2$, in particular ψ is holo (an affine map is holomorphic).

$\Rightarrow$ Summing up, $\mathbb{C}/\Gamma$ is a Riemann surface.

$$\to \mathbb{C}/\Gamma = \pi(\underbrace{\{\lambda\omega_1 + \lambda\omega_2 \mid 0 \leq \lambda \leq 1, 0 \leq \mu \leq 1\}}_{\text{CPT}})$$

Cont. images of cpt. is cpt. Thus $\mathbb{C}/\Gamma$ is cpt.

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*) We called them su(itable) without giving a precise def. (in the lect.).

Definition: X, Y Riemann surfaces. $f: X \to Y$ is called holomorphic (biholomorphic), if $\forall x_0 \in X$ $\forall$ map φ around x_0 and $\forall$ map ψ around $f(x_0)$, the map

$$\varphi \circ f \circ \psi^{-1} : \mathbb{C} \rightarrowtail \mathbb{C}$$ is holomorphic (biholomorphic).

Remark: (U, φ) in atlas... coordinate systems (or maps / sometimes φ itself is called a map.

$\psi = \varphi_1 \circ \varphi_2^{-1}$ are called transition fctions
(This is classical in the manifold theory.)

Notation: $\mathcal{O}(X, Y) = \{ f: X \to Y \mid f \text{ holomorphic} \}$
$\mathcal{O}(X, \mathbb{C}) =: \mathcal{O}(X)$, $\mathbb{C}$ with std holom. str.
$\mathcal{O}(X)$ is an algebra over $\mathbb{C}$ (or just a ring).
$f, g \in \mathcal{O}(X) \Rightarrow fg \in \mathcal{O}(X)$ [Easy lecture]
$f+g \in \mathcal{O}(X) \Leftarrow f+g \in \mathcal{O}(X)$
~~fg (?)~~ $f+g \in \mathcal{O}(X)$
$(f+g)h = fh + gh \, (\underset{+}{=} hf + hg = h(f+g))$

—X—

Remark: $f: X \to Y$, X, Y Riem. surf, f holom. in x_0 iff. $\exists U$ open connected s.t. $f|_U$ is ~~open~~ holomorphic.
This make sense since x_0 has a coordinate system around itself (U, φ). Take the conn. component containing x_0. φ homeo $\Rightarrow$ φ restr. to the conn. comp. is homeo again. Transitions are holomorphic
$\Rightarrow$ connected comp is a Riemann surface itself!
$f|_U : U \to Y$ holom. make sense.

2.1. Basic properties

Thm. (Riemann's removable singularities): $f : U \setminus \{z_0\} \subseteq \mathbb{C} \to \mathbb{C}$ holomorphic and bounded $\Rightarrow \exists \tilde{f} : U \to \mathbb{C}$ holomorphic extension of f.

Proof: $\exists a_n \in \mathbb{C}, n \in \mathbb{Z}$ $f_{|D_\varepsilon(z_0)\setminus\{z_0\}} = \sum_{n=-\infty}^{+\infty} a_n (z-z_0)^n$

bounded $\Rightarrow a_n = 0 \; \forall n < 0$ (otherwise limit of $f \to \infty$ in z_0 $\Rightarrow$ unb.)

$f_{|D_\varepsilon(z_0)\setminus\{z_0\}} = \sum_{n=0}^{+\infty} a_n (z-z_0)^n$ $\#$

Formally: $\tilde{f}(z_0) := a_0$,

$\tilde{f} = f$ outside z_0; $\tilde{f}$ is holo on $D_\varepsilon(z_0)$ (given by reg. L. series + Cauchy $\frac{d}{dz}\sum = \sum\frac{d}{dz}$) and $\tilde{f}$ is holo on $U \setminus D_\varepsilon(z_0)$ because it equals f, which is holo there. □

Thm. (Riemann's removable sing. on Riemann surfaces): Let U be open in a Riemann manifold X and $z_0 \in U$. Then $f : U \setminus \{z_0\} \to \mathbb{C}$ holomorphic bounded implies f has a holomorphic extension.

Proof: X Riem. surf., $\mathcal{U}$ holom. structure on X $\Rightarrow$ $\exists$ atlas in $\mathcal{U}$ $\Rightarrow \exists (U_1, \varphi)$, $z_0 \in U_1$.

$\bar{f} := f \circ (\varphi^{-1})_{|\varphi(U \cap U_1 \setminus \{z_0\})}$ is holomorphic (on $\mathbb{C}$, we have the identity as the map). Thus $\bar{f} : \varphi(U \cap U_1) \setminus \{\varphi(z_0)\} \to \mathbb{C}$ is bounded and holomorphic ($|\bar{f}| \leq |f|$ on $\varphi(U_1 \cap U_1) \setminus \{\varphi(z_0)\}$). Thus $\bar{f}$ has removable singularities

$\Rightarrow \exists \tilde{\bar{f}} \in \mathcal{O}(\varphi(U \cap U_1))$ extending $\bar{f}$

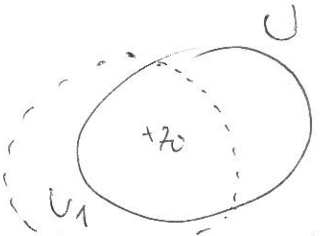
by Riem. rem. sg.!

$$\tilde{\tilde f} = \begin{cases} \tilde f \text{ on } U\cap U_1 & \text{holomorphic} \\ f \text{ on } U\setminus U_1 & \text{holomorphic}\end{cases}$$

Thus $\tilde{\tilde f}$ is holomorphic. □

<u>Remark:</u>

↑ small analysis of holomorphicity of $\tilde{\tilde f}$.



Notation as in the proof.

$\tilde{\tilde f} : U \to \mathbb{C}$, $w\in U$ ~~and~~ $\Rightarrow \exists U_2 \ni w$ & $\psi : U_2 \to \mathbb{C}$, $U_2\cap U \neq \emptyset$

Is $\tilde{\tilde f}\circ\psi^{-1} : \mathbb{C} \to \mathbb{C}$ holomorphic

$u \in \mathrm{Im}(\psi)$

a) $\psi^{-1}(u) \in \underbrace{U_1\cap U}$ ← holomorphic since $\tilde f$ is hol.

$\tilde{\tilde f}\circ\psi^{-1}(u) = \tilde f\circ\psi^{-1}(u)$

b) $\psi^{-1}(u)\in U\setminus U_1$

$\tilde{\tilde f}\circ\psi^{-1}(u) = \underbrace{f\circ\psi^{-1}}_{\text{holo since } f \text{ is holom.}}(u)$

c) $\psi^{-1}(u)$ outside of U $\Rightarrow$ does not contribute to the question of $\tilde{\tilde f}\circ\psi^{-1}$ holom.

It "goes" from $\tilde{\tilde f}\circ\psi^{-1} : \psi(U_2\cap U) \to \mathbb{C}$

<u>Thm</u> (Identity thm): Let $U\subseteq\mathbb{C}$ be a domain, $f,g\in\mathcal{O}(U)$ and $A\subseteq U$ have a limit point in U. If $f=g$ on A $\Rightarrow$

$\Rightarrow f=g$ on U.

Proof: $\emptyset$ std. cplx. an. txtbooks

<u>Remark</u>: 1) The limit point is an ∂U is not sufficient.

$U=\mathbb{C}\setminus\{0\}$, $A=\{z_n=\frac{1}{\pi n} \mid n\in\mathbb{N}\}$, $f(z)=\sin\frac{1}{z}$

$f_{|A} = 0_{|A}$, but $f = 0$ on U.

2) V closed in $\mathbb{C}$, connected, $A \subseteq V$ limit point in V

$f = g$ on A, f, g holo on $V \overset{\uparrow}{\Longrightarrow} \exists\, \tilde{f}, \tilde{g} \in \mathcal{O}(U)$ $\tilde{f}_{|V} = f$, $\tilde{g}_{|V} =$

def in $\mathbb{C}$ $\quad \underset{\text{open}}{U \supseteq V}$

$= g$; A limit in $U \Rightarrow$

$\overset{\uparrow}{\Rightarrow} \tilde{f} = \tilde{g}$ on $U \Rightarrow \tilde{f} = \tilde{g}$ on V.

Identity thm'

(Identity thm holds in closed sets if

* the limit point is in them)

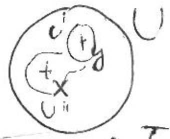
Thm. (Identity thm for Riemann surfaces): Let X, Y be Riemann surface and $f, g \in \mathcal{O}(X, Y)$. If $f = g$ on a subset $A \subseteq X$ having a limit poin in X, then $f = g$ on X.

Proof: $G = \{x \in X \mid \exists \underset{\text{open}}{\text{neighb.}}\ U \ni x \text{ s.t. } f_{|U} = g_{|U}\}$ ((coo. neighb are not necessary, we may do intersections

a) G is open. Picture not necessary (!)

$x \in G \Rightarrow \exists U \ni x \quad f_{|U} = g_{|U}$. Take $V = U$ and $y \in V$

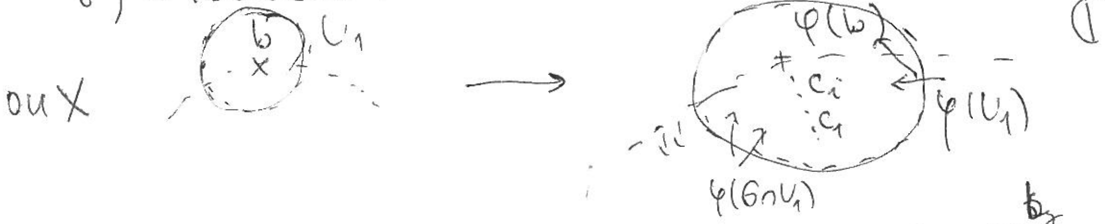
$\Rightarrow y$ has a neighb. e.g. U s.t. $f_{|U} = g_{|U}$! $\Rightarrow y \in G$.



((one can take $V = U'$, ~~but~~ or U', but it is not nec.))

(There is no need to make the neighborhoods smaller.)

b) G is closed. Let $b \in \partial G$. $\exists (U_1, \varphi)$, $b \in U_1$ map U_1 open



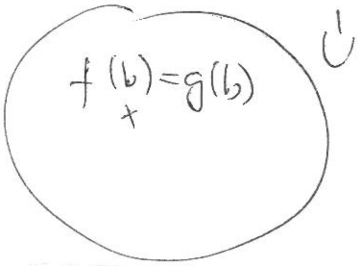
$c_i \to \varphi(b)$ non-constant, $c_i \in \varphi(G \cap U_1)$ $\exists$ by def. of boundary

$b_i := \varphi^{-1}(c_i) \quad \lim b_i = b$

f and g holom → esp. cont. $\Longrightarrow \lim f(b_i) = \lim g(b_i)$

$\qquad \qquad \qquad \qquad \qquad \quad \overset{\|}{f(b)} \qquad \quad \overset{\|}{g(b)}$

on Ψ: Take (U_1, Ψ^{*}) map around $f(b) = g(b)$



$$\bar{f} := \psi \circ f \circ \varphi^{-1} : \varphi(U_1 \cap G) \to \mathbb{C}$$
$$\bar{g} := \psi \circ g \circ \varphi^{-1} : \varphi(U_1 \cap G) \to \mathbb{C}$$

$\bar{f} = \bar{g}$ on $\varphi(U_1 \cap G)$. But $b \in U_1 \wedge b \in \partial G$

$\Rightarrow U_1 \cap G \neq \emptyset$

$+ U_1$ and G open $\Rightarrow \varphi(U_1 \cap G)$ open non-empty

def of boundary by non-empty inters.

It has limit point in $\varphi(U_1)$.

By Id. thm.' : $\bar{f} = \bar{g}$ on $\varphi(U_1)$. Thus,

$f = g$ on $U_1 \Rightarrow b \in U_1$ is an element of G. Thus G is closed (contains its boundary).

c) $G \neq \emptyset$: $\underline{a}$ be a limit point of A; (U_2, φ') a map around $\underline{a}$. $\varphi'(\mathring{A})$ *) has a limit point $\varphi'(\underline{a})$. Take a map Ψ_1 around $f(\underline{a}) = g(\underline{a})$ (f & g cont. equal on A; equal on its limit point) & the old kamarads' $\Psi_1 \circ f \circ \varphi'^{-1}$, $\Psi_1 \circ g \circ \varphi'^{-1}$ (in another fashion). They equal on $\varphi'(\mathring{A})$, so (identity thm') they equal on $\varphi'(U_2)$. Cons. $f = g$ on $U_2 \Rightarrow \underline{a} \in G$.

Summing-up, $G \neq \emptyset$, open, closed $\Rightarrow X = G$ since X is connected (if $X \setminus G \neq \emptyset \Rightarrow X = \underbrace{X \setminus G}_{\text{open}, \neq \emptyset} \cup G$ ↖ closed, ← open, $\neq \emptyset$ ↯) $\square$

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*) $\mathring{A} = A \cap U_2$ so that we can map them by φ'.

Recall application: $\sin(z+w) = \sin z \cos w + \cos z \sin w$

Definition: X Riemann surface, $Y \subseteq X$ open, $Y' \subseteq Y$. $f : Y' \to \mathbb{C}$ is meromorphic on Y if

1) $Y \setminus Y'$ isolated set in X
2) $f : Y' \to \mathbb{C}$ holomorphic
3) $\forall p \in Y' \setminus Y$ $\lim |f(x)| = +\infty$

Points in $Y' \setminus Y$ poles of f.
Set of merom. fctions $\mathcal{M}(Y)$.

Remark: • (U, ψ) coord. system, $\psi(p) = 0$, $p \in U$
f meromorphic on $U \Rightarrow \exists V \ni p$, $V \subseteq U$
$$f_{|V} \circ \psi^{-1}(z) = \sum_{k=-l}^{+\infty} c_k z^k, \quad l \in \mathbb{N}$$

Examples: 1) $p(z) = z^n + c_{n-1} z^{n-1} + \dots + c_n$, $n \geq 1$
$p \in \mathcal{O}(\mathbb{C})$
$p \in \mathcal{M}(\mathbb{P}^1\mathbb{C})$, $\mathbb{C} \subseteq \mathbb{P}^1\mathbb{C}$,
∞ is a pole of p

2) $e^{\frac{1}{z}}$ is not merom. on $\mathbb{C}$, although $e^{\frac{1}{z}}{}_{|\mathbb{C}\setminus\{0\}}$ is holomorphic! 3) trial.

Connection: mero & holo.

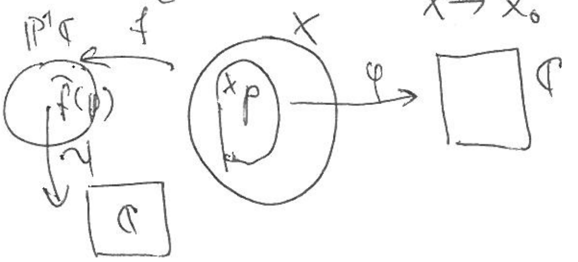
Thm.: X Riem. surf., $f \in \mathcal{M}(X)$, P set of poles of $f \Rightarrow$

1) $\tilde{f} = f$ on $X \setminus P$, $\tilde{f} = \infty$ on P is a holom. fction $\tilde{f} : X \to \mathbb{P}^1\mathbb{C}$

2) $f : X \to \mathbb{P}^1\mathbb{C}$ holom. $\Rightarrow f = +\infty$ on X or $f^{-1}(\infty)$ is isolated and $f : X \setminus f^{-1}(\infty) \to \mathbb{C}$ is meromorphic on X.

Proof: 1) $\tilde f$ is the continuous extension of f ($\lim_{x\to x_0}|f(x)|=+\infty$, $x_0\in P$)

$\tilde f: X\to\mathbb{P}^1\mathbb{C}$



- $P\cap D_\varphi=\emptyset$: $\psi\circ\tilde f\circ\varphi^{-1}:\mathbb{C}\hookrightarrow\mathbb{C}$ holomorphic

- $P\cap D_\varphi\neq\emptyset$: $\psi\circ\tilde f\circ\varphi^{-1}:\mathbb{C}\hookrightarrow\mathbb{C}$ continuous $\Rightarrow$ bounded on a compact subset of $\operatorname{Im}\varphi$; Moreover $\psi\circ\tilde f\circ\varphi^{-1}:\mathbb{C}\setminus\varphi(P)\hookrightarrow\mathbb{C}$ holomorphic. Thus R.R.S. $\Rightarrow$ it is holomorphic on its def. domain.

Riem. removing sg.

$\Rightarrow$ $\tilde f$ is holomorphic.

[Rem.: Why we cannot use RRS ~~at the~~ for $\tilde f: X\to\mathbb{P}^1\mathbb{C}$ directly?]

2) $f^{-1}(\infty)$ is d. Supp. y $f^{-1}(\infty)$ has limit point $\Rightarrow$ $f=\infty$ (R.I.T.). Thus $f: X\setminus f^{-1}(\infty)\to\mathbb{C}$

Riem. id. thm.

F. Example:

is holomorphic (restriction + $\mathbb{P}^1\mathbb{C}\setminus\{\infty\}=\mathbb{C}$) $\Rightarrow$

f is merom. on X $\Leftarrow$ $\lim_{x\to x_0}f(x)=\infty$, $x_0\in f^{-1}(\infty)$. ∎

Remark: $f\in\mathcal{M}(X)\rightsquigarrow\tilde f\in\mathcal{O}(X,\mathbb{P}^1\mathbb{C})$

$f\in\mathcal{O}(X,\mathbb{P}^1\mathbb{C})\setminus\{\text{const. }\infty\}\rightsquigarrow f\in\mathcal{M}(X)$

"Meromorphic seem as holomorphic"

~~(For you, this is the [illegible] of sg. X)~~

Theorem: X, Y Riemann surfaces and $f: X \to Y$ holomorphic map. Then for any $x_0 \in X$ $\exists (\varphi, U)$, $x_0 \in U$ and (ψ, V) around $f(x_0)$ s.t.

"rep of holom. ftions"

(1) $\varphi(x_0) = 0$, $\psi(f(x_0)) = 0$

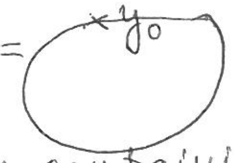
(2) $f(U) \subseteq V$

(3) $\exists k \geq 0$ $(\psi \circ f \circ \varphi^{-1})(z) = z^k$ $\forall z \in \varphi(U)$.

Proof: • (U_1, φ_1) around x_0: set $\underline{U_1'} = U_1$, $\varphi_1'(z) = \varphi_1(z) - \varphi(x_0)$

• (V_1, ψ_1) around $f(x_0)$: set $V_1' = V_1$, $\psi_1'(z) = \psi_1(z) - \psi_1(f(x_0))$

• preliminary: $f(U_1') \cap V_1' \supseteq \tilde{V} \leftarrow$ any open cont. y_0

✓ but what if $f(U_1') =$  $\Rightarrow$ we cannot take $\tilde{V}$

also told at the lecture

Aside step: $\tilde{V}$ any open containing y_0 and in $\overset{\text{def domain}}{D(\psi)}$. Take $\tilde{U} = f^{-1}(\tilde{V}) \cap U_1'$ this is open! Moreover $f(\tilde{U}) \subseteq \tilde{V} \subseteq D(\psi)$. This work! We do it later as promised since we must still take / make U_1' smaller (to fulfill (3)).

• Certainly, we may take $\tilde{f} := \psi_1' \circ f \circ \varphi_1'^{-1}$

$\tilde{f}(0) = 0$ and $\tilde{f}$ is holo (in a neighb. of $0 \in \mathbb{C}$)

- $f = \text{const} \Rightarrow$ thm. is satisfied: $k = 0$
- $f \neq \text{const.} \Rightarrow \tilde{f}$ is not const. (R.I.T.) (esp. not const. zero). Thus $\exists k$ ~~maximal~~ s.t. $\tilde{f}(z) = z^k g(z)$ on an open set $\underline{U_1''} \subseteq \mathbb{C}$ & $g(0) \neq 0$. We take maximal such $\underline{k}$.

From now on, f is non-constant.

• Take root $g(z) = h(z)^k$ (for chosen k) on $U_1''' \subseteq \mathbb{C}$.

• Define $\alpha(z) = z h(z)$.

$$\alpha'(0) = h(0) + z h'(z)\big|_{z=0} = h(0) = g(0)^{1/k} \neq$$

By implicit fction thm. (or directly by diff. of inverse of a holomorphic function) $\exists U^{(iv)}$ open on which, we may invert $\underline{\alpha}$. The inverse is known to be holomorphic (why?)

- Set $\varphi := \alpha \circ \varphi'_{1|U}$ where $U := \underbrace{\left(\underset{\mathbb{C}}{\varphi_1'(U_1') \atop \cap} \cap \underset{\mathbb{C}}{U^{(iv)} \atop \cap}\right)}$.

Compute:

$$(\psi \circ f \circ \varphi^{-1})(z) = (\underbrace{\psi \circ f \circ \varphi^{-1}_{1|U}}_{z^k g(z)} \circ \alpha^{-1})(z) = (z\,h)^k (\underbrace{\alpha^{-1}(z)}_{\text{inverse to } \underline{zh}}) =$$

$$= z^k \quad \left(\text{As } \sin^k(\arcsin x) = x^k \ \ldots\right)$$

Formally, we should check $f(U) \subseteq V_1$. Set $V := f(U) \cap V_1'$.

Since $f = \underset{\text{open}}{\underset{\uparrow}{\psi^{-1}}} \circ \underset{\text{open}}{\underset{\uparrow}{z^k}} \circ \underset{\text{open}}{\underset{\uparrow}{\varphi}}$ is open, we know that V is open.

$f(U) \subseteq V$ is „evident" $\square$

Remark: We cannot give up checking the openess of V, and say 'take any V open in $f(U)$ which contains y_0'. Since what if $y_0 \in \partial f(U)$?..

Remark: The openess ~~argument~~ "$\psi^{-1} \circ z^k \circ \varphi$" is open is more enlightned in the next proof.

If any case, we use f, g open $\Rightarrow f \circ g$ open. (topolog. fact, not so difficult)

''Note: The neighborhood 'puzzle' from lecture is solved in these notes.

Thm: Any $f: X \to Y$ non-constant holomorphic is open

either by the balls ~~and inter~~ as at the lecture **Or**

Proof: We want $U \subseteq X \Rightarrow f(U)$ is open, thus $\forall y_0 \in f(U)\ \exists$ $V \ni y_0$ $V \subseteq f(U)$.

$y_0 \in f(U) \Rightarrow x_0 \in U$ s.t. $f(x_0)=y_0$. By prev thm. $\exists (\varphi, \overline{U}_*)$ around x_0 and $(\psi, \overline{V}_*)$ s.t. $\psi \circ f \circ \varphi^{-1} = z^k$.

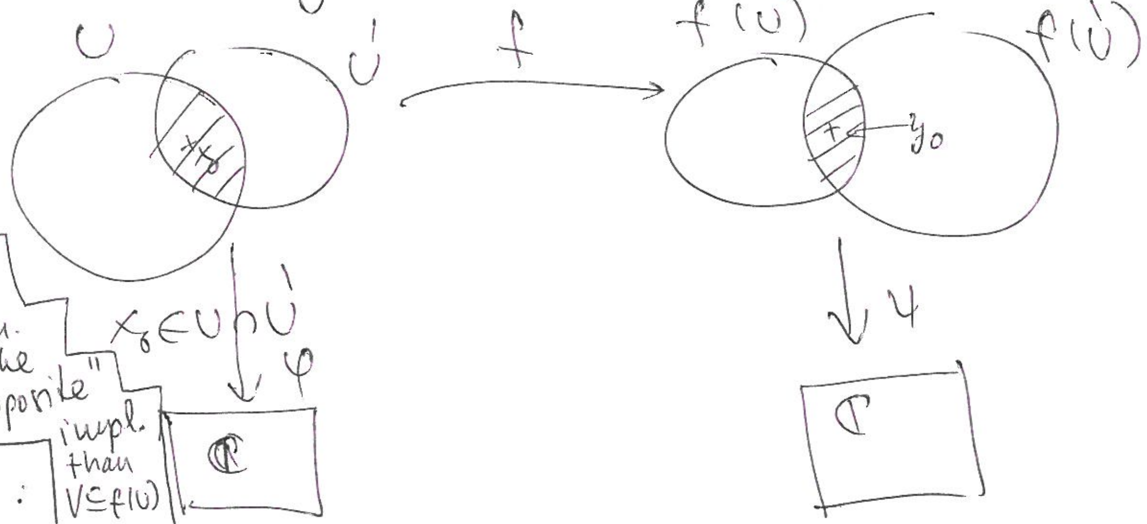
z^k is open (easy)

$$(\Delta)\ f_{|U \cap \bar{U}} = \psi^{-1} \circ z^k \circ \underbrace{\varphi_{|U \cap \bar{U}}}_{(\text{btw.} = \varphi_{|U})}$$

open as well ((since $\overbrace{\varphi \text{ cont}, \varphi^{-1} \text{ cont}}^{\varphi \text{ homeo}} \Rightarrow \varphi$ open
ψ^{-1} open $\Leftarrow \psi$ cont)). Take $V := f(\acute{U} \cap U)$. $(\Downarrow)$

It is open by (Δ); $y_0 \in f(\acute{U} \cap U)$ and $V \subseteq f(U)$. □

Remark:



Remark: $f(U) \subseteq V$ from (2) of prev. thm. is the "opposite" impl. than $V \subseteq f(U)$

Further Consequences:

Thm.: Any $f: X \to Y$ inj. holomorphic $\Rightarrow f$ biholomorphic (onto $f(X)$).

Pf.: Injective $\Rightarrow$ around any x_0, $k=1$ — but z^1 is biholo $\Rightarrow$ $f_{|U} = \varphi \circ z^1 \circ \psi^{-1}$ is biholo on U.
Since holom. is a local (moreover point-wise on open sets) property $\Rightarrow$ f is biholomorphic on "whole" X

□

Thm. (maximum principle): $f \in \mathcal{O}(X)$ non-constant $\Rightarrow$ $|f|$ does not achieve its maximum.

Proof: Y $\exists x_0$ $|f(x_0)| = \sup\{|f(x)|, x \in X\}$ (1)

$K := \{z \in \mathbb{C} \mid |z| \leq |f(x_0)|\}$ closed ball (disc)

$f(X) \subseteq K$ by (1)

$f(X)$ open by prec. thm. $\Rightarrow f(X) \subseteq K^\circ$.

Now $f(x_0) \in f(X)$ and
$f(x_0) \in \partial K$. Thus $f(x_0) \in K^\circ \wedge f(x_0) \in \partial K$ ↯

□

Thm.: f holomorphic non-constant $f : X \to Y$ & X cpt.
Then f is surjective (and Y is cpt).

Proof: $f(X)$ open since f open (a thm. above)
$f(X)$ cpt. since X cpt (cont. images of cpts are cpts)
$\Rightarrow f(X)$ is closed. (Since Y is connected) we have $f(X) = Y$. Esp. f onto and Y cpt. □

Remark: Other argumentation: one of
$f(X)$ op. by above thm.
$f(X)$ closed since f cont $\Rightarrow f(X) = Y$ since Y conn.
Thus. f is onto. Now, cont. im. of cpt are cpts.

Remark: Holom. non-constant ~~[illegible]~~ maps of cpt. must be into compacts! (esp. not into $\mathbb{C}$ and, additionally, ~~[illegible]~~ onto compacts)

[„Logical" consequence : ~~[illegible]~~ Holom. ~~maps~~ functions of compacts into $\mathbb{C}$ must be constant]

[Also possible: Holom map on compact set ~~have max~~

Analytical point: f holom ⇒ f cont ⇒

|f| cont ⇒ |f| achieves maximum ⇒ f is constant

~~Suppo~~ Let us formulate it and show it as cons.

Thm.: $f \in \mathcal{O}(X)$, X cpt. ⇒ $f = \text{const.}$

Proof: $^{\wedge}$ $f(X) = \mathbb{C}$ is compact ↯

Suppose f non-const.

Liouville-thm: ~~$f : \mathbb{C} \to \mathbb{C}$ bounde~~

$f \in \mathcal{O}(\mathbb{C})$ bounded $\Rightarrow f = \text{const.}$

Proof: $\bar{f} : \mathbb{P}^1 \to \mathbb{P}^1 \quad \bar{f}(\infty) = \lim_{z \to \infty} f(z)$

But f is bounded $\Rightarrow \bar{f}(\infty) \in \mathbb{C}$ Rieman Removabily

Thus $\bar{f} : \mathbb{P}^1 \to \mathbb{C}$ holo, $\mathbb{P}^1$ cpt. $\Rightarrow \bar{f} = \text{const}$

otherwise violates ($\mathbb{C}$ non-comp) ☐

Fundam. thm. alg: $\forall n \geq 1 \ \forall c_i \in \mathbb{C} \ f(z) = \sum_{i=0}^{n} c_i z^i$; $c_n \neq 0$

$\Rightarrow \exists x_0 \in \mathbb{C} \ f(x_0) = 0$

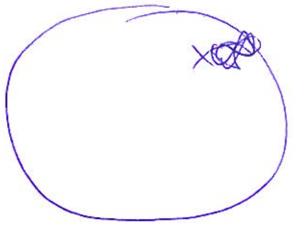
Proof: $\bar{f}(z) = f(z)$, $z \in \mathbb{C}$

$\bar{f}(z) = \lim_{z \to +\infty} f(z) = +\infty \quad (c_n \neq 0 \wedge n \geq 1)$

wy?

$\bar{f} : \mathbb{P}^1\mathbb{C} \to \mathbb{P}^1\mathbb{C}$ cont. holomorphic

$\Rightarrow \bar{f}$ is surjective (0 has preimage) ☐



$f : D \to \mathbb{C}$ bounded

Thm.: $f \in \mathcal{O}(X)$, X cpt. $\Rightarrow f = \text{const.}$

Proof: $\mathbb{C}$ non-compact $\square$

Remark: ~~By Hodge thy $\mathcal{O}(X) = \operatorname{Ker}\bar\partial \oplus \operatorname{Im}\bar\partial^* \oplus \operatorname{Harm}^0(X)$~~

f holo $\Rightarrow \Delta f = 0$

Dolbeault ↓ čech/Lera

Hodgethy $\operatorname{Harm}(X) \simeq \operatorname{Ker}\bar\partial / \operatorname{Im}\bar\partial_{-1} \simeq \check{H}^0(X) \simeq H^0_{sing}(X) \simeq \mathbb{C}$

$\{f \text{ holo}\} \leftrightarrow \mathbb{C}$ linearly.

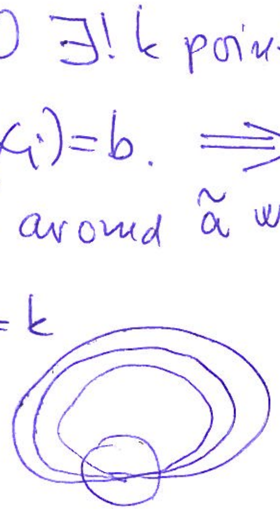
Definition: $f: X \to Y$, $x_0 \in X$. We say that f attains $f(x_0)$ with mult. k around x_0 iff $F(z) = z^k$ as in thm. on local behaviour.

Remark: $f(z) = z^2(z-1)$ f attains 0 around zero with multiplicity 2 (Find it acc. to proof) f attains 0 around 1 with multipl. 1.

Remark: ! $\forall b \in \operatorname{Im}(F)$ $\exists U_0 \ni 0$ $\exists!$ k points in U_0 $\{x_1, \dots, x_k\}$ $F(x_i) = b$. $\Longrightarrow$

f attains $f(\hat{a}) = \tilde{b}$ around $\tilde{a}$ with mult. k

$\exists \tilde{U}_0 \subseteq X: \#\{x \in \tilde{U}_0 \mid f(x) = \tilde{b}\} = k$



$z \mapsto z^3$

DRAWINGS

~~**Thm**: $f: X \to Y$ holom non-const on X cpt. $\Rightarrow$ Y is cpt and f is surj.~~

~~**Proof**: $f(X)$ is open; $f(X)$ is cpt $\Rightarrow$ it is closed $\Big\}$ $\Rightarrow$ $f(X)$ is surjective~~

~~$\Rightarrow f(X) = Y \Rightarrow$ f is surj. + Y cpt (cont. image of cpts are cpts). $\square$~~

~~**Thm.**: $f \in \mathcal{O}(X)$, X cpt. $\Rightarrow f = \text{const}$.~~

~~**Proof**: f surj. $\Rightarrow f(X) = \mathbb{C}$ $\leftarrow$ non-compact.~~

Definition: $f: X \to Y$, ~~We and $y \in f(X)$~~ $x_0 \in X$, We say that f attains $f(x_0)$ around x_0 with multiplicity k iff

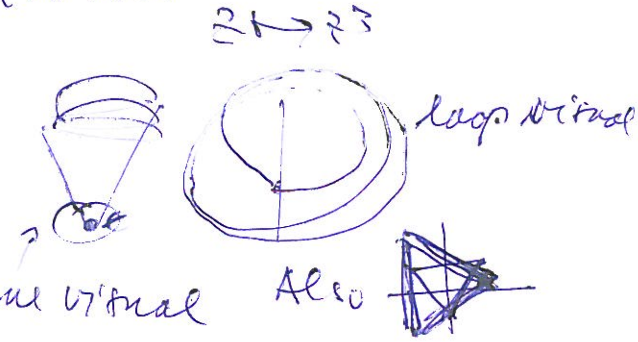
$$F(z) = z^k.$$

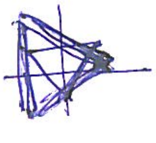
Remark: since $\forall\, b \in \mathrm{Im}(F)\ \exists\, U_0 \ni 0\ \exists!\, k$ points in U such $\forall\, \{x_1, \dots, x_k\} \in U$ $F(x_i) = b$.

Then by the thm. on local representation of holomorphic functions $\boxplus$

$f(a) = b$ attains with mult $k \Rightarrow \exists\, U \ni a$

$$\#\{x \in U \mid f(x) = b\} = k.$$



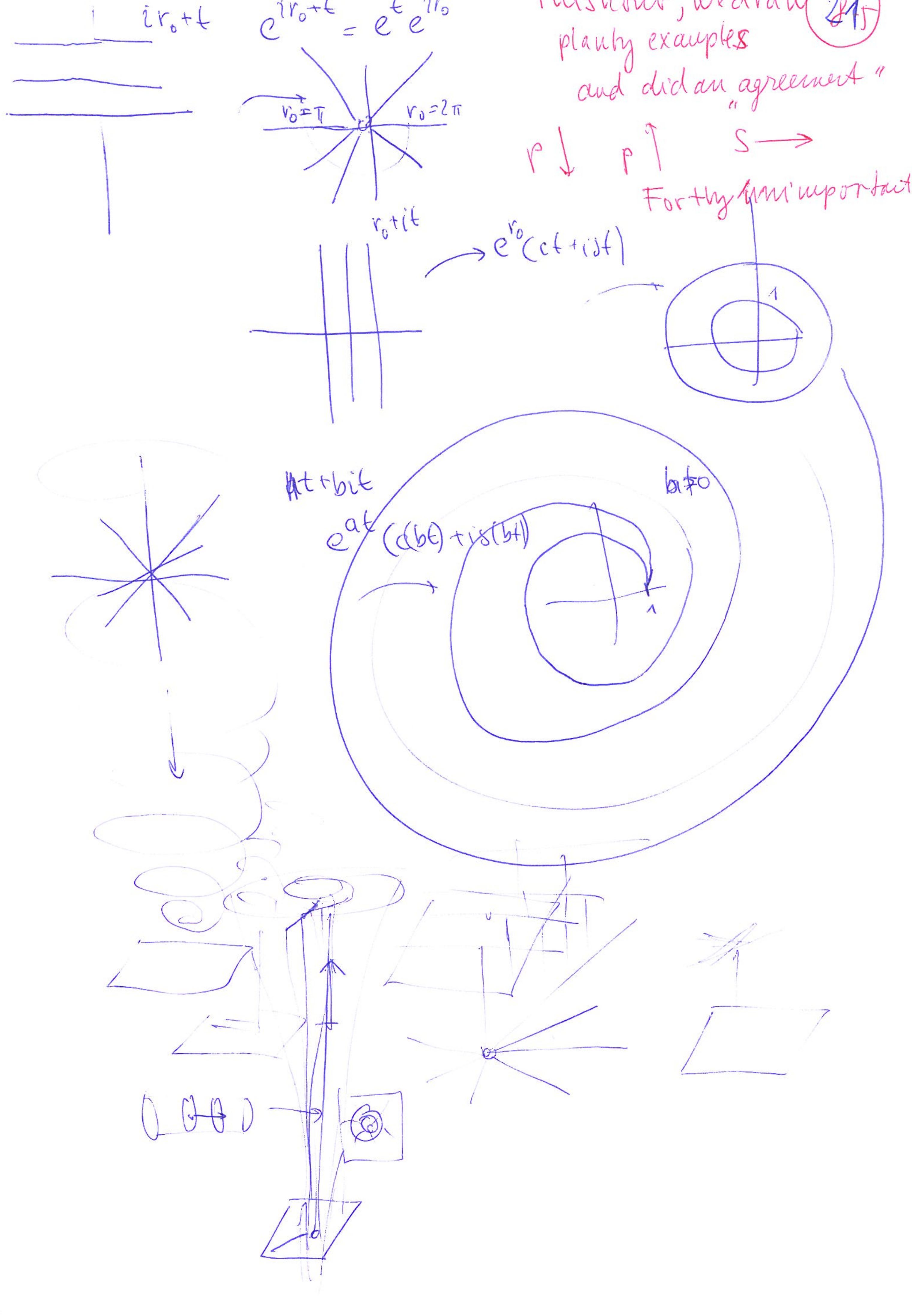
Also 

21

This hour, we draw planty examples and did an „agreement"

$\rho \downarrow$ $\rho \uparrow$ $S \longrightarrow$

For thy unimportant



PRAWINGS

$z \mapsto z^3$

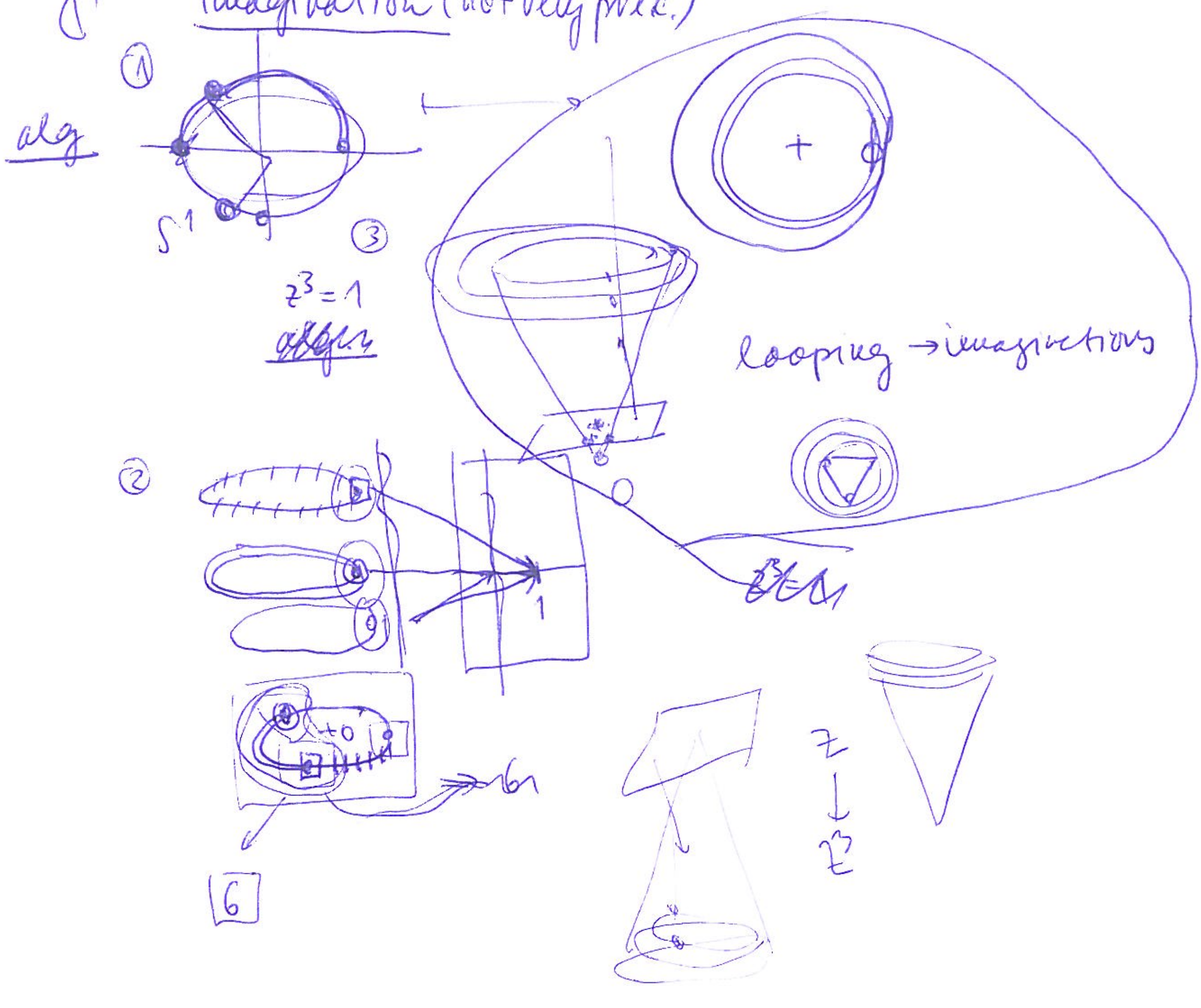
$(\cos t, \sin t) \longmapsto c(3t) + i\sin(3t) = (\cos(3t), \sin(3t))$

I

~~Moivre!~~ Moivre

$(\cos t + i\sin t)^3 = \cos^3 + 3\cos^2 + \dots = \cos(3t) + i\sin(3t)$

Imagination (not very prec.)



We also computed this example (similar at exam, also may ?

For $f(z) = \overset{k}{z^2} - 1$ calculate multipli-city k of attaining 0 at 1. Calculate maps s.t. $F = z^k$. [+ some homographies])

ⓐ23

First (we follow the proof of representability by polynomials) : $\underset{\psi_1 = Id,}{\wedge}$ $\varphi_1'(z) = z - \underset{x_0}{\overset{\shortparallel}{1}}$ φ_1' does not mean derivation

$$\psi_1'(z) = \psi_1(z) - \psi_1(0) = z - 0 = z, \text{ where } \psi_1 = Id.$$

"no manifolds + actually"

$\varphi_1'^{-1}$: $w = z - 1$
$z = w - 1 \Rightarrow w = z + 1$

$$\tilde{f} = \psi_1' \circ f \circ \varphi_1'^{-1}$$

$$\tilde{f}(w) = \underset{f(z+1)}{~~f(w)~~} = (z+1)^2 - 1 = z^2 + 2z = z(z+2) \Longrightarrow$$

$\boxed{k = 1}$ immediately

We can continue calculating the maps as in the proof

$$\tilde{f}(z) = z\,g(z) \Rightarrow g(z) = z + 2 \Rightarrow h(z) = z + 2$$

$\alpha(z) = z(z+2)$ [BTW $\alpha'(0) = 2 \neq 0$]

not derivation

$$\varphi(z) = (\alpha \circ \varphi_1')(z) = (z-1)(z-1+2) = (z-1)(z+1) = z^2 - 1 \text{ locally at } ①!$$

wy C

$f \circ \varphi^{-1} = id$ $\psi \circ f \circ \varphi^{-1} = \psi \circ (z^2 - 1) \circ (z \mapsto z^2 - 1)^{-1} =$
$\psi = \psi_1' = \psi_1 = Id$ $= \underline{z}.$

Compute domain of this map around 1

HW: Take $z^2 - 2z$ at 0 and calc k and maps s.t.

$$F = \psi \circ f \circ \varphi^{-1}$$

$$F(z) = z^k$$

Eg.

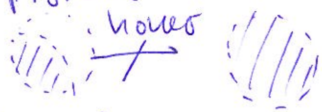
(*) $w = z^2 - 1 \Rightarrow z = \sqrt{w+1}$, $w = \sqrt{z+1}$
$z^2 - 1 \mapsto (\sqrt{z+1})^2 - 1 = z$)

Definition : Let $p: Y \to X$ be a continuous map between topological spaces Y and X. ~~p~~ p is called a covering if $\forall x \in X\ \exists U\ \exists n$ s.t. $p^{-1}(U) = \bigcup_{i=1}^{n} V_i$ for open set V_i, $i = 1, \dots, n$ and $p|_{V_i} : V_i \xrightarrow{\sim} U$.

Example : 1) $p_2 : z \mapsto z^2$ $\mathbb{C}^\times \to \mathbb{C}^\times$ is a covering prove using ~~taking~~ the neighborhoods

2) $D \hookrightarrow \mathbb{C}$ is not a covering ($|z| = 1$)
↑
problem

id :



Definition : $p : Y \to X$ is called discrete map if $\forall x \in X$ $p^{-1}(\{x\})$ is discrete.

Thm. : $p : Y \to X$ non-constant holo-map $\Rightarrow$ p is open and discrete.

Proof : open was ✓
discrete ? supp now ↯ discrete $\Rightarrow p^{-1}(\{x'\})$ of a point x' ~~has~~ is not discrete set $\Rightarrow$ has ~~accum~~ limit point $\Rightarrow$ p is constantly equal to x' (by Riem. identity thm.) (~~acc~~. □

<u>Definition</u>: $p: Y \to X$ non-const holo. $y \in Y$ branch point iff no neighb $V \ni y$ exists s.t. $p|_V$ is injective.
No branch points = unbranchanded.

(Remark: Branch poin is also called ramification point)

Thm.: No branch point $\Rightarrow$ local homeo.

Proof: $p: Y \to X$, $y \in Y$ not branched point $\Rightarrow$

$\exists V \ni y$ such that $p|_V$ injective

$$p|_V \ \underset{\text{holo}}{\overset{}{\underset{\Uparrow}{\text{cont.}}}}, \underset{\text{holo}}{\underset{\Uparrow}{\text{open}}} \Rightarrow p|_V \text{ homeo. } \text{~~onto~~}$$

$\Leftarrow$ triv

Remark: Converse is true as well. („the same")

Example: $p_k: \mathbb{C} \to \mathbb{C}$ $p_k(z) = z^k$, $k > 1 \Rightarrow$

p_k branches at 0

Think of: y_0 is branched point of p iff x_0 takes y_0 with mult ≥ 2 around x_0.

:-)

Definition: Let $p : Y \to X$ be continuous map between topol. spaces. We say that p has the curve lifting property if $\forall u : [0,1] \to X$ (curve) and each $y_0 \in Y$ s.t. $p(y_0) = u(0)$ $\exists \hat{u} : [0,1] \to X$ $\hat{u}(0) = y_0 \wedge p \circ \hat{u} = u$.

$$\begin{tikzcd} & Y \arrow[d, "p"] & \ni y_0 \arrow[d, mapsto] \\ {[0,1]} \arrow[ur, dashed, "\hat{u}"] \arrow[r, "u"'] & X & x_0 = p(y_0) = u(0) \end{tikzcd}$$

- Any $\hat{u}$ s.t $p \circ \hat{u} = u$ lifting.
- From now on, each top sp. is Hausdorff.

Thm: Let $p : Y \to X$ be a covering. Then ~~p~~ has the curve lifting property. Moreover, the lifting is unique in the foll. sense: If $\hat{u}_1(t_0) = \hat{u}_2(t_0)$ for two liftings of $u \Rightarrow \hat{u}_1 = \hat{u}_2$.

Proof (existence): Let $p(y_0) = \hat{u}(0)$.

cov: $\forall t \in [0,1]$ $\exists U_t \ni u(t)$ s.t. $p_{|p^{-1}(U_t)} : p^{-1}(U_t) \xrightarrow{\sim} \bigcup_{i=1}^{n_t} U_{t,i}$.

and $p|U_{t,i} \xrightarrow{\sim} U_t$

$u([0,1]) \subseteq \{U_t \mid t \in [0,1]\}$ covering

$u([0,1])$ cpt. $\Rightarrow \exists$ $\{U_{t_1}, \dots, U_{t_k}\}$ finite subcover (covering $u([0,1])$). Suppose $0 = t_0 < t_1 < \dots < t_{k-1} < t_k = 1$.

(instead of $p|Z : Z \to Y$, we write $p|Z \to Y$)

at least

Taking $\tau_k = \inf \{\tilde{\tau}_k \in [0,1] \mid u(\tilde{\tau}_k) \cap U_{t_k} \neq \emptyset\}$

Defines a sequence $0 = \tau_0 < \tau_1 < \dots < \tau_k = 1$

1) $u([\tau_k, \tau_{k+1}]) \subseteq U_{t_{k+1}}$ from constr.

2) $p^{-1}(U_{t_k}) = \bigcup_{i=1}^{n_{t_k}} U_{t_k,i} \wedge p|U_{t_k,i} \xrightarrow{\sim} U_{t_k}$

Construct $\hat{u}|[0,t_k] \to X$ $\hat{u}(0) = y_0$. $k=0$ triv.

Supp. $\hat{u}|[0,t_{k-1}]$ constructed; $y_{k-1} = \hat{u}(\tau_{k-1})$. We know

$$p(y_{k-1}) = u(\tau_{k-1}) \in U_{t_k} \Rightarrow \exists i = 1, \dots, n_{\tau_k} : y_k \in V_{t_k,i}$$

$$\varphi = \left(p\; U_{t_k} \longrightarrow V_{t_k,i}\right)^{-1} \qquad \begin{tikzcd} V_{t_k,i} \arrow[d, shift left, "p"] \\ U_{t_k} \arrow[u, shift left, "\varphi"] \end{tikzcd} \qquad \hat{u}|[t_{k-1}, t_k] = \varphi \circ u([t_{k-1}, t_k]).$$

cont lift of u on $[t_0, t_k]$, and so on. $\square$ (~~and so on~~

(uniqueness: Forster 4.8 Thm p. approx 22 just if you want.)

Remark: We did only the idea ~~with coverings~~ and did not chose every thing as we wanted. Especially the induction was not done. I will test the proof only if you want it.

Example:

$$\begin{tikzcd} \mathbb{C}^\times \arrow[d, "p"', "z^2 \mapsfrom z"] & \\ \mathbb{C}^\times & {[0,1]} \arrow[l, "u"] \arrow[ul, dashed, "\hat{u}"'] \end{tikzcd}$$

$p(z) = p_2(z) = z^2$

$u(t) = e^{2\pi i t}$

$\hat{u}(t) = e^{\pi i t}$

loop lifts to non-loop

Fibers have the same number of points (sometimes!

Thm: Let $p : Y \to X$ be a covering of a path-connected X. Then $\forall x_0, x_1 \in X$ $\# p^{-1}(x_0) = \# p^{-1}(x_1)$ (cardinality is preserved).

Proof: $u : [0,1] \to X$, $u(0) = x_0$, $u(1) = x_1$

$p^{-1}(x_0) \ni y$ choose any

Prec. thm $\exists \hat{u}_y : [0,1] \to Y$ $\hat{u}_y(0) = y$

$\varphi(y) := \hat{u}_y(1)$, $\varphi : p^{-1}(x_0) \to p^{-1}(x_1)$

injectivity $\hat{u}_{y_1}(1) = \hat{u}_{y_2}(1) \underset{\substack{\uparrow \\ \text{uniqueness of liftings}}}{\Rightarrow} \hat{u}_1 = \hat{u}_2$

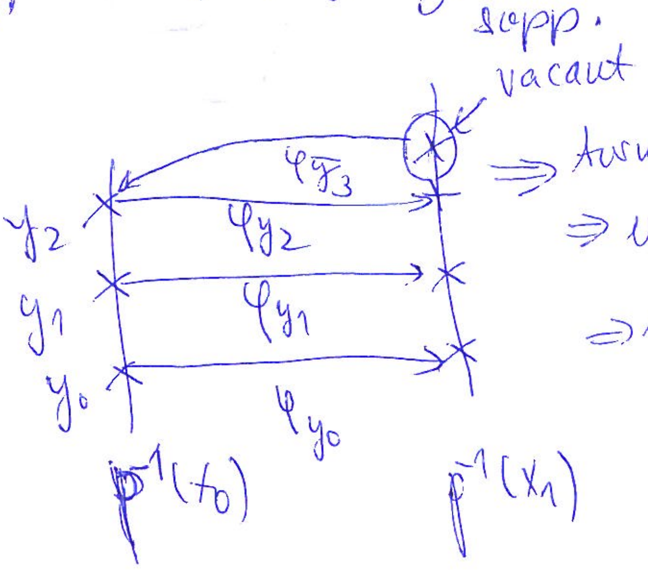
$\Rightarrow y_1 = y_2$

surj.: $\bar{u} : [0,1] \to X$ opposite to u $(\bar{u}(t) = u(1-t))$

$\bar{y} \in p^{-1}(x_1)$; $\hat{\bar{u}}_{\bar{y}}$ lift of $\bar{u}_{\bar{y}}$ $\bar{\varphi}(\bar{y}) = \hat{\bar{u}}_{\bar{y}}(1)$, $\bar{\varphi}\ p^{-1}(x_1) \to p^{-1}(x_0)$

If $\hat{\bar{u}}_{\bar{y}}(1) = y \Rightarrow \underset{\text{Im}}{\wedge}\hat{\bar{u}}_{\bar{y}} = \text{Im}\, \hat{u}_y$ esp.

$\bar{\varphi}(\varphi(y)) = y$, and $\varphi(\bar{\varphi}(y)) = y$ similarly. $\square$



$\Rightarrow$ turn in $\bar{u}_3$ to the opposite

$\Rightarrow u_2(0) = \bar{\bar{u}}_3(0)$

$\Rightarrow u_2 = \bar{\bar{u}}_3$ ↯

Definition: Locally cpt = each point has a cpt. neighborhood.

$p: Y \to X$ proper = X, Y loc. compact $\wedge$ preimage of compacts are compact sets

Note: Locally cpt. for Hausdorff spaces only.

Proper maps —— || —— well.

Lemma: 1) $p: Y \to X$, Y cpt $\Rightarrow$ p proper

2) $p: Y \to X$ proper $\Rightarrow$ p closed mapping

3) p proper discrete $\Rightarrow$ $p^{-1}(\{x\})$ finite

4) p proper discrete: $\forall x \in X$ $\forall V$ neighb of $p^{-1}(x)$ $\exists U$ neighb of x s.t. $p^{-1}(U) \subseteq V$.

Proof: 1) U cpt in X $\overset{\text{(Hausdorff)}}{\Rightarrow}$ U closed $\overset{p \text{ cont.}}{\Rightarrow}$ $p^{-1}(U)$ closed

closed in cpt is cpt $\Rightarrow$ $p^{-1}(U)$ cpt.

2) V closed in Y $\quad X \setminus p(V)$ open?

$x \in X \setminus p(V)$, $\bar{U}$ cpt. neighb of x ; p proper

$p^{-1}(\bar{U})$ cpt. $\Rightarrow$ $E := V \cap p^{-1}(\bar{U})$ cpt.

$p(E)$ cpt $\left(\begin{matrix}\text{cont. im. of} \\ \text{cpt are compacts}\end{matrix}\right)$ $\Rightarrow$ $X \setminus p(E)$ open

↯ $x \in p(E) \Rightarrow x = p(r)$, $r \in V$ ↯ $x \in X \setminus p(V)$.

Thus $X \setminus p(E)$ is an open neighb. of x.

3) $p^{-1}(\{x\})$ cpt.

If $p^{-1}(\{x\})$ infinite $\Rightarrow \exists (a_n)_{n \in N}$ monotone.

$p^{-1}(\{x\})$ cpt $\Rightarrow$ $(a_{n_i})_{i \in N}$ converges to b ↯ discreteness (in b)

Sheaves and their cohomology

Construction: • Let X be a topological space and $top(X)$ the set of open sets on X.

• Let us define the category $\mathcal{T}_X$.

$Ob(\mathcal{T}_X) = top(X)$ objects

$Mor(\mathcal{T}_X) = \{ i^U_V \mid \forall U, V \in top(X), U \subseteq V \}$, where $i^U_V(u) = u$ $\forall u \in U$.

• Let $\mathcal{G}$ be the category of groups/rings/modules/algebras. For definiteness the small category (to work in set thy).

Definition: Any contravariant functor $\mathcal{F}$ from $\mathcal{T}_X$ to $\mathcal{G}$ is called a presheaf (of groups/rings/mods/...) on X.

Remark: Functor + contravariance

$\mathcal{F}(U)$ is a group/ring/... $\forall U \in Ob(\mathcal{T}_X) = top(X)$

$\mathcal{F}(i^U_V)$ is a group/ring/... homomorphism of $\mathcal{F}(V) \longrightarrow \mathcal{F}(U)$ ("changes" $i^U_V : U \to V$, $\mathcal{F}(i^U_V) : \mathcal{F}(V) \to \mathcal{F}(U)$!)

$\mathcal{F}(i^V_W \circ i^U_V) = \mathcal{F}(i^U_V) \circ \mathcal{F}(i^V_W)$ contravariance on morphism + $\mathcal{F}(i^U_U) = id_{\mathcal{F}(U)}$ (behaviour on identity morphism on U)

since in "our case" $i^V_W \circ i^U_V = i^U_W$, we have

$$\mathcal{F}(i^U_W) = \mathcal{F}(i^U_V) \circ \mathcal{F}(i^V_W).$$

Notation: $\rho^V_U = \mathcal{F}(i^U_V)$ so called restriction morphism (for $\mathcal{F}$, U, V)

Example: 1) X a top. space $\mathcal{F}(U)=\mathcal{C}(U)=\{f: U\to\mathbb{R} \mid$ continuous fctions on f $\}$; values $\mathbb{C}$ are also "allowed" f is a

$\mathcal{F}(i^U_V): \mathcal{F}(U)\to\mathcal{F}(V)$, $\mathcal{F}(i^U_V)(g)=g_{|U}$, $U\subseteq V$, $g\in\mathcal{F}(V)$. Thus ρ^V_U is the restr. to U & of a function.

Test: $\mathcal{F}(i^U_W)(g)= g_{|U}$

$[\mathcal{F}(i^U_V)\circ\mathcal{F}(i^V_W)](g)=\mathcal{F}(i^U_V)(\mathcal{F}(i^V_W)g)=$

$=\mathcal{F}(i^U_V)(g_{|V})=(g_{|V})_{|U}=g_{|U}$ since $U\subseteq V$.

Thus RHS=LHS and $\mathcal{F}(\mathcal{C})$ is a presheaf.

2) X Riemann surface $\mathcal{F}(U)=\mathcal{O}(U)(=\{f: U\to\mathbb{C} \mid$ f holom. on U$\}$, $\mathcal{F}(i^U_V)g=g_{|U}$ $\forall g\in\mathcal{O}(U)$ (restr. of holom. is holom.)

! **Definition**: Let $\mathcal{F}$ be a presheaf of groups/rings/... on a topological space X. $\mathcal{F}$ is called a sheaf of groups/rings/... on X if $\forall U\in top(X)$ $\forall I$ set, $\forall (U_i)_{i\in I}\in top(X)$ s.t. $\bigcup_{i\in I} U_i=U$,

identity axiom (I) $\forall f,g\in\mathcal{F}(U)$ $\forall i\in I$ $\rho^U_{U_i}(f)=\rho^U_{U_i}(g)\Rightarrow f=g$

glueing axiom (II) $\forall i\in I$ $\forall f_i\in\mathcal{F}(U_i)\Big[\big((\forall j\in I)(\rho^{U_i}_{U_i\cap U_j}(f_i)=\rho^{U_j}_{U_i\cap U_j}(f_j)\big)\Rightarrow$

$\Rightarrow \exists f\in\mathcal{F}(U)$ s.t. $\rho^U_{U_i}(f)=f_i$ $\forall i\in I\Big]$

Example: 1) X Hausdorff $\Rightarrow \mathcal{F} = \mathcal{C}$ is a sheaf; thing X a $\mathcal{C}^\infty$-manif

2) **Non-example:** X a topol. space, G a group ③

$\# G \geq 2$. Define $\underline{G}(U) := G \ \forall U \in top(X) \ U \neq$

$\underline{G}(\emptyset) = \{0\} \leftarrow$ the trivial group.

$\rho^V_U = id_G \ \forall U \neq \emptyset, V \supseteq U$ (esp. $V \neq \emptyset$)

$\rho^V_\emptyset = \mathbb{0}$ (the zero morphism $\{0\} \to \{0\}$ $\mathbb{0}(0) = 0$)

Suppose $X \supseteq U_1 \cup U_2 \wedge U_1 \cap U_2 = \emptyset, U_1, U_2 \in top($

, e.g. $X = \{a, b\}$ discr. top., and $U_i \neq \emptyset \ i = 1,2$.

Let $g_1, g_2 \in G \ g_1 \neq g_2$ and set $U := U_1 \cup U_2$

$U_1 \neq \emptyset, U_2 \neq \emptyset$

$\mathcal{F}(U_1) = G, \ \mathcal{F}(U_2) = G$

$\rho^{U_1}_{U_1 \cap U_2}(g_1) = \mathbb{0}$ (necessarily since $U_1 \cap U_2 = \emptyset$)

$\rho^{U_2}_{U_1 \cap U_2}(g_2) = \mathbb{0}$

So assumptions of II are satisfied. But

$\underset{\in \underline{G}(U)}{\exists g}$ s.t. $\rho^U_{U_1}(g) = g_1$ and $\rho^U_{U_2}(g) = g_2$?

$\rho^U_{U_1}(g) = g$; $\rho^U_{U_2}(g) = g$. Thus $g_1 = g = g_2$ ↯

$(g_1 \neq g_2)$.

Definition: Let G be a group. We set $\underline{G}^{(U)} = \{ f \mid f$ is a locally constant function on U with values in $G \}$, $U \in top(X)$.

$\underline{G}(i^U_V)(f) = f_{|U} \ \forall f \in \underline{G}(V)$. Then $\underline{G}$ is called ~~[illegible]~~ the constant sheaf

Cohomology of sheaves – construction

$\mathcal{F}$ be a sheaf on t.s. X and $\mathcal{U} = (U_i)_{i \in I}$ be a covering of X
(by cov. we mean open covering)

$$C^q(\mathcal{F}, \mathcal{U}) = \prod_{(i_0, \dots, i_q) \in \underline{I^{q+1}}} \mathcal{F}(U_{i_0} \cap \dots \cap U_{i_q})$$

q-cocycles
product of groups
two Cartesian products: $\prod$ and I^{q+1} $\overset{\shortparallel}{}$ $(\underbrace{I \times \dots \times I}_{(q+1)\text{-ti}})$

$\underline{\delta^0 :} \; C^0(\mathcal{F}, \mathcal{U}) \to C^1(\mathcal{F}, \mathcal{U})$, $C^0(\mathcal{F}, \mathcal{U}) = \prod_{i_0 \in I} \mathcal{F}(U_{i_0})$

$$(a_i)_{i \in I} = \{ a : I \to \bigcup_{i_0 \in I} \mathcal{F}(U_{i_0}) \mid a(i_0) \in \mathcal{F}(U_{i_0}) \; \forall i_0 \in I \}$$

$$\delta(a_i)_i = (b_{ij})_{i,j \in I} \in \prod_{(i_0, i_1) \in I^2} \mathcal{F}(U_{i_0} \cap U_{i_1})$$

$$(b_{ij})_{(i,j) \in I^2} = \{ b : I \times I \to \bigcup_{(i_0, i_1) \in I^2} \mathcal{F}(U_{i_0} \cap U_{i_1}) \mid b(i_0, i_1) \in \mathcal{F}(U_{i_0} \cap U_{i_1}) \; \forall (i_0, i_1) \in I^2 \}$$

$b_{ij} := a_i^{-1} a_j$

We mean $\underset{\underset{\mathcal{F}(U_i \cap U_j)}{\cap}}{\rho^{U_i}_{U_i \cap U_j}(a_i^{-1})} \; \underset{\underset{\mathcal{F}(U_i \cap U_j)}{\cap}}{\rho^{U_j}_{U_i \cap U_j}(a_j)}$

or $a_i^{-1}|_{U_i \cap U_j} \cdot a_j|_{U_i \cap U_j}$ sometimes one writes.

$\underline{\delta^1 :} \; C^1(\mathcal{F}, \mathcal{U}) \to C^2(\mathcal{F}, \mathcal{U})$

$$\delta^1(b_{ij})_{ij} = (c_{ijk})_{ijk \in I}$$

$$c_{ijk} = b_{jk} b_{ik}^{-1} b_{ij}$$

a) δ^0, δ^1 maps $\underline{\text{into}}$ $C^1(\mathcal{F}, \mathcal{U})$, $C^2(\mathcal{F}, \mathcal{U})$ (def. of cart. prod.

b) δ^i are group homom.

C^i forms a group How?
Prove that δ^i's are group homoms.

Item b): δ^i are group homomorphisms if $\mathcal{F}$ is a sheaf of abelian groups/rings (with resp. to +)/ algebras (also w.r.t. +).

The group str. on $C^{q+1}(\mathcal{F},\mathcal{U})$ is given by

$$(a_{i_0\dots i_q})_{i_0,\dots,i_q\in I} + (b_{i_0\dots i_q})_{i_0,\dots,i_q\in I} :=$$

$$(a_{i_0\dots i_q} + b_{i_0\dots i_q})_{i_0,\dots,i_q\in I}\,.$$

$\forall i=0,1:$

c) $B^i(\mathcal{F},\mathcal{U}) = \operatorname{Im}\delta^{i-1} \subseteq C^i(\mathcal{F},\mathcal{U})$ co-boundaries

$Z^i(\mathcal{F},\mathcal{U}) = \operatorname{Ker}\delta^i \subseteq C^i(\mathcal{F},\mathcal{U})$ co-cycles (Ko-Zyklen), where $\delta^{-1}=0$

$$\check{H}^i(\mathcal{F},\mathcal{U}) := \frac{Z^i(\mathcal{F},\mathcal{U})}{B^i(\mathcal{F},\mathcal{U})}$$

i-th Čech cohomology

for $\mathcal{F}$ and $\mathcal{U}$.

d) Def. make sense: $\delta^1\delta^0=0$ ($\Rightarrow \operatorname{Im}\delta^0 \subseteq \operatorname{Ker}\delta^1$); $\delta^0\delta^{-1}=0$ (cause $\delta^{-1}=0$:-)), thus one can divide Z^i by B^i.

$$\delta^1(\delta^0(a_i)_i) = \delta^1(b_{ij})_{i,j} \qquad b_{ij}=a_j-a_i$$

$$\delta^1(b_{ij})_{i,j} =: c_{ijk}\,,\; c_{ijk} = b_{jk} - b_{ik} + b_{ij} =$$

$$= b_k - b_j - b_k + b_i + b_j - b_i = e$$

It works! also in the non-commutative case (»surprisingly«).

e) • $(b_{ij})_{ij} = \delta^{1}(a_i)_i \Rightarrow b_{ij} = a_j - a_i$ (we say that $(b_{ij})_{ij}$ splits; 1-co-boundary splits)

• $\delta^{0}(a_i)_{i\in I} = 0 \Rightarrow a_j - a_i = 0$ on $U_i \cap U_j$

$$\xRightarrow[\exists a\in \mathcal{F}(X)]{\text{Id. axiom}} a_i = a_{|U_i} \Rightarrow \boxed{Z^0(\mathcal{F},\mathcal{U}) = \mathcal{F}(X)}$$

• $\delta^{1}(b_{ij})_{i,j} = 0 \Rightarrow (c_{ijk})_{i,j,k} = [\delta^{1}(b_{ij})_{i,j}]_{i,j,k}$ satisfies

$c_{ijk} = b_{jk} - b_{ik} + b_{ij} \Rightarrow$

$\Rightarrow 0 = b_{jk} - b_{ik} + b_{ij}$ (we shall see that this implies $b_{ii} = 0$ & $b_{ij} = -b_{ji}$ (antisymmetry of 1-co-cycles) next time

---

f) Note that $\check{H}^0(\mathcal{F},\mathcal{U}) = \dfrac{Z^0(\mathcal{F},\mathcal{U})}{B^0(\mathcal{F},U)} =$

$$\cong \frac{\mathcal{F}(X)}{\operatorname{Im}\delta^{-1}} = \frac{\mathcal{F}(X)}{\{0\}} \cong \mathcal{F}(X)$$ by the prec. item.

Last time: $\mathcal{F}$ sheaf, $\mathfrak{U}$ covering of X, $\check{H}^i(\mathcal{F},\mathfrak{U})$ was defined for $i=0,1$.

<u>Definition</u>: Let $\mathfrak{U},\mathfrak{V}$ be coverings of X. We say that $\mathfrak{V}$ is <u>finer</u> than $\mathfrak{U}$ if $\forall V \in \mathfrak{V}\ \exists i\ V \subseteq U_i$ for $U_i \in$ and write $\mathfrak{V} \leq \mathfrak{U}$.

In this situation ($\leq$ is a directed system, so-called directed system, we can set

$$H^i(X,\mathcal{F}) := \varinjlim_{\mathfrak{U}} H^i(\mathcal{F},\mathfrak{U})$$ which is the quotient space $\bigcup_{\mathfrak{U}} H^i(\mathcal{F},\mathfrak{U})/\simeq$, where $\simeq$ is defined as follows:

Let $[\xi] \in \check{H}^i(\mathcal{F},\mathfrak{U}_1)$, $[\eta] \in \check{H}^i(\mathcal{F},\mathfrak{U}_2)$. We say $\xi \simeq \eta$ if there exists $\mathfrak{V} \leq \mathfrak{U}_1$ and $\mathfrak{V} \leq \mathfrak{U}_2$ s.t. $\forall W \in \mathfrak{V}$: $\xi_{|W} = \eta_{|W}$. The correctness of the definition (?) and the fact that $\simeq$ is an equivalence to be found in Forster, p. 98-99.

Recall: In the case of $i=0$, we found already that
$$H^0(\mathcal{F},\mathfrak{U}) \cong \mathcal{F}(X) \text{ (for any } \mathfrak{U}).$$

Summing-up, we defined $H^i(X,\mathcal{F})$ for $i=0,1$ so called Čech cohomology groups (also sheaf cohomology groups).

Results important for computing Čech cohomologies

Let X be a Riemann surface. Then $\mathcal{E}^\infty$ be the sheaf of smooth functions, i.e

$$\mathcal{E}^\infty(U) = \{ f : U \to \mathbb{C} \mid f \text{ is } \mathcal{E}^\infty \}$$

$\mathcal{E}^\infty(i^U_V) f = f_{|V}$ $\forall U \subseteq V$ restriction morphism is restriction of maps.

Result: $H^1(X, \mathcal{E}^\infty) = 0$

Result: X simply connected $\Rightarrow$

$$H^1(X, \underline{\mathbb{C}}) = 0$$
$$H^1(X, \underline{\mathbb{Z}}) = 0$$

Note: For any group G, the sheaf $\underline{G}$ was (already) defined.

Definition: $\mathfrak{U} = (U_i)_{i \in I}$ is called good if $\forall i \in I$

! $\underline{\underline{H^1(U_i, \mathcal{F}) = 0}}$ (sometimes Leray's cov.).

! **Theorem (Leray)**: $\mathcal{F}$ be a sheaf of abelian groups on a space X and $\mathfrak{U}$ be a good covering. Then

$$H^1(X, \mathcal{F}) \cong H^1(\mathfrak{U}, \mathcal{F}).$$

Proof: omitted (p. 101, Forster).

Example: $X = \mathbb{C}^\times (:= \mathbb{C} \setminus \{0\})$, $\mathcal{F} = \underline{\mathbb{Z}}$.

$\mathcal{U} = (U_1 = \mathbb{C}^\times \setminus \mathbb{R}_-, U_2 = \mathbb{C}^\times \setminus \mathbb{R}_+)$

$(I = \{1,2\})$. $\mathcal{U}$ is good (e.g., cause U_i are star-shaped...) By Leray's thm [e re:]

$H^1(\mathbb{C}^\times, \underline{\mathbb{Z}}) = H^1(\mathcal{U}, \underline{\mathbb{Z}})$. "What's $H^1(\mathcal{U}, \underline{\mathbb{Z}})$"

$Z^1(\mathcal{U}, \underline{\mathbb{Z}}) \ni (a_{ij})_{i,j \in \{1,2\}}$

a) $\delta(a_{ij})_{i,j} = (b_{klm})_{k,l,m}$

$b_{klm} = a_{lm} - a_{km} + a_{kl}$

Thus $a_{lm} - a_{km} + a_{kl} = 0$

$k = l = m = i \Rightarrow a_{ii} - a_{ii} + a_{ii} = 0 \Rightarrow$

$a_{ii} = 0$ (this is general!, for Z^1)

$m = k = i,\ l = j \Rightarrow a_{ji} - a_{ii} + a_{ij} = 0$

$\Rightarrow -a_{ij} = a_{ji}$.

Also general! $\sim \boxed{(a_{ij})_{i,j} \in Z^1 \text{ is alternating}}$

In particular,

knowing only a_{12} determines a_{11}, a_{22}, a_{21}, i.e., the whole co-cycle!

$Z^1(\mathcal{U}, \underline{\mathbb{Z}}) = \{ (a_{ij})_{i,j\in\{1,2\}} \mid \delta(a_{ij})_{i,j} = 0 \} \cong$

$\cong \langle a_{12} \in \underline{\mathbb{Z}}(U_1\cap U_2)\rangle$

→ We spoke about locally constant f-tions which is the same.

$H^{\pm} = \{ z \mid \operatorname{Im} z \gtrless 0\}$

$\underline{\mathbb{Z}}(U_1\cap U_2) = \underline{\mathbb{Z}}(H^+\cup H^-)$

$\cong \underline{\mathbb{Z}}\times\mathbb{Z}$ ($\cong$ $\mathbb{Z}$-valued cont f-tions on H^+ an $\mathbb{Z}$-valued cont f-tions on H^-)

$C^0(\mathcal{U},\underline{\mathbb{Z}}) = \{(b_i)_{i\in\{1,2\}} \mid b_1 \in \underset{\underset{\mathbb{Z}(\ )}{\nearrow}}{U_1},\ b_2\in \mathbb{Z}(U_2)\} \cong \mathbb{Z}\times\mathbb{Z}$

$\operatorname{Im}\delta^0 = ?$ $\delta^0((b_i)_i)_{12} = b_2 - b_1$ (co-~~cycles~~ chains)

Under the isomorphisms above $\delta^0((b_1,b_2)) = (b_2-b_1, -b_2+b_1)$

Thus $B^1 \cong \Delta^- = \{(a,-a) \mid a\in\mathbb{Z}\}$

$$H^1(\mathbb{C}^\times, \underline{\mathbb{Z}}) = \frac{Z^1(\mathcal{U},\underline{\mathbb{Z}})}{B^1(\mathcal{U},\underline{\mathbb{Z}})} \cong \frac{\mathbb{Z}\times\mathbb{Z}}{\Delta} \underset{\uparrow}{\cong} \mathbb{Z}$$

Define this iso! [

We also mentioned that $C^1(\mathbb{C}^\times,\mathcal{U}) \cong \mathbb{Z}^6$

„In general": $p_i \neq p_j\ \forall i\neq j$ $p_i\in\mathbb{C}$. Then

$$H^1(\mathbb{C}\setminus\{p_1,\dots,p_n\}) \cong \mathbb{Z}^n.$$

Results:

Finiteness thm: X cpt Riemann surface. Then $\dim H^1(X,\mathcal{O}) =: g$ and is called the $\boxed{\text{genus}}$ of X, is finite.

Proof: Weyl ~~approach~~ on Laplace ~~„correction"~~ and gen. of Poincaré

~~Thm~~ for closed forms (so-called Dolbeault lemma). ⑤
lemma

Part of Hodge theory $\square$

Remark : Plural of genus is genera or genuses.

4.2. Divisors

Definition: X R.s. A divisor on X is a map $D : X \to \mathbb{Z}$ such that $\forall K$ cpt in X $D_{|K}$ is non-zero only in finitely many points.

(Remark: In particular supp D does not have a limit point.)

Definition: $\mathrm{Div}(X) := \{D \mid D \text{ divisor on } X\}$ with $(D_1+D_2)(x) = D_1(x)+D_2(x)$ i.e., group structure

Definition: X R.s., $f \in \mathcal{M}(X)$, $a \in X$

$$\mathrm{ord}_a f = \begin{cases} 0 & \text{holomorphic and non-zero at } a \\ k & \text{zero of order } k \text{ at } a \\ -k & \text{pole of order } k \text{ at } a \\ \infty & f \equiv 0 \text{ on a neighb. of } a \end{cases}$$

$(f) = \{x \mapsto \mathrm{ord}_x f \mid x \in X\}$ (is a divisor $\forall f \not\equiv 0$)

Remark: $\mathrm{ord}_a f$ does not depend on the map!

X compact Riemann surface

~~Def~~ Definition : $D \in \mathrm{Div}(X)$, $\deg : \mathrm{Div}(X) \to \mathbb{Z}$

$$\deg(D) = \sum_{x \in X} D(x)$$ degree of divisor

EXAMINATION PART „STOPS".

Definition : X R. surface, $D \in \mathrm{Div}(X)$ $\forall U \subseteq X$ open

$$\mathcal{O}_D(U) = \{f \in \mathcal{M}(U) \mid \mathrm{ord}_x(f) \geq -D(x) \ \forall x \in U\}$$

Theorem: X be a compact Riemann surface and D be a divisor on X. Then $H^i(X,\mathcal{O}_D)$ are finite dimensional and

$$\dim H^0(X,\mathcal{O}_D)-\dim H^1(X,\mathcal{O}_D)=$$
$$=1-g+\deg D.$$

Proof: "Induction + ~~homology of the~~ Čech cohomology is a cohomology theory (=: satisfies the Steenrods axioms)."
see Hatcher

□

Application: X cpt. Riemann surface of genus g and $p\in X$. There exists $f\in\mathcal{M}(X)$ s.t. $\mathrm{ord}_p(f)\leq g+1$ and $f\in\mathcal{O}(X\setminus\{p\})$.

Proof: $D(p)=g+1$, $D(x)=0\ \forall x\neq p$.

$$\dim H^0(X,\mathcal{O}_D)-\dim H^1(X,\mathcal{O}_D)=$$
$$=1-g+\deg D=1-g+g+1=2.$$

$$\Rightarrow \dim H^0(X,\mathcal{O}_D)\geq 2$$

$$H^0(X,\mathcal{O}_D)=\mathcal{O}_D(X)=$$

pole at most g+1.

$$=\left\{f\in\mathcal{O}(x)\ \middle|\ \begin{array}{l}\mathrm{ord}_p f\geq -g-1\\ \mathrm{ord}_x f=0\ \forall x\neq p\end{array}\right\}$$

⇕
holo out of p.

What remains: ⑦

① Uniformization thm ($\supseteq$ classif. of 2-dim cpt manifolds.

② First "Notes on „abelian integrals"; by this I mean:

a) Weierstraß fuction

b) Theta funtions only

[illegible]

Definition: $\mathrm{Pic}(X)=\mathrm{Div}(X)/\mathrm{Div}_p(X)$, where

$\mathrm{Div}_p(X)=\{(f)\mid f\in\mathcal{M}(X)\setminus\{0\}\}$ so-called

principal divisors.

So-called Picard group.