Ellipticity of the symplectic twistor complex

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Plan of the talk

1. Segal-Shale-Weil representation

2. Symplectic spinor valued forms $\mathbf{E}$ and their decomposition

3. Howe duality for SSW acting on $\mathbf{E}$

4. Fedosov manifolds

5. Structure of the curvature tensor acting on symplectic spinor valued forms

6. Symplectic twistor complex

7. Ellipticity of the symplectic twistor complex

Segal-Shale-Weil representation

$(\mathbb{V},\omega)$ real symplectic vector space of dimension $2l$

$\mathbb{L},\mathbb{L}'$ Lagrangian subspaces of $(\mathbb{V},\omega)$ such that

$\mathbb{V}=\mathbb{L}\oplus\mathbb{L}'$

$G:=Sp(\mathbb{V},\omega)\simeq Sp(2l,\mathbb{R})$ symplectic group

$K:=$ maximal compact subgroup of G, $K\simeq U(l)$

$\pi_1(G)\simeq\pi_1(K)\simeq\mathbb{Z}\Longrightarrow\exists\ 2:1$ covering of G

$\lambda:\tilde{G}\stackrel{2:1}{\to}G$ $\tilde{G}=:Mp(\mathbb{V},\omega)\simeq Mp(2l,\mathbb{R})$ metaplectic group

($\tilde{G}$ is not simply connected)

Segal-Shale-Weil representation

There exists a distinguished unitary representation of the metaplectic group, the so called Segal-Shale-Weil representation.

$$
\begin{array}{ccc}
Mp(\mathbb{V},\omega) = \tilde{G} & & \\
\lambda \downarrow & \searrow^{\mathsf{SSW}} & \\
Sp(\mathbb{V},\omega) = G & \underset{\nexists}{\longrightarrow} & \mathcal{U}(L^2(\mathbb{L}))
\end{array}
$$

Thus, SSW : $\tilde{G} \to \mathcal{U}(L^2(\mathbb{L}))$ ("true" representation of $\tilde{G} = Mp(\mathbb{V},\omega)$). The horizontal arrow represents a projective ("non-true") representation of $G = Sp(\mathbb{V},\omega)$.
Call $L^2(\mathbb{L})$ - the space of symplectic spinors.

Segal-Shale-Weil representation SSW

1. SSW is unitary; other names: oscillator, metaplectic, symplectic spinor rep.

2. SSW does not descend to a representation of the symplectic group

3. $L^2(\mathbb{L}) \simeq L^2_+(\mathbb{L}) \oplus L^2_-(\mathbb{L})$ = direct sum decomposition into irreducibles; $L^2_\pm(\mathbb{L})$ - even/odd square Lebesgue integrable complex valued functions

4. Inventors: Weil (number thy), Berezin (quantum mechanics of many particle systems)

5. Related topics: Schrödinger representation of the Heisenberg group, Stone-von Neumann theorem.

Highest weight module properties of SSW

Set $\mathbf{S} := L^2(\mathbb{L})$, $\mathbf{S}_\pm := L^2_\pm(\mathbb{L})$

Harish-Chandra underlying $(\mathfrak{g}, K)$-module

a) $\mathfrak{g} = \mathfrak{mp}(2l, \mathbb{R}) \simeq \mathfrak{sp}(2l, \mathbb{R})$, K maximal compact in $Mp(\mathbb{V}, \omega)$, $K \simeq \lambda^{-1}(U(l))$

b) $HC(\mathbf{S}) \simeq \mathbb{C}[z^1, \ldots, z^l]$, where $\mathfrak{mp}(2l, \mathbb{R})$ acts via Dixmier 'realization'

Supersymmetry: $\mathfrak{g}' = \mathfrak{so}(\mathbb{V}', B)$, $\mathbb{V}'$ complex $2l$ dim. vector space, B a $\mathbb{C}$-bilinear form on $\mathbb{V}'$. Then the space of (orthogonal) spinors $\mathbf{S}' = \bigoplus_{k=0}^{l} \bigwedge^k \mathbb{U}$, $\mathbb{U}$ isotropic in $(\mathbb{V}, B)$.

Highest weight of $\mathbb{S}_\pm$ is $(-\frac{1}{2}, \ldots, -\frac{1}{2}, -\frac{1}{2})$ and $(-\frac{1}{2}, \ldots, -\frac{1}{2}, -\frac{3}{2})$ (in the 'standard' basis).

Decomposition of symplectic spinor valued forms

1. Take
$$\rho : \tilde{G} \to \mathsf{Aut}(\bigwedge^{\bullet} \mathbb{V}^* \otimes \mathbf{S}),$$
defined by
$$\rho(g)(\alpha \otimes s) := \lambda(g)^{*\wedge}(\alpha) \otimes SSW(g)s,$$
where $g \in \tilde{G}$, $\alpha \otimes s \in \bigwedge^{\bullet} \mathbb{V}^* \otimes \mathbf{S}$.

2. **Topology**: Hilbert tensor product topology. Then ρ is admissible and of finite length representation.

Aim: Decompose $\mathbf{E} := \bigwedge^{\bullet} \mathbb{V}^* \otimes \mathbf{S}$ into irreducibles.

Decomposition theorem

For $i = 0, \ldots, l$, set $m_i = i$. For $i = l+1, \ldots, 2l$, set $m_i := 2l - i$.

Set $\Xi := \{(i,j) | i = 0, \ldots, 2l; j = 0, \ldots, m_i\}$.

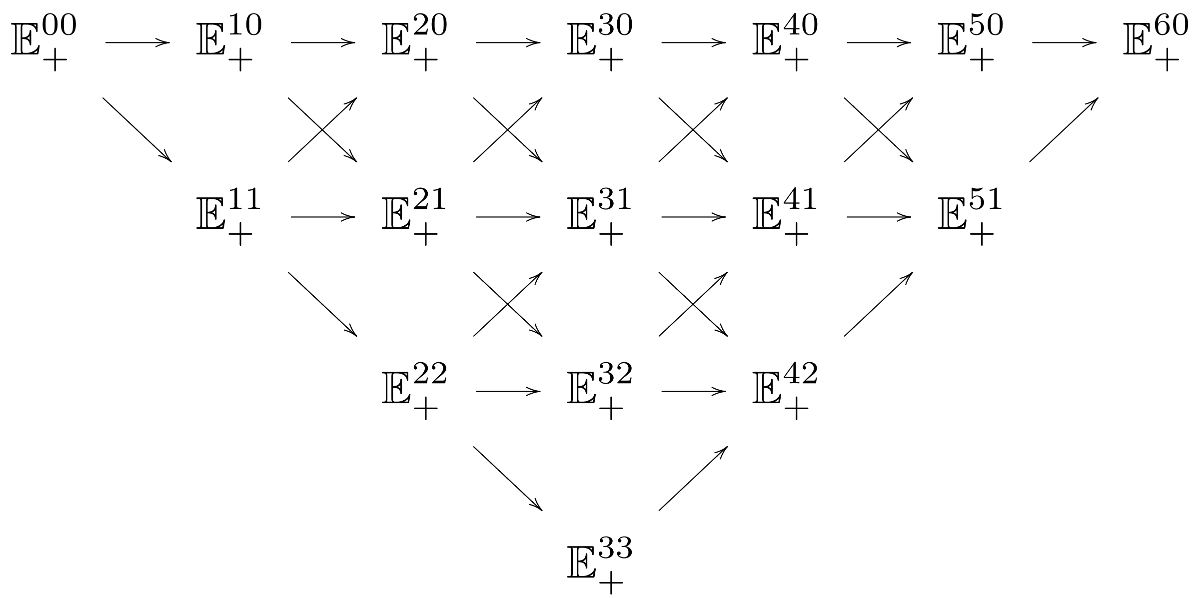
Example: For $l = 2$, the set $\Xi = \{(0,0), (1,0), (1,1), (2,0), (2,1), (2,2), (3,0), (3,1), (4,0)\}$.

Theorem: For $i = 0, \ldots, 2l$, the following decomposition

$$\bigwedge^i \mathbb{V}^* \otimes \mathbf{S}_\pm \simeq \bigoplus_{(i,j) \in \Xi} \mathbf{E}_\pm^{ij} \text{ holds.}$$

Visualization of the decomposition theorem

$$\mathbb{S}_+ \quad \mathbb{V}\otimes\mathbb{S}_+ \quad \bigwedge^2\mathbb{V}\otimes\mathbb{S}_+ \quad \bigwedge^3\mathbb{V}\otimes\mathbb{S}_+ \quad \bigwedge^4\mathbb{V}\otimes\mathbb{S}_+ \quad \bigwedge^5\mathbb{V}\otimes\mathbb{S}_+ \quad \bigwedge^6\mathbb{V}\otimes\mathbb{S}_+$$



Highest weight description

The infinitesimal structure of the Harish-Chandra $(\mathfrak{g}, K)$-module $\mathbb{E}^{ij}_{\pm}$ of $\mathbf{E}^{ij}_{\pm}$ satisfies

$$\mathbb{E}^{\pm}_{ij} \simeq L(\underbrace{\frac{1}{2}, \ldots, \frac{1}{2}}_{j}, \underbrace{-\frac{1}{2}, \ldots, -\frac{1}{2}}_{l-j-1}, -1 + \frac{1}{2}(-1)^{i+j+sgn(\pm)}),$$

$sgn(\pm) := \pm 1, (i, j) \in \Xi.$

Howe-type duality

1. Schur duality for $G := GL(\mathbb{V})$

$$\rho_k : G \longrightarrow \mathsf{Aut}(\mathbb{V}^{\otimes k})$$

$$\rho_k(g)(v_1 \otimes \ldots \otimes v_k) := gv_1 \otimes \ldots \otimes gv_k,$$

$g \in G, v_i \in \mathbb{V}$, $i = 1, \ldots, k$. Tensor representation.

$$\sigma_k : \mathfrak{S}_k \longrightarrow \mathsf{Aut}(\mathbb{V}^{\otimes k})$$

$$\sigma_k(\tau)(v_1 \otimes \ldots \otimes v_k) := v_{\tau(1)} \otimes \ldots \otimes v_{\tau(k)},$$

$\tau \in \mathfrak{S}_k$, $v_i \in \mathbb{V}$, $i = 1, \ldots, k$. Permutation representation.

Easy:

$$\sigma_k(\tau)\rho_k(g) = \rho_k(g)\sigma_k(\tau)$$

$g \in G, \tau \in \mathfrak{S}_k$. The representations commute.

Not so easy = Schur duality: $T\rho_k(g) = \rho_k(g)T \Rightarrow T \in \mathbb{C}[\sigma_k(\mathfrak{S}_k)]$ (the group algebra of the group $\sigma_k(\mathfrak{S}_k)$). $\mathfrak{S}_k$ is called the Schur dual of $GL(\mathbb{V})$ for $\mathbb{V}^{\otimes k}$.

Leads to Young diagrams. Combinatorial structure of $\mathfrak{S}_k$ translates into a combinatorial structure of the representations of $GL(\mathbb{V})$.

2. Another type of duality: spinor valued forms

Group: $\tilde{G} = Spin(\mathbb{V}, B)$

Space: $\bigwedge^\bullet \mathbb{V} \otimes \mathbb{S}$, where $\mathbb{S}$ is the space of (orthogonal) spinors

$\mathrm{End}_{\tilde{G}}(\bigwedge^\bullet \mathbb{V} \otimes \mathbb{S}) := \{T : \bigwedge^\bullet \mathbb{V} \otimes \mathbb{S} \to \bigwedge^\bullet \mathbb{V} \otimes \mathbb{S} |$for all $g \in G$ $T\rho(g) = \rho(g)T\}$. - Commutant - old-fashioned

Result: $\mathrm{End}_{\tilde{G}}(\bigwedge^\bullet \mathbb{V} \otimes \mathbb{S}) = \langle \sigma(\mathfrak{sl}(2, \mathbb{C})) \rangle$ for certain representation σ of $\mathfrak{sl}(2, \mathbb{C})$. Thus, $\mathfrak{sl}(2, \mathbb{C})$ is a Howe type dual of $Spin(\mathbb{V}, B)$ on $\bigwedge^\bullet \mathbb{V} \otimes \mathbb{S}$.

Leads to a systematic treatment of some questions on Dirac operators and their higher spin analogues.

3. Further example of Howe duality in geometry

Duality between $U(n)$ and $\mathfrak{sl}(2, \mathbb{C})$ when acting on (p, q)-forms of a complex vector space. Lefschetz decomposition on Kähler manifolds.

Howe duality for symplectic spinor valued forms

Group $\tilde{G} = Mp(\mathbb{V}, \omega)$ acting on $\mathbf{E} := \bigwedge^{\bullet} \mathbb{V}^* \otimes \mathbf{S}$ via ρ.

Result:

$$\mathrm{End}_{\tilde{G}}(\mathbf{E}) \simeq \sigma(\mathfrak{osp}(1|2)),$$

where $\mathfrak{osp}(1|2)$ is the ortho-symplectic Lie super-algebra and σ is a super Lie algebra representation.

Defining relations of $\mathfrak{osp}(1|2)$

Ortho-symplectic super Lie algebra $\mathfrak{osp}(1|2) = \langle f^+, f^-, h, e^+, e^- \rangle$.

Relations

$$[h, e^\pm] = \pm e^\pm \qquad [e^+, e^-] = 2h,$$

$$[h, f^\pm] = \pm \frac{1}{2} f^\pm \qquad \{f^+, f^-\} = \frac{1}{2} h,$$

$$[e^\pm, f^\mp] = -f^\pm \qquad \{f^\pm, f^\pm\} = \pm \frac{1}{2} e^\pm,$$

The representation σ of $\mathfrak{osp}(1|2)$

Consider the following mapping.

$$\sigma : \mathfrak{osp}(1|2) \to \mathrm{End}(\bigwedge^{\bullet} \mathbb{V}^* \otimes \mathbf{S})$$

$$\sigma(f^+)(\alpha \otimes s) := \frac{\imath}{2}\epsilon^i \wedge \alpha \otimes e_j.s$$

$$\sigma(f^-)(\alpha \otimes s) := \frac{1}{2}\omega^{ij}\iota_{e_i}\alpha \otimes e_j.s,$$

where $\alpha \otimes s \in \bigwedge^{\bullet} \mathbb{V}^* \otimes \mathbf{S}$.

For other elements use the relations of $\mathfrak{osp}(1|2)$, e.g., $H = \sigma(h) = \sigma(2\{f^+, f^-\}) = 2\{F^+, F^-\} = 2(F^+F^+ + F^+F^+)$ an endomorphism of $\mathbf{E}$.

The image $\sigma(\mathfrak{osp}(1|2)) = \mathrm{End}_{\tilde{G}}(\mathbf{E})$.

1. F^+ rising in a horizontal way

2. F^- lowering in a horizontal way.

Back to the picture.

Geometric part

(M, ω) symplectic manifold of dimension $2l$. $\mathcal{R}$ bundle of symplectic bases in TM.

1. $\mathcal{R} := \{(e_1, \ldots, e_{2l})$ is a symplectic basis of $(T_m, \omega_m) | m \in M\}$.

2. $p_1 : \mathcal{R} \to M$, the foot-point projection, is a principal $Sp(2l, \mathbb{R})$-bundle.

3. $p_2 : \mathcal{P} \to M$ be a principal $Mp(2l, \mathbb{R})$-bundle.

4. $\Lambda : \mathcal{P} \to \mathcal{R}$ be a surjective bundle morphism over the identity on M.

Definition: We say that $(\mathcal{P}, \Lambda)$ is a metaplectic structure if

$$
\begin{array}{ccccc}
Mp(2l,\mathbb{R}) \times \mathcal{Q} & \longrightarrow & \mathcal{Q} & & \\
 & & & \searrow^{p_2} & \\
\Big\downarrow \lambda\times\Lambda & & \Big\downarrow \Lambda & & M \\
 & & & \nearrow_{p_1} & \\
Sp(2l,\mathbb{R}) \times \mathcal{P} & \longrightarrow & \mathcal{P} & &
\end{array}
$$

commutes. The horizontal arrows are the actions of the respective groups.

Symplectic spinors

$$\mathcal{S} := \mathcal{P} \times_{\mathsf{meta}} \mathbf{S}.$$

Elements of $\Gamma(M, \mathcal{S})$ - symplectic spinor fields (Kostant)

Symplectic connection = torsion-free affine connection ∇ satisfying $\nabla\omega = 0$. It gives rise to a principal bundle connection Z on $p_1 : \mathcal{R} \to M$. Take a lift $\hat{Z}$ of Z to the metaplectic structure $p_2 : \mathcal{P} \to M$. Consider the associated covariant derivative on $\mathcal{S} \implies$ symplectic spinor derivative $\nabla^{\mathcal{S}}$.

Remark. With help of $\nabla^{\mathcal{S}}$, one can define the symplectic Dirac operator and do, e.g., harmonic analysis for symplectic spinors (Katharina Habermann in '90).

Manifolds admitting a metaplectic structure:

1.) phase spaces $(T^*N, d\theta)$, N orientable,

2.) complex projective spaces $\mathbb{P}^{2k+1}\mathbb{C}$, $k \in \mathbb{N}_0$,

3.) Grassmannian $Gr(2,4)$ e.t.c.

4.) Calabi-Yau manifolds

Fedosov manifolds

1. (M, ω) symplectic manifold

2. ∇ symplectic connection (no uniqueness)

$\Longrightarrow (M, \omega, \nabla)$ Fedosov manifold

Classical definition

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

$X, Y, Z \in \mathfrak{X}(M)$.

Symplectic curvature

1. $\sigma_{ij} = R^k{}_{ikj}$ - symplectic Ricci tensor

2. $2(l+1)\widetilde{\sigma}^{\nabla}_{ijkl} = \omega_{il}\sigma_{jk} - \omega_{ik}\sigma_{jl} + \omega_{jl}\sigma_{ik} - \omega_{jk}\sigma_{il} + 2\sigma_{ij}\omega_{kl}$

3. $W^{\nabla} = R^{\nabla} - \widetilde{\sigma}^{\nabla}$ - symplectic Weyl tensor

Let us call a Fedosov manifold (M, ω, ∇) of Ricci type, if $W^{\nabla} = 0$.

Symmetries of W and $\widetilde{\sigma}$ - via harmonic tensors (symplectic analogue of Weyl formulas, Zholebenko)

Symplectic curvature R

$R_{ijkl} = -R_{jikl}$
$R_{ijkl} = R_{ijlk}$
$R_{ijkl} + R_{jkli} + R_{klij} + R_{lijk} = 0$ (do not get all Bianchi)

Symplectic Ricci tensor: $\sigma_{ij} = \sigma_{ji} \Longrightarrow$ no symplectic scalar curvature

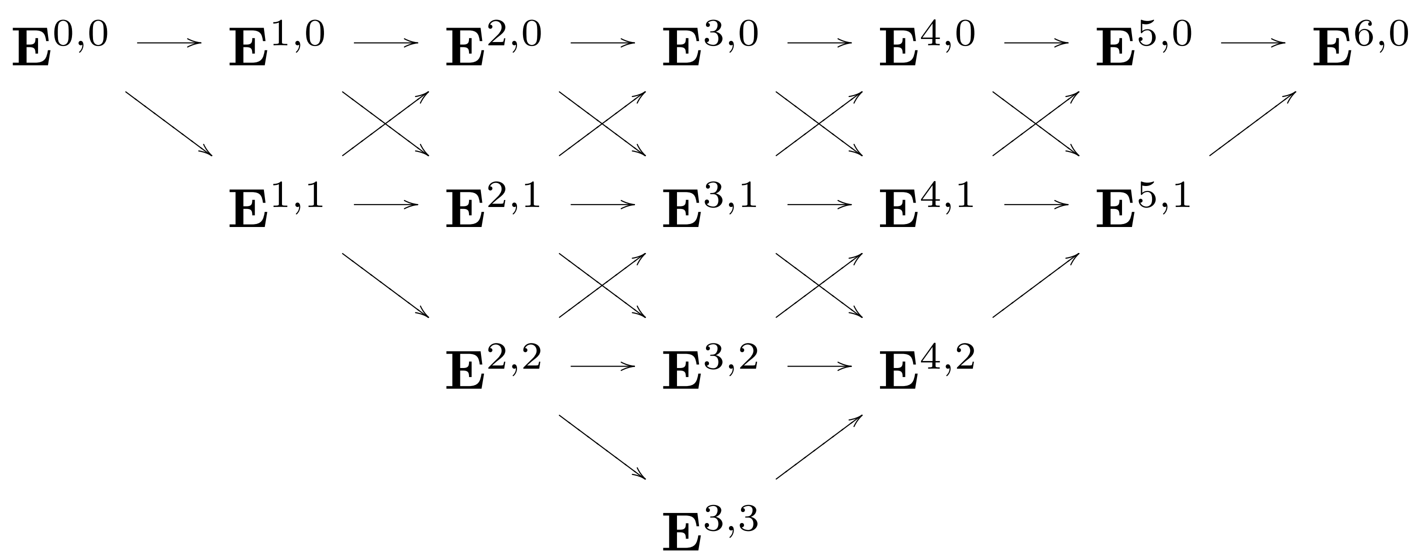
Symplectic Weyl tensor W

The same symmetries as R
+ completely trace-free

Theorem: (M, ω, ∇) symplectic manifold admitting a metaplectic structure. Then

$$d^{\nabla^S} : \Gamma(M, \mathcal{E}^{ij}_{\pm}) \to \Gamma(M, \mathcal{E}^{i+1,j-1}_{\pm} \oplus \mathcal{E}^{i+1,j}_{\pm} \oplus \mathcal{E}^{i+1,j+1}_{\pm}),$$

where $\mathcal{E}^{ij}_{\pm}$ is the associated bundle to the principal $Mp(\mathbb{R}, 2l)$-bundle via ρ.


$\mathbf{E}^{0,0}$ $\mathbf{E}^{1,0}$ $\mathbf{E}^{2,0}$ $\mathbf{E}^{3,0}$ $\mathbf{E}^{4,0}$ $\mathbf{E}^{5,0}$ $\mathbf{E}^{6,0}$
$\mathbf{E}^{1,1}$ $\mathbf{E}^{2,1}$ $\mathbf{E}^{3,1}$ $\mathbf{E}^{4,1}$ $\mathbf{E}^{5,1}$
$\mathbf{E}^{2,2}$ $\mathbf{E}^{3,2}$ $\mathbf{E}^{4,2}$
$\mathbf{E}^{3,3}$

Complex of symplectic twistor operators

Definition: For $i = 0, \ldots, 2l$, set $T_i := p^{i+1,m_{i+1}} \circ d^{\nabla^S}_{|\mathcal{E}^{im_i}}$. **Symplectic twistor operator.**

Theorem:(SK,09) Let (M^{2l}, ω, ∇) be a Fedosov manifold admitting a metaplectic structure. If $l \geq 2$ and the symplectic Weyl tensor field $W^{\nabla} = 0$, then

$$0 \longrightarrow \Gamma(M, \mathcal{E}^{00}) \xrightarrow{T_0} \Gamma(M, \mathcal{E}^{11}) \xrightarrow{T_1} \cdots \xrightarrow{T_{l-1}} \Gamma(M, \mathcal{E}^{ll}) \longrightarrow 0 \text{ and}$$

$$0 \longrightarrow \Gamma(M, \mathcal{E}^{ll}) \xrightarrow{T_l} \Gamma(M, \mathcal{E}^{l+1,l+1}) \xrightarrow{T_{l+1}} \cdots \xrightarrow{T_{2l-1}} \Gamma(M, \mathcal{E}^{2l,2l}) \longrightarrow 0$$

are complexes.

Core: Computing the action of W and σ on E. Not only a Howe duality - because σ is not "living" in the ""infinitesimal world"" (explain).

Ellipticity of the symplectic twistor complex

Theorem: Let (M, ω, ∇) be a Fedosov manifold of Ricci type admitting a metaplectic structure. Then the truncated symplectic twistor complexes

$$0 \longrightarrow \Gamma(M, \mathcal{E}^0) \xrightarrow{T_0} \Gamma(M, \mathcal{E}^1) \xrightarrow{T_1} \cdots \xrightarrow{T_{l-2}} \Gamma(M, \mathcal{E}^{1-1}) \text{ and}$$

$$\Gamma(M, \mathcal{E}^l) \xrightarrow{T_l} \Gamma(M, \mathcal{E}^{l+1}) \xrightarrow{T_{l+1}} \cdots \xrightarrow{T_{2l-1}} \Gamma(M, \mathcal{E}^{2l}) \longrightarrow 0$$

are elliptic.

Proof. Only commutation Howe type relations + Cartan lemma (on exterior systems). □

Folge: (Reduced) cohomologies of symplectic twistor complexes are finite dimensional. One has Hodge for this complex.

Ellipticity in other instances

Stein, Weiss: ellipticity for generalized gradients (Casmir computations + Weyl character formulas)

Baston: inverse question - similar methods

Schmid: Casimir + ""combinatorics"" (ell. for symmetric spaces of inner type)

deRham: easy representation theory of $O(n)$ or $GL(n, \mathbb{R})$, direct Cartan lemma

Dolbeault: easy representation theory of $U(n)$ (compact real form of $GL(n, \mathbb{R})$)

Hotta: generalizes Schmid (Bott-Borel-Weil + homology algbera)