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Symplectic spin geometry

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I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.

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In .... date .... Marie Holíková

Název práce: Symplektická spin geometrie

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Abstrakt: Symplektický Diracův operátor a symplecktický twistorový operátor jsou symplektickými analogiemi Diracova a twistorového operátoru v spin-Riemannově geometrii. V práci je věnována pozornost základním aspektum těchto dvou operátorů. Konkrétně, detailně studujeme jádro symplektického twistorového operátoru na symplektickém vektorovém prostoru dimenze $2n$. Ukazuje se, že prostor řešení je symplektickou analogií klasického ortogonálního případu. Dále, na příkladu $2n$-dimenzionálního toru ukážeme závislost prostoru řešení symplektického Diracova a twistorového operátoru na výběru metaplektické struktury. Navíc zkonstruujeme symplektická zobecnění klasické theta funkce pro symplektický Diracův operátor. V práci se zabýváme symplektickou Cliffordovou analýzou pro symplektický Diracův operátor, s důrazem na reálný symplektický prostor dimenze 2. Studujeme symetrie prvního řádu symplektického Diracova operátoru, symplektickou analogii Fisherova produktu, a sestrojíme báze symplektických monogenik v reálné dimenzi 2 resp. jejich rozšíření na symplektické prostory vyšší dimenze.

Klíčová slova: symplektická spin geometrie, symplektický Diracův operátor, Symplektický twistorový operátor, Symplektická Cliffordova analýza.

Title: Symplectic spin geometry

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Abstract: The symplectic Dirac and the symplectic twistor operators are symplectic analogues of classical Dirac and twistor operators appearing in spin-Riemannian geometry. Our work concerns basic aspects of these two operators. Namely, we determine the solution space of the symplectic twistor operator on the symplectic vector space of dimension $2n$. It turns out that the solution space is a symplectic counterpart of the orthogonal situation. Moreover, we demonstrate on the example of $2n$-dimensional tori the effect of dependence of the solution spaces of the symplectic Dirac and the symplectic twistor operators on the choice of the metaplectic structure. We construct a symplectic generalization of classical theta functions for the symplectic Dirac operator as well. We study several basic aspects of the symplectic version of Clifford analysis associated to the symplectic Dirac operator. Focusing mostly on the symplectic vector space of the real dimension 2, this amounts to the study of first order symmetry operators of the symplectic Dirac operator, symplectic Clifford-Fourier transform and the reproducing kernel for the symplectic Fischer product including the construction of bases for the symplectic monogenics of the symplectic Dirac operator in real dimension 2 and their extension to symplectic spaces of higher dimension, respectively.

Keywords: Symplectic spin geometry, Symplectic Dirac operator, Symplectic twistor operator, Symplectic Clifford analysis.

Contents

Introduction

The subject of our thesis concerns various aspects of the theory of symplectic Dirac and symplectic twistor operators - symplectic spinor analogues of classical Dirac and twistor operators.

Many problems and questions in differential geometry of Riemannian spin manifolds are based on analytic and spectral properties of the classical Dirac and twistor operators acting on spinor valued fields. In particular, there is a quite subtle relation between the geometry and the topology of a given manifold and the spectra and solution spaces of Dirac and twistor operators. See e.g., [1], [18] and references therein. In the Riemannian geometry, the twistor equation appears as an integrability condition for the canonical almost complex structure on the twistor space and it plays a prominent role in conformal differential geometry due to its bigger symmetry group. In physics, its solution space defines infinitesimal isometries on the riemannian super manifolds.

Symplectic geometry naturally originates in physics and provides a language for Hamiltonian description of classical mechanics, similarly to the use of Riemannian geometry in the Lagrangian description of classical mechanics.

The research in the symplectic spinor geometry was initiated by D. Shale [37] and B. Konstant [28] in the context of quantization. Symplectic spinors were introduced in [28] and their algebraic properties were studied in [5]. Geometric aspects of the theory of symplectic Dirac operator D_s were studied by K. Habermann, [21], who introduced the symplectic Dirac operator. The lecture notes [22] contains a comprehensive introduction to the symplectic spinor geometry and symplectic Dirac operators. A symplectic parallel to the twistor operator were then introduced and considered from various perspectives in [24], [31], [32].

The content of this thesis is as follows. In the first five chapters, basic terms are summarized as well as properties needed for understanding of the results discussed in the next chapters.

The first chapter is preliminary and contains basic notions as the ones of symplectic vector spaces, symplectic groups and symplectic Lie algebras. Their relation with symplectic Clifford algebras are recalled here.

The second chapter gives a brief excursion into the function theory, namely the Schwartz space together with its distinguished basis of Hermite functions. The rest of the chapter offers a framework for Fréchet function spaces and nuclear function spaces. We recall their topological tensor product as well.

The Segal-Shale-Weil representation of the metaplectic group is introduced in Chapter 3. It is a faithful infinite-dimensional unitary representation which plays the same role as the spinor representation of the spin group plays in Riemannian spin geometry. In order to deal with symplectic spinors as elements of the representation space, the symplectic Clifford multiplication is introduced. It allows us to multiply symplectic spinors by vectors.

Chapter 4 contains a brief introduction of several geometrical aspects, namely the notion of a symplectic manifold and basics on the theory of bundles and connections on associated vector bundles. If there is a double covering of the symplectic frame bundle of symplectic manifold (M, ω), the so called metaplectic bundle, one can induce the Segal-Shale-Weil representation to obtain the symplectic spinor or Konstant bundle. This bundle is of infinite rank and its fibres are modules isomorphic to the Segal-Shale-Weil representation.

In Chapter 5, we conclude the introductory part of the thesis with definitions of the symplectic Dirac D_s and the symplectic twistor operator T_s. See the monograph [18] for a comparison with the Dirac operator in Riemannian geometry.

Chapters 6 and 7 are based on two published articles [12], [13] with a few minor changes. The aim of these chapters is to study the symplectic twistor operator T_s in the context of the metaplectic Howe duality, see [9], and consequently to determine its solution space on the canonical symplectic space $(\mathbb{R}^{2n}, \omega)$. From the analytic point of view, T_s represents an overdetermined system of partial differential equations and acts on the space of polynomials with values in the Segal-Shale-Weil representation. From the point of view of representation theory, T_s is $\mathfrak{mp}(2n, \mathbb{R})$-equivariant. The solution space was described using the interaction of T_s with the Howe dual pair $(\mathfrak{mp}(2n, \mathbb{R}), \mathfrak{sl}(2))$, where $\mathfrak{sl}(2)$ is generated by D_s, X_s. As we shall see, concerning T_s, there is a substantial difference between the situation for $n = 1$ (Chapter 6) and $n > 1$ (Chapter 7).

The operators acting on the space of symplectic monogenics are studied in Chapter 8. The symplectic monogenics are elements of the solution space of the symplectic Dirac operator on the canonical symplectic manifold $(\mathbb{R}^{2n}, \omega)$. Decomposing $\mathrm{Pol}(\mathbb{R}^{2n}, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R}^n)$ into $\mathfrak{mp}(2n, \mathbb{R})$-submodules with the use of the metaplectic Howe duality, we discuss the lift of the $\mathfrak{mp}(2n, \mathbb{R})$-symmetry of the space of symplectic monogenics to a representation space of a bigger Lie algebra.

In the harmonic analysis on $\mathbb{R}^n$, the classical Fourier transform can be encoded by the exponential of the operator $(\Delta - ||x||^2)$, i.e. a difference of the Laplace operator Δ and $||x||^2$, the square of the operator of the norm of $x \in \mathbb{R}^n$. There are analogous results in harmonic analysis for finite groups based on Dunkl operators or Dirac operator, cf. [7], [8] and [10]. Chapter 9 discusses rudiments of symplectic Clifford-Frourier transform by an investigation of an eigenfunction decomposition of the difference of D_s and X_s.

A symplectic Fischer product on polynomial symplectic spinors $\mathrm{Pol}(\mathbb{R}^2, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R})$ is defined in Chapter 10. This product is a symplectic analogue of the Fischer product on complex polynomials valued in the classical Clifford algebra.

Symplectic monogenics, discussed many times throughout the thesis, are the elements of $\mathfrak{mp}(2, \mathbb{R})$-submodules of the Fischer decomposition of polynomial spinors contained in the kernel of the symplectic Dirac operator. Constructions of two different basis of symplectic monogenics on $(\mathbb{R}^2, \omega)$ are given in the first part of the Chapter 11. These bases are then used to determine the behaviour of the symplectic Fischer product. Moreover, one of the basis is constructed in such a way that the symmetries of the symplectic Dirac operator act by the scalar on the set of basis elements.

In Chapter 12, we describe a recursive construction of symplectic monogenics on $(\mathbb{R}^{2n}, \omega)$ from the symplectic monogenics on $(\mathbb{R}^2, \omega)$ and $(\mathbb{R}^{2(n-1)}, \omega)$.

It is a classical result in Riemannian spin geometry that spectral properties of Dirac operator depend on the choice of a spin-structure, cf. [18]. A nice example of this phenomenon appears in [17], proving the dependence of its solution space on the choice of spin structures in the case of the real tori of an arbitrary dimension. The main theme of Chapter 13 is to study analogous phenomenon in the context of symplectic geometry and related symplectic Dirac operator. The computations show that on even dimensional tori, the solution space for the symplectic Dirac operator depends on the choice of the metaplectic structure used to define it. The question of existence of a solution is in the symplectic case more subtle than in the Riemannian geometry due to the restriction to a specific function class.

Another aspect of symplectic Dirac operator on symplectic tori is related to a construction of symplectic theta functions, regarded as a specific class of functions in the solution space of the symplectic Dirac rather than classical Dolbeault operator, cf. [34], [35]. In Chapter 14, we present several classes of symplectic theta functions and their basic properties.

1. Preliminaries

1.1 Symplectic vector space

In the first section we define a symplectic vector space and its various subspaces and recall their basic properties.

Definition 1.1.1. The *symplectic vector space* is a pair (V, Ω), where V is a vector space of finite dimension over the field of real numbers $\mathbb{R}$ and $\Omega : V \times V \to \mathbb{R}$ is a non-degenerate skew symmetric bilinear form.

Let Ω be a non-degenerate skew symmetric bilinear form on a $2n$ dimensional vector space V, $n \in \mathbb{N}$. Then there exists a basis $e_1, \dots, e_{2n}$ of the vector space V such that

$$\Omega(e_j, e_k) = 0 = \Omega(e_{n+j}, e_{n+k}) \quad \text{and} \quad \Omega(e_j, e_{n+k}) = \delta_{j,k}. \tag{1.1}$$

for every $j, k = 1, \dots, n$. The symbol $\delta_{j,k}$ denotes the Kronecker delta.

Definition 1.1.2. The *symplectic basis* of the symplectic vector space (V, Ω) is a basis $\{e_1 \dots, e_n, e_{n+1}, \dots, e_{2n}\}$ with the property (1.1).

Then the symplectic form Ω is in the symplectic basis of (V, Ω)

$$\Omega(v, u) = (-v-) \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} | \\ u \\ | \end{pmatrix},$$

where u, $v \in V$ and I stands for $n \times n$ identity matrix.

Example 1. Let us take the vector space $\mathbb{R}^{2n}$ with the basis

$$e_1 = (1, 0, \dots, 0), \dots, e_j = (0, \dots, 0, \overbrace{1}^{j}, 0, \dots, 0), \dots, e_{2n} = (0, \dots, 0, 1)$$

and the form Ω such that it is represented by matrix $\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ in this basis. Then $(\mathbb{R}^n, \Omega)$ is called the *canonical symplectic vector space* and Ω on $\mathbb{R}^{2n}$ is called the *canonical symplectic form.*

It will cause no confusion if we use Ω to designate the canonical symplectic form of the canonical symplectic vector space $(\mathbb{R}^{2n}, \Omega)$ and any symplectic form on a general symplectic vector space (V, Ω), since it is always obvious from the framework whether it is the former or latter case.

Not all subspaces of a symplectic space looks the same.

Definition 1.1.3. A subspace Y of a symplectic vector space (V, Ω) is called the *symplectic subspace*, if $\Omega|_Y$ is non-degenerate. If $\Omega|_Y \equiv 0$, then a subspace Y is called the *isotropic subspace.*

For instance the linear span of vectors e_1, e_{n+1} from Example 1 is a symplectic subspace and the linear span of vectors e_1, e_2 is an isotropic subspace.

A subspace Y of a symplectic vector space (V, Ω) can be determined by its symplectic orthogonal complement

$$Y^\Omega = \{v \in V \mid \Omega(v, u) = 0 \text{ for all } u \in Y\}.$$

A subspace Y of a symplectic vector space (V, Ω) is a symplectic subspace if and only if $Y \cap Y^\Omega = \{0\}$. Or equivalently if and only if $Y \oplus Y^\Omega = V$. Whereas a subspace Y is an isotropic subspace if and only if $Y \subseteq Y^\Omega$. If Y is an isotropic subspace, then $\dim Y \leq \frac{1}{2} \dim V$.

Definition 1.1.4. The *Lagrangian subspace* Y of a symplectic vector space is a subspace that satisfies $Y = Y^\Omega$.

The Lagrangian subspace is isotropic. In particular a subspace Y of a symplectic vector space (V, Ω) is Lagrangian if and only if $\dim Y = \frac{1}{2} \dim V$. Especially, every basis $e_1, \ldots, e_n$ of a Lagrangian subspace Y could be extended to the basis $e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}$ of V.

Let us denote by $\mathbb{R}^n = \langle e_1, \ldots, e_n \rangle$ and $(\mathbb{R}^n)' = \langle e_{n+1}, \ldots, e_{2n} \rangle$ linear spans of the first n and the second n elements of basis of $(\mathbb{R}^{2n}, \Omega)$. Then $\mathbb{R}^n$ and $(\mathbb{R}^n)'$ are called *canonical Lagrangian subspaces of* $(\mathbb{R}^{2n}, \Omega)$.

1.2 Symplectic group

The term symplectic group can refer to two different, but related groups, sometimes denoted by $\mathrm{Sp}(2n, \mathbb{R})$ and $\mathrm{Sp}(n)$. The symplectic group $\mathrm{Sp}(n)$ consists of linear mappings of the space of quaternions $\mathbb{H}^n$, but it will not be used in this thesis.

Definition 1.2.1. The *symplectic group* $\mathrm{Sp}(2n, \mathbb{R})$ is a group of all automorphisms of $(\mathbb{R}^{2n}, \Omega)$ preserving the canonical symplectic form Ω

$$\Omega : (x, y) \mapsto \sum_{j=1}^{n} x_j y_{n+j} - \sum_{j=1}^{n} x_{n+j} y_j,$$

where $x = (x_1, \ldots, x_{2n}), y = (y_1, \ldots, y_{2n}) \in \mathbb{R}^{2n}$.

Thus the elements of the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$ are automorphisms $A \in \mathrm{GL}(2n, \mathbb{R})$ such that for every $v, w \in \mathbb{R}^{2n}$ satisfy

$$\Omega(Av, Aw) = \Omega(v, w).$$

Let us denote

$$J_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

the matrix of the canonical symplectic form with respect to some symplectic basis.

A maximal compact subgroup of the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$ is

$$\mathrm{Sp}(2n, \mathbb{R}) \cap \mathrm{O}(2n, \mathbb{R}),$$

where $\mathrm{O}(2n,\mathbb{R}) = \{A \in \mathrm{GL}(2n,\mathbb{R}) \,|\, AA^T = A^T A = I\}$ is a group of all real orthogonal matrices. See, e.g., [16, Prop. 4.5] If we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ by the mapping $(x, y) \mapsto x + iy$, $x \in \mathbb{R}^n$, $y \in (\mathbb{R}^n)'$, then it is easy to realise

$$\mathrm{Sp}(2n,\mathbb{R}) \cap \mathrm{O}(2n,\mathbb{R}) \cong \mathrm{U}(n),$$

where $\mathrm{U}(n) = \{A \in M_n(\mathbb{C}) \,|\, \overline{A^T} A = I\}$ is the unitary group of $\mathbb{C}^n$ with canonical hermitian form. The symbol $M_n(\mathbb{C})$ denotes the set of complex $n \times n$ matrices. Therefore the maximal compact subgroup of $\mathrm{Sp}(2n,\mathbb{R})$ is the unitary group $\mathrm{U}(n)$. Based on this, the following property is for instance proved in [16, Prop. 4.8].

Proposition 1.2.1. The symplectic group $\mathrm{Sp}(2n,\mathbb{R})$ is path connected and its fundamental group is $\mathbb{Z}$.

Let us define following sets, which are related to a structure of the symplectic group.

$$\mathcal{D} = \left\{ \begin{pmatrix} D & 0 \\ 0 & (D^{-1})^T \end{pmatrix} \middle| \, D \in \mathrm{GL}(n,\mathbb{R}) \right\}$$
$$\mathcal{N} = \left\{ \begin{pmatrix} I & N \\ 0 & I \end{pmatrix} \middle| \, N \in \mathrm{GL}(n,\mathbb{R}), N = N^T \right\}$$
$$\overline{\mathcal{N}} = \left\{ \begin{pmatrix} I & 0 \\ N & I \end{pmatrix} \middle| \, N \in \mathrm{GL}(n,\mathbb{R}), N = N^T \right\}$$

A direct computation shows that the sets $\mathcal{D}, \mathcal{N}$ and $\overline{\mathcal{N}}$ are subgroups of $\mathrm{Sp}(2n,\mathbb{R})$. Moreover it holds that

$$\overline{\mathcal{N}}\mathcal{D}\mathcal{N} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n,\mathbb{R}) \,|\, \det A \neq 0 \right\}.$$

In [16, Prop. 4.10] is proved following property of the symplectic group.

Proposition 1.2.2. The symplectic group $\mathrm{Sp}(2n,\mathbb{R})$ is generated by $\mathcal{D} \cup \mathcal{N} \cup \{J_0\}$ or it is generated by $\mathcal{D} \cup \overline{\mathcal{N}} \cup \{J_0\}$.

1.3 Symplectic Lie algebra

Definition 1.3.1. The *symplectic Lie algebra* of the symplectic group has the matrix realization

$$\mathfrak{sp}(2n,\mathbb{R}) = \{A \in M_{2n}(\mathbb{R}) \,|\, A^T J_0 + J_0 A = 0\}.$$

Elements of the symplectic Lie algebra are endomorphisms A of the canonical symplectic vector space $(\mathbb{R}^{2n}, \Omega)$ satisfying

$$\Omega(Av, w) + \Omega(v, Aw) = 0,$$

for every $v, w \in \mathbb{R}^{2n}$.

A matrix realization of the $\mathfrak{sp}(2n,\mathbb{R})$ is

$$\begin{aligned} X_{jk} &= E_{j,k} - E_{n+k,n+j}, \\ Y_{jj} &= E_{j,n+j}, \\ Y_{jk} &= E_{j,n+k} + E_{k,n+j} \text{ for } j \neq k, \\ Z_{jj} &= E_{n+j,j}, \\ Z_{jk} &= E_{n+j,k} + E_{n+k,j} \text{ for } j \neq k, \end{aligned} \tag{1.2}$$

where $j,k = 1,\dots,n$ and $E_{j,k}$ is the $2n \times 2n$ matrix with 1 on the intersection of the j-th row and the k-th column and zero otherwise. Thus $\mathfrak{sp}(2n,\mathbb{R})$ is a linear span of matrices X_{jk}, Y_{jk}, Z_{jk} where $j,k = 1,\dots,n$, see, e.g. [19].

There is another useful representation of the symplectic Lie algebra $\mathfrak{sp}(2n,\mathbb{R})$ on complex valued polynomials in $2n$ real variables $\mathrm{Pol}(\mathbb{R}^{2n},\mathbb{C})$. Let $x_1,\dots,x_n$, $y_1,\dots,y_n$ be variables on $\mathbb{R}^{2n}$, we will denote by ∂_{x_j} and ∂_{y_j} partial differentiations instead of $\frac{\partial}{\partial x_j}$ and $\frac{\partial}{\partial y_j}$, $j,k = 1,\dots,n$. Then for $j,k = 1,\dots,n$ following endomorphisms of $\mathrm{Pol}(\mathbb{R}^{2n},\mathbb{C})$

$$\begin{aligned} X_{jk} &= x_j\partial_{x_k} - y_k\partial_{y_j}, \\ Y_{jj} &= x_j\partial_{y_j}, \\ Y_{jk} &= x_j\partial_{y_k} + x_k\partial_{y_j} \text{ for } j \neq k, \\ Z_{jj} &= y_j\partial_{x_j}, \\ Z_{jk} &= y_j\partial_{x_k} + y_k\partial_{x_j} \text{ for } j \neq k. \end{aligned} \tag{1.3}$$

form a faithful representation of the symplectic Lie algebra.

Definition 1.3.2. The *symplectic Clifford algebra* $Cl_s(\mathbb{R}^{2n},\Omega)$ is an associative unital algebra over $\mathbb{C}$ given by the quotient of the tensor algebra $T(\mathbb{R}^{2n})$ by a two-sided ideal $I \subset T(\mathbb{R}^{2n})$ generated by

$$v \cdot w - w \cdot v = -i\Omega(v,w)$$

for all $v,w \in \mathbb{R}^{2n}$, where Ω is the symplectic form and $i \in \mathbb{C}$ is the complex unit.

Let $\mathfrak{a}(2n)$ denote a subspace of the symplectic Clifford algebra $Cl_s(\mathbb{R}^{2n},\Omega)$ spanned by $v \cdot w + w \cdot v$ for $v,w \in \mathbb{R}^{2n}$. The following proposition, proved in [22, Lem. 1.1.6], relates the symplectic Clifford algebra and the symplectic Lie algebra.

Lemma 1.3.1. The space $\mathfrak{a}(2n)$ is a Lie subalgebra of $Cl_s(\mathbb{R}^{2n},\Omega)$. It is isomorphic to the symplectic Lie algebra $\mathfrak{sp}(2n,\mathbb{R})$.

The symplectic Clifford algebra $Cl_s(\mathbb{R}^{2n},\Omega)$ is isomorphic to the Weyl algebra W_{2n} of differential operators with polynomial coefficients on $\mathbb{R}^n$. The Weyl algebra is an associative algebra generated by $\{q_1,\dots,q_n,\partial_{q_1},\dots,\partial_{q_n}\}$, where q_j denotes multiplication operator by q_j, and partial differentiation by q_j is denoted by ∂_{q_j}, for $j = 1,\dots,n$. See [5] for more details about the Weyl algebra, also referred to as the common symplectic Clifford algebra.

The symplectic Lie algebra $\mathfrak{sp}(2n,\mathbb{R})$ can be realized as a subalgebra of the symplectic Clifford algebra, i.e. also as a subalgebra of W_{2n}.

Further, the symplectic Lie algebra $\mathfrak{sp}(2n, \mathbb{R})$ is realized by operators on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, for definition see Section 2.1,

$$\begin{aligned}
&X_{jk} = q_j \partial_{q_k} + \frac{1}{2}\delta_{jk}, \\
&Y_{jj} = -\frac{i}{2} q_j^2, \\
&Y_{jk} = -i q_j q_k \quad \text{for } j \neq k, \\
&Z_{jj} = -\frac{i}{2} \partial_{q_j}^2, \\
&Z_{jk} = -i \partial_{q_j} \partial_{q_k} \;\; \text{for } j \neq k,
\end{aligned} \tag{1.4}$$

where i is the imaginary unit and δ_{jk} denotes the Kronecker delta.

2. Introduction to function theory

In the present chapter, we review some basics of (Schwartz, Fréchet, nuclear) function theory and topological vector spaces needed in the definition of symplectic spinors.

We start by introducing the Schwartz function space and its distinguished orthonormal basis given by Hermite functions. We treat a wider framework of Fréchet function space, whose distinguished example is the Schwartz function space. Then we pass to the notion of topological tensor product and nuclear function spaces.

For a complete introduction with more details, proofs and examples we recommend books [4], [39].

2.1 Schwartz space

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ - the space of rapidly decreasing functions on $\mathbb{R}^n$ is a subspace of square integrable function $L^2(\mathbb{R}^n)$ considered with Hilbert scalar product $\langle f, g\rangle = \int\limits_{\mathbb{R}^n} f(x)\overline{g}(x)\,\mathrm{d}x$, for $f, g \in L^2(\mathbb{R}^n)$.

Moreover $\mathcal{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ for $0 < p < \infty$.

Definition 2.1.1. The *Schwartz space* is the function space

$$\mathcal{S}(\mathbb{R}^n) = \{f \in \mathcal{C}^\infty(\mathbb{R}^n) \mid \|f\|_{\alpha,\beta} < \infty \text{ for all } \alpha, \beta \in \mathbb{N}_0^n\},$$

where $\mathcal{C}^\infty(\mathbb{R}^n)$ denotes the space of all complex valued smooth functions on $\mathbb{R}^n$ and the semi-norms are defined by

$$\|f\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} |x^\alpha D_\beta f(x)|$$

with $x^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$. The symbol D_β denotes differentiation in relevant variables and of a given order, i.e., $D_\beta = \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}}$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n$, and $\alpha, \beta \in \mathbb{N}_0^n$ are multiindexes.

The Schwartz space decomposes into two parts, of odd and even functions. Let us denote by $\mathcal{S}(\mathbb{R}^n)^-$ odd part and by $\mathcal{S}(\mathbb{R}^n)^+$ even part of $\mathcal{S}(\mathbb{R}^n)$.

Definition 2.1.2. The *Fourier transformation* $\mathcal{F}f$ of a complex valued integrable function $f \in L^1(\mathbb{R}^n)$ is defined by

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = \int\limits_{\mathbb{R}^n} e^{-i2\pi\langle x,\xi\rangle} f(x)\,\mathrm{d}x.$$

The *inverse Fourier transformation* $\mathcal{F}^{-1}f$ of f is

$$(\mathcal{F}^{-1}f)(x) = \check{f}(x) = \int\limits_{\mathbb{R}^n} e^{i2\pi\langle x,\xi\rangle} f(\xi)\,\mathrm{d}\xi,$$

where x, $\xi \in \mathbb{R}^n$ and $\langle\cdot,\cdot\rangle$ is the canonical inner product on $\mathbb{R}^n$.

Theorem 2.1.1. The Fourier transformation $\mathcal{F}$ and the inverse Fourier transformation $\mathcal{F}^{-1}$ are mutually inverse continuous endomorphisms when restricted to $\mathcal{S}(\mathbb{R}^n)$.

2.2 Hermite functions and polynomials

The basic tool needed to work with vector spaces in an explicit way is its basis. We shall briefly expose one specific choice of a complete orthogonal system for $\mathcal{S}(\mathbb{R})$ given by Hermite functions and associated Hermite polynomials. Here, we consider $\mathcal{S}(\mathbb{R})$ as a pre-Hilbert space with respect to the scalar product on $L^2(\mathbb{R})$. This collection yields a complete orthogonal system of $L^2(\mathbb{R})$-space weighted by e^{-x^2}.

Definition 2.2.1. For $n \in \mathbb{N}_0$ we define the n-th Hermite polynomial by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

An explicit form is by [33]

$$H_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k n!}{k!(n-2k)!} (2x)^{n-2k},$$

where $\lfloor \cdot \rfloor$ is the floor function. In particular, the first several Hermite polynomials are

$$\begin{aligned} H_0(x) &= 1, \\ H_1(x) &= 2x, \\ H_2(x) &= 4x^2 - 2, \\ H_3(x) &= 8x^3 - 12x, \\ H_4(x) &= 16x^4 - 48x^2 + 12. \end{aligned}$$

Since the action of the operator $-e^{x^2} \frac{d}{dx} e^{-x^2}$ and $e^{\frac{x^2}{2}} \left(x - \frac{d}{dx}\right) e^{-\frac{x^2}{2}}$ on an arbitrary function $f(x)$ provides the same result, i.e.

$$-e^{x^2} \frac{d}{dx}(e^{-x^2} f(x)) = 2x f(x) - \frac{d}{dx} f(x) = e^{\frac{x^2}{2}} \left(x - \frac{d}{dx}\right) e^{-\frac{x^2}{2}} f(x),$$

we infer an equivalent definition of Hermite polynomial

$$H_1(x) = e^{\frac{x^2}{2}} \left(x - \frac{d}{dx}\right) e^{-\frac{x^2}{2}}$$

and by the induction principle we obtain

$$H_n(x) = e^{\frac{x^2}{2}} \left(x - \frac{d}{dx}\right)^n e^{-\frac{x^2}{2}}.$$

Definition 2.2.2. For $n \in \mathbb{N}_0$, the normalized *n-th Hermite function* is

$$\psi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{-\frac{x^2}{2}} H_n(x).$$

Equivalently,

$$\psi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi})}} \left(x - \frac{d}{dx}\right)^n e^{-\frac{x^2}{2}}.$$

2.3 Fréchet spaces

Let us now highlight the concept of topology on a vector space defined by a family of semi-norms. As a topological vector space, we always suppose a Hausdorff topological vector space.

Definition 2.3.1. Let X be a vector space and $\mathcal{P} = \{p_i \,|\, i \in I \subset \mathbb{N}\}$ be a family of semi-norms on X. *An open ball defined by the family of semi-norms $\mathcal{P}$ of radius* $\varepsilon = \{\varepsilon_i\}_{i\in I}$ and *center* $x_0 \in X$ is the set of all points $x \in X$ such that $\varepsilon_i > p_i(x - x_0), \varepsilon_i > 0,\ i \in J$ for every finite subset $J \subset I$. The basis of the *topology defined by the family of semi-norms $\mathcal{P}$* is generated by all open balls defined by the family of semi-norms $\mathcal{P}$.

There exists a translation invariant metric $\rho_{\mathcal{P}}$ such that topology defined by $\rho_{\mathcal{P}}$ on a vector space X is identical with the topology defined by countable family of semi-norms $\mathcal{P} = \{p_i \,|\, i \in I \subset \mathbb{N}\}$, with the property that for every $0 \neq x \in X$ there exists $i \in I$ that $p_i(x) \neq 0$.

Definition 2.3.2. A topological vector space (X, τ) is said to be *locally convex* if there is a basis of neighborhoods in X consisting of convex sets. The topology τ is then called the *locally convex topology.*

A topological vector space with topology defined by a countable family of semi-norms is locally convex. Conversely, it can be proved that the topology of a locally convex vector space can always be defined by a family of semi-norms. See [4, p. 424].

Definition 2.3.3. A topological vector space X is the *Fréchet space* if X is complete and its topology is induced by a countable family of semi-norms.

Equivalently, a topological vector space is a Fréchet space if it is complete, locally convex and its topology can be defined by a translation invariant metric.

Example 2. The space $\mathbb{R}^n$ with classical Euclidean metric is a Fréchet space.

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing functions is a Fréchet space with family of semi-norms from the Definition 2.1.1,

$$\|f\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} |x^\alpha D_\beta f(x)|$$

with multiindexes $\alpha, \beta \in \mathbb{N}_0$.

Another class of Fréchet spaces are the function spaces with suitable family of semi-norms. For example, $\mathcal{C}^\infty(\mathbb{R})$ with the family of semi-norms $\|f\|_{k,n} = \sup\{|D^{(k)}f(x)| \,|\, x \in [-n, n]\}$ for all $n, k \in \mathbb{N}_0$, is a Fréchet space.

Definition 2.3.4. Let V and W be two Fréchet spaces and $\{p_i\}_{i\in\mathbb{N}_0}$ be a family of semi-norms defining a structure of the Fréchet space on W. A mapping $F : V \to W$ is said to be *smooth* if $p_i \circ F : V \to \mathbb{C}$ is smooth for each $i \in \mathbb{N}_0$.

2.4 Topological tensor products

In this section, X and Y are two locally convex topological vector spaces and $X \otimes Y$ their tensor product. Let us remind that for us, topological vector spaces X, Y are Hausdorff.

Definition 2.4.1. We call *π-topology* or *projective topology* on $X \otimes Y$ the strongest locally convex topology on this vector space for which the mapping $(x, y) \mapsto x \otimes y$ of $X \times Y$ into $X \otimes Y$ is continuous. Let $X \otimes_\pi Y$ denote the space $X \otimes Y$ with projective topology. The completion of $X \otimes_\pi Y$ is called the *projective tensor product* of X and Y and it will be denoted by $X \hat{\otimes}_\pi Y$.

A subset of $X \otimes Y$ is a neighbourhood of zero in π-topology if and only if its preimage under $(x, y) \mapsto x \otimes y$ contains a neighbourhood of zero in $X \times Y$, i.e. if it contains a set of the form $U \otimes V = \{x \otimes y \in X \otimes Y \,|\, x \in X, y \in Y\}$ where U and V are a neighbourhoods of zero in X and Y respectively.

Let us denote by X'_σ, Y'_σ the weak duals to X and Y. A subset $H \subseteq L(E, F)$ of linear mappings between two locally convex space E, F is called *equicontinuous* if for each neighborhood V of the zero vector in F there exists a neighborhood U of the zero vector in E such that $g(U) \subseteq V$ for each $g \in H$.

A bilinear mapping $f : X \times Y \to G$ with partial mappings $f_x : y \mapsto f(x, y)$ and $f_y : x \mapsto f(x, y)$ is said to be *separately continuous* if both f_x and f_y are continuous. We denote by $B(X, Y)$ the space of continuous bilinear mappings of $X \times Y$ into scalar field $\mathbb{C}$ (or $\mathbb{R}$) and $\mathcal{B}(X, Y)$ the space of separately continuous bilinear mappings of $X \times Y$ into scalar field. Let us notice that $B(X'_\sigma, Y'_\sigma)$ is isomorphic to $X \otimes Y$.

Definition 2.4.2. Let us call *ε-topology* or *injective topology* on $X \otimes Y$ the topology induced by $B(X'_\sigma, Y'_\sigma)$ considered as a vector subspace of $\mathcal{B}(X'_\sigma, Y'_\sigma)$, the space of separately continuous bilinear forms on $X'_\sigma \times Y'_\sigma$ equipped with the topology of uniform convergence on the product of equicontinuous subsets of X' and equicontinuous subsets of Y'. Let $X \otimes_\varepsilon Y$ denote the space $X \otimes Y$ with injective topology. The *injective tensor product* of X and Y is the completion of $X \otimes_\varepsilon Y$ and we will denote it by $X \hat{\otimes}_\varepsilon Y$.

Nuclear space is a space with a lot of of good properties of finite-dimensional vector space.

Definition 2.4.3. A locally convex topological vector space X is *nuclear* if and only if for every locally convex topological vector space Y, the topological vector spaces $X \hat{\otimes}_\pi Y$ and $X \hat{\otimes}_\varepsilon Y$ are isomorphic.

Consequently, for a nuclear space X we write just $X \hat{\otimes} Y$ instead of $X \hat{\otimes}_\pi Y$ or $X \hat{\otimes}_\varepsilon Y$.

Example 3. The space $\mathrm{Pol}(\mathbb{C}^n)$ of polynomials in n variables is nuclear space. Its topology is the inductive limit of inner product topologies on a finite dimensional spaces of polynomials of a fixed maximal order. For details, see [39, p. 526].

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and the space of smooth functions $\mathcal{C}^\infty(U)$ on any open subset $U \subset \mathbb{R}^n$ are nuclear.

Let us take $f_m \in \mathcal{S}(\mathbb{R}^m)$, $f_n \in \mathcal{S}(\mathbb{R}^n)$. From algebraic point of view, we can insight the following structure by pullbacks of projections. Denote

$$\pi_m : \mathbb{R}^{m+n} \to \mathbb{R}^m, \quad \pi_n : \mathbb{R}^{m+n} \to \mathbb{R}^n$$

two canonical projections. Then the product of pullbacks

$$(\pi_m^* f_m)(\pi_n^* f_n) \in \mathcal{S}(\mathbb{R}^{m+n}).$$

The tensor product $\mathcal{S}(\mathbb{R}^m) \otimes \mathcal{S}(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n)$. Moreover we have the following for completion of the tensor product.

Theorem 2.4.1. There is a canonical isomorphism

$$\mathcal{S}(\mathbb{R}^m)\hat{\otimes}\mathcal{S}(\mathbb{R}^n) \cong \mathcal{S}(\mathbb{R}^{m+n}).$$

The reason for the existence of the isomorphism is that the Schwartz space is both Fréchet and nuclear space. See [39] for more details of the isomorphism.

3. Segal-Shale-Weil representation

In this section, we want to present some facts from representation theory needed to introduce the Segal-Shale-Weil representation. This representation plays the same role as the spin representation of the spin group - the double cover of the orthogonal group, used in the Riemannian geometry. The symplectic group $\mathrm{Sp}(2n,\mathbb{R})$ has a double covering called metaplectic group $\mathrm{Mp}(2n,\mathbb{R})$.

The Segal-Shale-Weil representation is an infinite dimensional uniraty representation of the metaplectic group $\mathrm{Mp}(2n,\mathbb{R})$ on the space of all complex valued square integrable function $L^2(\mathbb{R}^n)$. We use mostly the Schwartz space $\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$, the space of smooth vectors of the Segal-Shale-Weil representation, as the symplectic spinor space in this thesis.

Let us recall $\mathbb{R}^n$ and $(\mathbb{R}^n)'$ two canonical Lagrangian subspaces of the canonical symplectic space $(\mathbb{R}^{2n},\Omega)$, such that $\mathbb{R}^n \oplus (\mathbb{R}^n)' = \mathbb{R}^{2n}$.

3.1 Heisenberg group and its Schrödinger representation

Sometimes by the term Heisenberg group is meant the group of upper triangular 3×3 matrices with ones on a diagonal. But there will be presented a more abstract definition of the Heisenberg group and its Schrödinger representation, which will be used to construct the Segal-Shale-Weil representation. For more details about the Schrödinger representation we refer to [16].

Definition 3.1.1. Let (V,ω) be a symplectic vector space. The *Heisenberg group* $\mathrm{H}(V)$ is a group with a support set $V \times \mathbb{R}$ and group operation

$$(v,x)\cdot(w,y) = \Big(v+w, x+y+\frac{1}{2}\omega(v,w)\Big),$$

where (v,x), $(w,y) \in V \times \mathbb{R}$. Let us denote $\mathrm{H}(n)$ the Heisenberg group $\mathrm{H}(\mathbb{R}^{2n})$ for the canonical symplectic vector space $(\mathbb{R}^{2n},\Omega)$.

Remark 1. The Heisenberg group $\mathrm{H}(n)$ can be represented by the following realization. Let be $v \in \mathbb{R}^n$, $w \in (\mathbb{R}^n)'$, i.e. $(v,w) \in \mathbb{R}^{2n}$, and $t \in \mathbb{R}$, then element $(v,w,t) \in \mathrm{H}(n)$ can be represented by a matrix

$$\begin{pmatrix} 1 & v & t \\ 0 & I & w \\ 0 & 0 & 1 \end{pmatrix}.$$

Here I denote unit $n \times n$ matrix, v is row and w is column n-vector. The set of such matrices is a group with respect to the classic matrix multiplication and an identity matrix as the unit element. The isomorphism with a group defined in the Definition 3.1.1 is

$$\begin{pmatrix} 1 & v & t \\ 0 & I & w \\ 0 & 0 & 1 \end{pmatrix} \mapsto \Big((v,w), t-\frac{1}{2}\langle v,w\rangle\Big).$$

The center of the Heisenberg group is

$$Z(\mathrm{H}(n)) = \{(v, x) \mid v = 0\} \cong \mathbb{R}.$$

Definition 3.1.2. Let G be a Lie group, W be a topological vector space and $\mathrm{Aut}(W)$ denote all continuous linear mappings from W onto W with continuous inverse. A homomorphism of groups $\mathfrak{r} : G \to \mathrm{Aut}(W)$ is called *representation* of a group G, if $\tilde{\mathfrak{r}} : G \times W \to W$ defined by $\tilde{\mathfrak{r}}(g, v) = \mathfrak{r}(g)v$ is continuous. The space W is called *G-module* or *representation space*.

The representation is called *unitary* if W is a Hilbert space and $\mathrm{Im}(\mathfrak{r}) \subseteq \mathrm{U}(W)$, where $\mathrm{U}(W)$ denotes a group of unitary operators on W, i.e. the group of operators A satisfying $\overline{A^T}A = \mathrm{Id}$, where Id is an identity operator.

Let us notice, that $G \times W$ is considered with the product topology. The set $\mathrm{Aut}(W)$ is a group, because a composition of two mapping with a continuous inversion, is continuous with continuous inversion and an unit element of the group is the identity mapping.

Let us remind that a mapping $\psi : M \to N$ between two manifolds of dimensions m, n with atlases $\{U^M_\alpha, \varphi^M_\alpha\}_{\alpha \in A}, \{U^N_\beta, \varphi^N_\beta\}_{\beta \in B}$ is called smooth ($\mathcal{C}^\infty$) if the mapping $\varphi^N_\beta \circ \psi \circ (\varphi^M_\alpha)^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is smooth for all α, β from index sets A, B of atlases of M and N.

A mapping $\psi : G \to W$, where G is a Lie group and W is a complex topological vector space, is smooth if for every $v^* \in W^*$ is $g \mapsto v^*\big(\psi(g)\big) \in \mathbb{C}$ a smooth mapping as a mapping from a manifold to the complex numbers. (A symbol W^* denotes the continuous dual of a space W.)

Definition 3.1.3. Let $\mathfrak{r}$ be a representation of a group G on a topological vector space W.

1) A space $W' \subset W$ is called *G-invariant* if $\mathfrak{r}(a)W' \subset W'$ for every $a \in G$.

2) A representation $\mathfrak{r}$ is called *irreducible* if the only G-invariant closed subspaces of W are $\{0\}$ and W.

Let $\{x_1, \dots, x_n\}$ be a coordinate chart on $\mathbb{R}^n$. Let us denote by $X = \sum_{j=1}^n X^j$ a sum of operators of multiplication by x_j, i.e. $X^j f = x_j f$. By $D = \sum_{j=1}^n D_j$ is denoted a sum of operators $D_j f = \frac{1}{2\pi i}\frac{\partial f}{\partial x_j}$ where $f \in L^2(\mathbb{R}^n)$. The operator D is obviously unbounded on $L^2(\mathbb{R}^n)$. Finely, Id means an appropriate identity operator.

Let us notice that for a suitable operator T, the exponential of T is

$$e^T = \sum_{j=0}^{\infty} \frac{T^j}{j!}.$$

Definition 3.1.4. The *Schrödinger representation* of $\mathrm{H}(\mathbb{R}^n \oplus (\mathbb{R}^n)')$ on the Hilbert space $L^2(\mathbb{R}^n)$ is a homomorphism $\mathfrak{r}_S : \mathrm{H}(\mathbb{R}^n \oplus (\mathbb{R}^n)') \to \mathrm{Aut}\big(L^2(\mathbb{R}^n)\big)$ defined by

$$\mathfrak{r}_S(p, q, t) = e^{2\pi i(t\,\mathrm{Id} + pD + qX)},$$

where $(p, q, t) \in \mathrm{H}(\mathbb{R}^n \oplus (\mathbb{R}^n)')$.

It holds that $e^{t\,\mathrm{Id}}f(x) = e^t f(x)$ and $e^{2\pi i(pD+qX)}f(x) = e^{2\pi iqx+\pi ipq}f(x+p)$, $x \in \mathbb{R}^n$, $f \in L^2(\mathbb{R}^n)$. The Schrödinger representation is thus given by

$$\big(\mathfrak{r}_S(p,q,t)f\big)(x) = e^{2\pi i(t+qx+\frac{1}{2}pq)}f(x+p),$$

where $(p,q,t) \in \mathrm{H}(\mathbb{R}^n \oplus (\mathbb{R}^n)')$, $f : \mathbb{R}^n \to \mathbb{C}$, $x \in \mathbb{R}^n$. It is continuous in the sense of Definition 3.1.2 as may be seen from the formula.

Let us notice that on elements of the center of the Heisenberg group the representation acts by

$$\mathfrak{r}_S(0,0,t)f = e^{2\pi it}f,$$

for every $t \in \mathbb{R}$ and $f \in L^2(\mathbb{R}^n)$. Hence an action of the central element is only multiplication by scalar. The following property is shown in [16, p. 22].

Proposition 3.1.1. The Schrödinger representation is unitary.

In the following, we show that the Schrödinger representation is an irreducible representation. To this aim we need several knowledge from representation theory of infinite dimension. For more details and proves, see [25].

Let us introduce a mapping $V : L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, sometimes called the *Fourier-Wigner transform*, given by

$$[V(f,g)](p,q) = \langle \mathfrak{r}_S(p,q,0)f, g\rangle,$$

where $f, g \in L^2(\mathbb{R}^n)$, $(p,q) \in \mathbb{R}^{2n}$ and $\langle\cdot,\cdot\rangle$ is the Hilbert scalar product on $L^2(\mathbb{R}^n)$. For more details about transform V, see [16].

The Fourier-Wigner transform satisfies

$$\langle V(f_1,g_1), V(f_2,g_2)\rangle = \langle f_1, f_2\rangle\overline{\langle g_1, g_2\rangle} \tag{3.1}$$

for every f_1, f_2, g_1 and $g_2 \in L^2(\mathbb{R}^n)$.

Definition 3.1.5. Two representations $\mathfrak{r}_1 : G \to \mathrm{Aut}(W_1)$ and $\mathfrak{r}_2 : G \to \mathrm{Aut}(W_2)$ are called *equivalent* if there exists a continuous linear isomorphism $T : W_1 \to W_2$, with continuous inverse satisfying

$$T \circ \mathfrak{r}_1(a) = \mathfrak{r}_2(a) \circ T$$

for every $a \in G$. The mapping T is called *intertwining* (sometimes *equivariant*) *operator*. If in additional, W_1 and W_2 are Hilbert spaces, $\mathfrak{r}_1$, $\mathfrak{r}_2$ are unitary representations and the operator T can be chosen unitary, then representations $\mathfrak{r}_1$, $\mathfrak{r}_2$ are called *unitary equivalent*.

The following theorem is a version of Schur Lemma for infinite dimensional representations and will be used in the construction of the Segal-Shale-Weil representation.

Theorem 3.1.2. (Schur) Let $\mathfrak{r}_1, \mathfrak{r}_2$ be two unitary irreducible representations of a Lie group G on a complex Hilbert spaces W_1 and W_2, respectively. If $\mathfrak{r}_1$ is not equivalent to $\mathfrak{r}_2$ then $\mathrm{Hom}_G(W_1, W_2) = 0$. If $\mathfrak{r}_1$ and $\mathfrak{r}_2$ are equivalent then $\mathrm{Hom}_G(W_1, W_2) \simeq \mathbb{C}$. In addition, if $\mathfrak{r}_1$ and $\mathfrak{r}_2$ are equivalent then every element of $\mathrm{Hom}_G(W_1, W_2)$ is possible to normalise by a multiplication by a scalar into an isometry.

Definition 3.1.6. Let G be a Lie group and $\mathfrak{r}$ be a representation of G on a Fréchet space W. A vector $w \in W$ is *smooth vector* for representation $\mathfrak{r}$ if $g \mapsto \mathfrak{r}(g)w \in W$, $g \in G$, is a smooth mapping $G \to W$.

The set of smooth vectors is a vector space due to the linearity of differentiation. Let us denote by W^0 a vector space of all smooth vectors in W.

Proposition 3.1.3. (Gårding) Let $W^0 \subseteq W$ be a vector subspace of smooth vectors of a representation space W of a unitary representation $\mathfrak{r}$. Then W^0 is a dense subspace of W.

Definition 3.1.7. Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras. A Linear mapping $\varphi : \mathfrak{g} \to \mathfrak{h}$ is called *homomorphism of Lie algebras* if

$$\varphi\big([X, Y]\big) = [\varphi(X), \varphi(Y)],$$

for every $X, Y \in \mathfrak{g}$. The homomorphism of the Lie algebra $\mathfrak{g}$ to $\mathfrak{gl}(V) = \mathrm{End}(V)$ is called *representation of* $\mathfrak{g}$.

For a proof of the following Proposition, see [25, Prop. 3.9].

Proposition 3.1.4. Let $\mathfrak{g}$ be a Lie algebra of a Lie group G. Let W^0 be a space of smooth vectors of a representation $\mathfrak{r}$ of a group G. A mapping $\mathrm{d}\mathfrak{r} : \mathfrak{g} \to \mathrm{End}(W^0)$ given by

$$\mathrm{d}\mathfrak{r}(X)w = \frac{d}{dt}\Big|_{t=0}[\mathfrak{r}(e^{tX})w],$$

where $X \in \mathfrak{g}$ and $w \in W^0$, is a representation of a Lie algebra $\mathfrak{g}$.

In general $W_0 \subsetneq W$, as it is possible to show on examples. However, it is possible to define a representation only on a smooth vectors of that representation.

Proposition 3.1.5. A vector space $W^0 \subseteq W$ of smooth vectors of a representation $\mathfrak{r}$ is closed on the action of the representation.

A space W^0 is closed not only on the action of the Lie algebra $\mathfrak{g}$, but also on the action of the group G.

Back to the Schrödinger representation. We recall the proof of the well known fact.

Proposition 3.1.6. The Schrödinger representation $\mathfrak{r}_S$ is irreducible.

Proof. Let $0 \neq \mathcal{M} \subseteq L^2(\mathbb{R}^n)$ be a closed invariant space for $\mathfrak{r}_S$ on $L^2(\mathbb{R}^n)$. Let $0 \neq f \in \mathcal{M}^0$, where $\mathcal{M}^0 \subseteq \mathcal{M}$ is subspace of smooth vectors. Let be $g \perp \mathcal{M}$. Due to choice of f is $g \perp f$. Hence $\mathcal{M}^0$ is closed not only on an action of $\mathrm{d}\mathfrak{r}_S$ (see Proposition 3.1.4), but also on $\mathfrak{r}_S$ (see Proposition 3.1.5), it holds that $g \perp e^{2\pi i(pD+qX)}f$ for every $(p, q) \in \mathbb{R}^{2n}$. (A scalar multiplication by $e^{2\pi it}$ in the formula of representation does not cased anything on perpendicularity.) Therefore $V(f, g) = 0$. By equation (3.1) is $0 = \|V(f, g)\|^2 = \|f\|^2\|g\|^2$. The assumption $f \neq 0$ gives $g = 0$, therefore is $\mathcal{M} = L^2(\mathbb{R}^n)$ hence $\mathcal{M}$ is closed. □

3.2 Stone-von Neumann Theorem

The Stone-von Neumann Theorem will be used to construct a projective representation of the symplectic group $\mathrm{Sp}(2n,\mathbb{R})$.

Theorem 3.2.1. (Stone-von Neumann) Let $\mathfrak{r}$ be an irreducible unitary representation of the Heisenberg group $\mathrm{H}(n)$ on a separable complex Hilbert space W such that

$$\mathfrak{r}(0,t)w = e^{-2\pi i t}w$$

for every $t \in \mathbb{R}$ and $w \in W$. Then $\mathfrak{r}$ is unitary equivalent to the Schrödinger representation $\mathfrak{r}_S$.

For proof of the Stone-von Neumann Theorem, see [16, Th. 1.50].

For every $g \in \mathrm{Sp}(2n,\mathbb{R})$, let us define a representation of the Heisenberg group $\mathfrak{r}_S^g$

$$\mathfrak{r}_S^g : \mathrm{H}(n) \to \mathrm{Aut}\left(L^2(\mathbb{R}^n)\right), \quad \mathfrak{r}_S^g = \mathfrak{r}_S \circ a^g,$$

where $\mathfrak{r}_S$ is the Schrödinger representation and a mapping $a^g : \mathrm{H}(n) \to \mathrm{H}(n)$ is defined by

$$a^g(v,t) = \big(g(v),t\big),$$

where $v \in \mathbb{R}^{2n}$ and $t \in \mathbb{R}$. The mapping a^g is a group homomorphism since

$$\begin{aligned} a^g\big[(v,t)\cdot(w,s)\big] &= a^g\Big(v+w, t+s+\frac{1}{2}\omega(v,w)\Big) = \Big(g(v+w), t+s+\frac{1}{2}\omega(v,w)\Big) \\ &= \Big(gv+gw, t+s+\frac{1}{2}\omega(gv,gw)\Big) = (gv,t)\cdot(gw,s) = a^g(v,t)\cdot a^g(w,s). \end{aligned}$$

for every (v,t),$(w,s) \in \mathrm{H}(n)$.

The mapping $\mathfrak{r}_S^g$ is a representation since it is a composition of two homomorphisms. Obviously,

$$\mathfrak{r}_S^g(0,s)f = \mathfrak{r}_S(0,s)f = e^{-2\pi i s}f$$

for every $s \in \mathbb{R}$ and $f \in L^2(\mathbb{R}^n)$. Hence a^g is an automorphism of the group $\mathrm{H}(n)$ and the Schrödinger representation is unitary and irreducible, the representation $\mathfrak{r}_S^g$ is unitary and irreducible. Continuity in the sense of Definition 3.1.2 is obvious.

All irreducible unitary representations of the Heisenberg group with the same action (up to scalar factor) of the center element of $\mathrm{H}(n)$ are unitary equivalent due to Stone-von Neumann Theorem 3.2.1 . It means that there exists an intertwining operator $U(g)$ such that

$$U(g) \circ \mathfrak{r}_S(v,s) = \mathfrak{r}_S^g(v,s) \circ U(g) \tag{3.2}$$

for every $(v,s) \in \mathrm{H}(n)$ and $g \in \mathrm{Sp}(2n,\mathbb{R})$.

Proposition 3.2.2. A mapping $g \mapsto U(g)$, $g \in \mathrm{Sp}(2n,\mathbb{R})$, induces a projective unitary representation $U : \mathrm{Sp}(2n,\mathbb{R}) \to \mathrm{U}\big(L^2(\mathbb{R}^n)\big)$, i.e. there exists a map $\gamma : \mathrm{Sp}(2n,\mathbb{R}) \times \mathrm{Sp}(2n,\mathbb{R}) \to S^1 = \mathrm{U}(1)$ such that for every $a,b \in \mathrm{Sp}(2n,\mathbb{R})$ it holds that

$$U(ab) = \gamma(a,b)U(a)U(b).$$

Proof. Take $a, b \in \mathrm{Sp}(2n, \mathbb{R})$ and $(v, s) \in \mathrm{H}(n)$ then by (3.2) are

$$U(a)\Big(U(b)\mathfrak{r}_S(v,s)U(b)^{-1}\Big)U(a)^{-1} = U(a)\mathfrak{r}_S^b(v,s)U(a)^{-1} = \mathfrak{r}_S^{ab}(v,s),$$

$$U(ab)\mathfrak{r}_S(v,s)U(ab)^{-1} = \mathfrak{r}_S^{ab}(v,s).$$

Operators $U(a)U(b)$ and $U(ab)$ intertwine the same representation. By Schur Lemma 3.1.2 it is possible to normalise both operators to be isometry therefore $U(ab) = \gamma(a,b)U(a)U(b)$ where $\gamma(a,b)$ is a complex unit. □

The group $\mathrm{Sp}(2n, \mathbb{R})$ is generated by a matrix J_0 and matrices of the form $\left(\begin{smallmatrix} D & 0 \\ 0 & (D^{-1})^T \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} I & N \\ 0 & I \end{smallmatrix}\right)$, where $D \in \mathrm{GL}(n, \mathbb{R})$ and N is symmetric square $n \times n$ matrix see Proposition 1.2.2. Therefore it is sufficient to determine a representation U on this generators. A mapping

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \mapsto \mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \tag{3.3}$$

$$\begin{pmatrix} D & 0 \\ 0 & (D^{-1})^T \end{pmatrix} \mapsto \big(f(x) \mapsto (\det D)^{\frac{1}{2}} f(D^T x)\big) \tag{3.4}$$

$$\begin{pmatrix} I & N \\ 0 & I \end{pmatrix} \mapsto \big(f(x) \mapsto e^{-i\frac{1}{2}\langle Nx, x\rangle} f(x)\big) \tag{3.5}$$

defines one of possible projective representation of the group $\mathrm{Sp}(2n, \mathbb{R})$. Where $\mathcal{F}$ denotes a Fourier transform and $\langle \cdot, \cdot \rangle$ denotes a canonical scalar product on $\mathbb{R}^n$. See [22] for more details.

3.3 Metaplectic group and Segal-Shale-Weil representation

The metaplectic group is a double covering of the symplectic group, hence we start with reminder of a term covering and its relation to fundamental group of a space. We refer an interested reared to a classical book [23]. Then we pass to Segal-Shale-Weil representation of the metaplectic group. As a sources of this topic we recommend [16], [22] and [40].

Definition 3.3.1. Let X be a topological space. The *covering space* of a space X is a space C with continuous surjective mapping $p : C \to X$, such that for every $x \in X$ there exists an open neighbourhood V of x such that $p^{-1}(V)$ is disjoint union of open sets in C, each of which is mapped homeomorphically onto V by the mapping p.

Let X be a connected space. The covering $p : C \to X$ of a space X is called *n-fold covering* of a space X, if for every $x \in V$ is $p^{-1}(V)$ disjoint union of $n \in \mathbb{N}$ open sets homeomorphic with V.

Lemma 3.3.1. Let X and C be path connected spaces and let $p : C \to X$ be a covering of X. Then an induced mapping $\pi_1 p : \pi_1(C) \to \pi_1(X)$ between fundamental groups $\pi_1(C)$ and $\pi_1(X)$ of spaces C and X is injective. In addition, the number of folds of $p : C \to X$ is equal to an index of the group $(\pi_1 p)\big(\pi_1(C)\big)$ in the group $\pi_1(X)$.

The symplectic group $\mathrm{Sp}(2n,\mathbb{R})$ is path connected and its fundamental group is $\pi_1\big(\mathrm{Sp}(2n,\mathbb{R})\big) = \mathbb{Z}$, see Section 1.2. Since the group $\mathbb{Z}$ has only one subgroup of index 2, there exists by Lemma 3.3.1 only one connected twofold covering of the group $\mathrm{Sp}(2n,\mathbb{R})$ up to a homomorphism. Let us denote this double covering

$$\lambda : \mathrm{Mp}(2n,\mathbb{R}) \to \mathrm{Sp}(2n,\mathbb{R}). \tag{3.6}$$

Definition 3.3.2. The covering space (3.6) of the symplectic group $\mathrm{Sp}(2n,\mathbb{R})$ is called the *metaplectic group* $\mathrm{Mp}(2n,\mathbb{R})$.

The metaplectic group is a Lie group, which does not have any faithful finite dimensional representation, therefore there does not exist any matrix realization of the group $\mathrm{Mp}(2n,\mathbb{R})$.

It was proved by a "cocycle computing" in [40] that the projective unitary representation U of the group $\mathrm{Sp}(2n,\mathbb{R})$, which is described in the end of the Chapter 3.2, lifts to a unitary representation of the metaplectic group $\mathrm{Mp}(2n,\mathbb{R})$. For following representation, see [40] and [22, Prop. 1.3.5].

Proposition 3.3.2. There exists a unique unitary representation

$$\mathfrak{m} : \mathrm{Mp}(2n,\mathbb{R}) \to \mathrm{U}(L^2(\mathbb{R}^n))$$

which satisfies

$$\mathfrak{m}(g) \circ \mathfrak{r}_S(v,s) = \mathfrak{r}_S\big(\lambda(g)v, s\big) \circ \mathfrak{m}(g)$$

for every $g \in \mathrm{Mp}(2n,\mathbb{R})$ and $(v,s) \in \mathrm{H}(n)$.

The unitary representation $\mathfrak{m}$ is called *Segal-Shale-Weil representation.* For proof of following two Propositions, see [16], [36] and references therein.

Proposition 3.3.3. The Segal-Shale-Weil representation $\mathfrak{m}$ of the metaplectic group is faithful, i.e. $\mathfrak{m} : \mathrm{Mp}(2n,\mathbb{R}) \to \mathrm{U}(L^2(\mathbb{R}^n))$ is injective. The representation $\mathfrak{m}$ decomposes into the sum of two inequivalent irreducible unitary representations, which are restrictions of $\mathfrak{m}$ to the subspaces of even and odd functions in $L^2(\mathbb{R}^n)$.

Proposition 3.3.4. The space of smooth vectors of the Segal-Shale-Weil representation $\mathfrak{m}$ is precisely the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. In particular $\mathcal{S}(\mathbb{R}^n)$ is $\mathfrak{m}$-invariant.

The elements of the representation space $L^2(\mathbb{R}^n)$ are called *symplectic spinors*, since they play the same role as elements of a spinor representation of the orthogonal group in the Riemannian case.

Let us define the symplectic Clifford multiplication, which allows us to multiply symplectic spinors by vectors as in the Riemannian case.

Definition 3.3.3. The *symplectic Clifford multiplication*

$$\cdot : \mathbb{R}^{2n} \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$$

is defined on the elements of base $\{e_1, \dots, e_{2n}\}$ of the symplectic vector space $(\mathbb{R}^{2n}, \Omega)$ for $f \in \mathcal{S}(\mathbb{R}^n)$, $q = (q^1, \dots, q^n) \in \mathbb{R}^n$ and $j = 1, \dots, n$ by

$$(e_j \cdot f)(q) = iq^j f(q),$$

$$(e_{n+j} \cdot f)(q) = \partial_{q^j} f(q),$$

where ∂_{q_j} denotes partial derivative with respect to q_j. We expand the symplectic Clifford multiplication on the other elements of $\mathbb{R}^{2n}$ linearly.

We will usually write $v \cdot w \cdot f$ instead of $v \cdot (w \cdot f)$ for $v, w \in \mathbb{R}^{2n}$ and $f \in \mathcal{S}(\mathbb{R}^n)$.

Next we will mention two properties of the symplectic Clifford multiplication, for more details and proofs we refer to [22]. The symplectic Clifford multiplication is an unbounded operator on $L^2(\mathbb{R}^n)$.

Lemma 3.3.5. For every $v, w \in (\mathbb{R}^{2n}, \Omega)$ and $f \in \mathcal{S}(\mathbb{R}^n)$, we have

$$v \cdot w \cdot f - w \cdot v \cdot f = -i\Omega(v, w)f. \tag{3.7}$$

Lemma 3.3.6. The symplectic Clifford multiplication is $\mathrm{Mp}(2n, \mathbb{R})$-equivariant, i.e. for every $g \in \mathrm{Mp}(2n, \mathbb{R})$, $v \in \mathbb{R}^{2n}$ and $f \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\big(\lambda(g)v\big) \cdot \mathfrak{m}(g)f = \mathfrak{m}(g)(v \cdot f).$$

In the end of this section, let us mention a metaplectic Lie algebra.

Definition 3.3.4. The *metaplectic Lie algebra* $\mathfrak{mp}(2n, \mathbb{R})$ is a Lie algebra of the metaplectic Lie group $\mathrm{Mp}(2n, \mathbb{R})$.

On the Lie algebra level is $\mathfrak{mp}(2n, \mathbb{R})$ isomorphic to $\mathfrak{sp}(2n, \mathbb{R})$ because they are tangent spaces at the unit element of the symplectic group and its double cover, the metaplectic group. The metaplectic Lie algebra $\mathfrak{mp}(2n, \mathbb{R})$ can be realized by homogeneity two elements in the symplectic Clifford algebra $Cl_s(\mathbb{R}^{2n}, \Omega)$ similarly to the symplectic Lie algebra, see Proposition 1.3.1.

4. Fibre bundle, connection and symplectic spinor bundle

The aim of this chapter is the introduction of geometric structures needed for the definition of symplectic spinor bundle. We will recall symplectic manifolds, fibre bundles, principal and associated bundles, connections and covariant derivation on relevant structures. For a comprehensive survey of these structures, we refer for example to monographs [4] or [27].

More informations and details about symplectic spinor bundle described in the fourth section are in the book [22].

4.1 Symplectic manifolds

Definition 4.1.1. The *symplectic manifold* (M, ω) is a smooth manifold of even dimension $2n$ with a skew-symmetric differential 2-form ω, which is closed, i.e. $\mathrm{d}\omega = 0$, and non degenerate, i.e. for every $m \in M$ there does not exist any non zero element $v \in T_m M$ such that $\omega_m(u, v) = 0$ for every $u \in T_m M$.

Example 4. Examples of symplectic manifolds:

- The real vector space $\mathbb{R}^{2n}$ with standard coordinates $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ and the symplectic form $\omega = \sum_{j=1}^{n} \mathrm{d}x_j \wedge \mathrm{d}y_j$. In this case we denote *real symplectic manifold* simply by $(\mathbb{R}^{2n}, \omega)$ when no confusion can arise.
- The sphere S^2 with any volume form.
- The cotangent bundle T^*M of an n-dimensional manifold M with the symplectic form $\omega_M = \sum_{j=1}^{n} \mathrm{d}x_j \wedge \mathrm{d}\xi_j$ where $x_1, \ldots, x_n$ are local coordinates on M and $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$ are appropriate coordinates on T^*M.

Definition 4.1.2. The *symplectomorphism* between two symplectic manifolds $(M_1,\, \omega_1)$, $(M_2,\, \omega_2)$ is a diffeomorphism $F : M_1 \to M_2$ satisfying $F^*\omega_2 = \omega_1$, where F^* is the cotangent mapping to the mapping F. Then manifolds $(M_1,\, \omega_1)$ and $(M_2,\, \omega_2)$ are called *symplectomorphic*.

Theorem 4.1.1. (Darboux) Every symplectic manifold is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega)$.

For details and proof we refer to [3, Th. 8.1].

4.2 Fibre bundles

We will work with fibre bundles with Fréchet manifolds as total spaces. Therefore the following definitions will be stated in this generality.

A Fréchet manifolds are infinity dimensional generalizations of classical n-dimensional differentiable manifolds.

Definition 4.2.1. Let M be a Hausdorff topological space with a countable basis of open sets. The *map* on M is a pair (U, φ), where $U \subset M$ is open and $\varphi : U \to V$ is a homeomorphism onto an open subset V of some fixed Fréchet space F. The *Fréchet atlas* on M is a set of maps $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ such that $M = \cup_{\alpha \in A} U_\alpha$ and every two maps are compatible, i.e. transition functions $\varphi_\alpha \circ \varphi_\beta^{-1}$ are diffeomorphisms of open subsets of Fréchet space F. The set A is an appropriate index set.

Definition 4.2.2. The *Fréchet manifold* M is a Hausdorff topological space with countable basis of open sets and Fréchet atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$. A *differentiable structure* on a Fréchet manifold is a maximal Fréchet atlas.

In the following we use the term manifold without Fréchet for a finite dimensional manifold.

Definition 4.2.3. Let E and F be Fréchet manifolds and M be a manifold with an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$. The *fibre bundle with fibre* F is E with a smooth surjective mapping $\pi : E \to M$ such that for every $m \in M$, there exists an open neighbourhood $m \in U_\alpha \subset M$ and a diffeomorphism $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ which preserves fibres, i.e. the following diagram is commutative

$$\begin{array}{ccc} E \supset \pi^{-1}(U_\alpha) & \xrightarrow{\psi_\alpha} & U_\alpha \times F \\ & \overset{\pi}{\searrow} & \downarrow p \\ & & U_\alpha, \end{array}$$

where p is a projection on the first component in the Cartesian product.

The space M is called the *base space of fibre bundle*, E is called the *total space* and F is the *fibre*. A couple (U_α, ψ_α) is called the *map of fibre bundle* or the *local trivialisation* of E.

Two local trivialisations (U_α, ψ_α) and (U_β, ψ_β) are *compatible* if $\psi_\alpha \circ \psi_\beta^{-1}$ is a diffeomorphism. It means that $(\psi_\alpha \circ \psi_\beta^{-1})(x, s) = \big(x, \bar{\psi}_{\alpha\beta}(x, s)\big)$, $x \in U_{\alpha\beta} = U_\alpha \cap U_\beta$, where $\bar{\psi}_{\alpha\beta} : (U_\alpha \cap U_\beta) \times F \to F$, is smooth and $\bar{\psi}_{\alpha\beta}(x, \cdot)$ is a diffeomorhism of F for every $x \in U_{\alpha\beta}$. A mapping $\psi_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{Diff}(F)$ with values in the group $\mathrm{Diff}(F)$ of all diffeomorphisms of F is called the *transition function of the fibre bundle.*

Transition functions satisfy $\psi_{\alpha\beta}(x)\psi_{\beta\gamma}(x) = \psi_{\alpha\gamma}(x)$ for every $x \in U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$ and $\psi_{\alpha\alpha} = \mathrm{Id}\,|_M$.

A set of local trivialisations $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in A}$ is called the *atlas of fibre bundle* if $\{U_\alpha\}_{\alpha \in A}$ is an open covering of M and every two local trivialisations are compatible. Two atlases of a fibre bundle are *equivalent* if their union is an atlas of the fibre bundle.

Definition 4.2.4. A fibre bundle $\pi : E \to M$ with fibre V, where V is a Fréchet space, is called the *vector bundle*, if $\pi^{-1}(m)$ is isomorphic to V (as a topological vector space) for every $m \in M$ and ψ_α (from Definition 4.2.3) is a linear map for every open $U_\alpha \subset M$.

A typical example of a vector bundle is the tangent bundle TM of an n dimensional manifold M

$$TM = \bigcup_{m \in M} T_m M$$

together with a projection $\pi : TM \to M$ which maps every vector $v \in T_m M$ to a point $m \in M$.

Definition 4.2.5. The *section* of a fibre bundle $\pi : E \to M$ is a smooth mapping $\sigma : M \to E$ such that $\pi \circ \sigma = \mathrm{Id}|_M$. The *local section* is a smooth mapping $\sigma : U \subset M \to E$ such that $\pi \circ \sigma = \mathrm{Id}|_U$, where U is an open subset of M. We will denote by $\Gamma(M, E)$ a set of all smooth sections of fibre bundle $\pi : E \to M$.

Definition 4.2.6. Let G be a Lie group and F be a Fréchet manifold, then the *G-bundle structure* consists of

1) a fibre bundle $\pi : E \to M$ with a fibre F,

2) a left action $l : G \times F \to F$ of the Lie group on a fibre F,

3) an atlas of the fibre bundle $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in A}$ with transition functions $\{\psi_{\alpha\beta}\}$ that acts on F by the action of the group G, i.e. $\{\psi_{\alpha\beta}\}$ is a family of transition functions $\psi_{\alpha\beta} : U_{\alpha\beta} \to G$, such that $\psi_{\alpha\beta}(x)\psi_{\beta\gamma}(x) = \psi_{\alpha\gamma}(x)$ for every $x \in U_{\alpha\beta\gamma}$, $\psi_{\alpha\alpha}(x)$ is an unit of the group G for every $x \in U_\alpha$, and $l(\psi_{\alpha\beta}(x), s) = \psi_{\alpha\beta}(x)s$ for any $s \in F$ and $x \in U_{\alpha\beta}$.

The group G is called the *structure group.*

Definition 4.2.7. The *principal G-bundle* is a G-bundle structure with a fibre $F = G$. The left action of the group G on the group G is given by a left translations.

Every principal G-bundle $\pi : E \to M$ has an uniquely determined right action (right translation) $r : E \times G \to E$ given by $\varphi_\alpha\big(r(\varphi_\alpha^{-1}(x, a), g)\big) = (x, ag)$, $x \in M$, $(x, a) \in E$, $g \in G$, such that it preserves fibres, i.e. $\pi\big(r(m, g)\big) = \pi(m)$ for all $m \in E$ and $g \in G$. Left and right translations commute. This right action is free, it means that whenever there exists $u \in E$ such that $g \in G$ satisfies $r(u, g) = u$, then g is an unit of the group G.

Example 5. The bundle of symplectic frames of a $2n$ dimensional symplectic vector space over a symplectic manifold (M, ω) is a principal $\mathrm{Sp}(2n, \mathbb{R})$-bundle $\pi_P : P \to M$. A fibre of this bundle at a point $m \in M$ is a set of all symplectic bases of a $2n$ dimensional vector space $P_m = \pi^{-1}(m)$ situated at the point m. This set is isomorphic to $\mathrm{Sp}(2n, \mathbb{R})$ and therefore it is possible to define differentiable structure on P. A structure group of this frame bundle is $\mathrm{Sp}(2n, \mathbb{R})$ which acts, as usual, on symplectic bases of the given symplectic vector space.

Definition 4.2.8. Let $\pi_1 : E_1 \to M_1$ be a principal G_1-bundle and $\pi_2 : E_2 \to M_2$ be a principal G_2-bundle. Let $f : M_1 \to M_2$ be a smooth mapping of manifolds. A mapping $\phi : E_1 \to E_2$ is called the *bundle homomorphism over f* if the following diagram commutes

$$\begin{array}{ccccc} E_1 \times G_1 & \xrightarrow{r_1} & E_1 & \xrightarrow{\pi_1} & M_1 \\ \downarrow{\scriptstyle \phi\times\Lambda} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle f} \\ E_2 \times G_2 & \xrightarrow{r_2} & E_2 & \xrightarrow{\pi_2} & M_2 \end{array}$$

where r_1 and r_2 are the relevant actions of the groups on the total spaces and $\Lambda : G_1 \to G_2$ is a homomorphism of groups.

Definition 4.2.9. Let $\pi : E \to M$ be a principal G-bundle, V be a Fréchet space and $\rho : G \to \mathrm{Aut}(V)$ be a representation of a group G. Let us define an equivalence on $E \times V$ by

$$(u, s) \sim \big(u \cdot g, \rho(g^{-1})(s)\big), \quad g \in G, \quad (u, s) \in E \times V.$$

Then the *associated bundle $E \times_\rho V$ to E by a representation ρ*, or *associated bundle $E \times_\rho V$* for short, is a space of equivalence classes of $\sim$.

There is a bundle structure over M with a standard fibre V on an associated bundle $E \times_\rho V$. This structure can be determined by local sections $\sigma : M \to E$ since for every $v \in V$ a mapping $\sigma : M \to E$ determines a section $\tilde{\sigma} : M \to E \times_\rho V$ given by $x \mapsto [(\sigma(x), v)]$, $x \in M$. We shall write $[\sigma(x), v]$ briefly.

We will denote by $\Gamma(M, \mathcal{V})$ the set of all smooth sections of an associated bundle $\mathcal{V} = E \times_\rho V$ over a manifold M.

For the proof of the following Proposition see [27, p. 94].

Proposition 4.2.1. A set of smooth sections $\Gamma(M, \mathcal{V})$ of associated bundle $\mathcal{V} = E \times_\rho V$ is isomorphic (as a vector space) to the set $\mathcal{C}^\infty(E, V)^G$ of all G-equivariant mappings with values in V, where

$$\mathcal{C}^\infty(E, V)^G = \{f : E \to V \mid f(pg) = \rho(g) f(p) \text{ for all } g \in G \text{ and all } p \in E\}.$$

4.3 Principal connection, associated connection and covariant derivative

We start with an affine connection and a definition of a symplectic connection on a symplectic manifold. Then we pass to the connection on a principal bundle.

Definition 4.3.1. The *affine connection* on a smooth manifold M is a mapping ∇ which maps an ordered couple of vector fields $X, Y \in \mathfrak{X}(M)$ to a smooth vector field $\nabla_X Y$, such that

1) ∇ is $\mathbb{R}$-bilinear,

2) $\nabla_{fX} Y = f \nabla_X Y$ for every smooth function f on M,

3) $\nabla_X (fY) = (Xf)Y + f(\nabla_X Y)$ for every smooth functions f on M (Leibniz rule).

Definition 4.3.2. The *symplectic connection* ∇ is an affine connection ∇ on a symplectic manifold (M, ω) such that $\nabla \omega = 0$, by which we mean

$$X(\omega(Y, Z)) = \omega(\nabla_X Y, Z) + \omega(Y, \nabla_X Z),$$

and its torsion is zero, i.e.

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = 0$$

for all vector fields $X, Y, Z \in \mathfrak{X}(M)$.

Unfortunately, there is no canonical unique choice of a symplectic connection as the Levi-Civita connection in Riemannian geometry, see [20] and [38].

Let $\pi : E \to M$ be a G-principal bundle and V be a Fréchet space. We will denote by ρ a representation $\rho : G \to \mathrm{Aut}(V)$ of the Lie group G. Let $\mathfrak{g}$ be a Lie algebra of the structure group G of the principal bundle $\pi : E \to M$.

Vertical vectors on a principal bundle $\pi : E \to M$ are such vectors from tangent space T_pE, at a point $p \in E$, that the tangent mapping $\pi_* : T_pE \to T_{\pi(p)}M$ is zero on them. We will denote by T_p^vE the space of all vertical vectors at the point $p \in E$. Any smooth distribution $E \ni p \to T_p^hE \subseteq T_pE$ that satisfies $T_pE = T_p^vE \oplus T_p^hE$ is called *horizontal distribution*.

The *fundamental vector field of an action of a Lie group* G on a principal bundle $\pi : E \to M$ is the field

$$\tilde{X}(p) = \frac{d}{dt}\big(p \exp(tX)\big)\Big|_{t=0},$$

where $X \in \mathfrak{g}$, $p \in E$. There is a linear isomorphism

$$\Phi : T_p^vE \ni \tilde{X}(p) \mapsto X \in \mathfrak{g},$$

and a vector X is called the *generator of a vector field* $\tilde{X}$.

We denote by $\mathcal{Z}_p$ the composition of the isomorphism Φ with a projection on the vertical part. Then $\mathcal{Z}_p$ is a differential 1-form on E with values in $\mathfrak{g}$.

$$\mathcal{Z}_p : T_pE = T_p^vE \oplus T_p^hE \to \mathfrak{g},$$
$$\mathcal{Z}_p : \tilde{X}(p) \oplus Y \mapsto X.$$

Definition 4.3.3. The *connection on a principal G-bundle* $\pi : E \to M$ with an action $r(g) : E \to E$, $g \in G$ of group G, is a 1-form $\mathcal{Z}$ on E with values in the vector space $\mathfrak{g}$, such that:

1. $\mathcal{Z}(\tilde{X}) = X$, where $\tilde{X}$ is the fundamental vector field of an action of the Lie group G on E.

2. The form $\mathcal{Z}_p$ depends smoothly on p.

3. $\mathcal{Z}_{r(g)(p)}\big(r(g)_* v\big) = Ad(g^{-1})\mathcal{Z}_p(v)$, where Ad is an adjoint representation $Ad : G \to \mathrm{Aut}(\mathfrak{g})$, $r(g)_*$ is the tangent mapping to $r(g)$, $g \in G$ and $v \in T_pE$.

Recall that the Lie algebra $\mathfrak{g}$ is the tangent space to manifold G at the unit e of the group G and $\mathrm{End}(V)$ is a tangent space to $\mathrm{Aut}(V)$ at $\rho(e) = \mathrm{Id}$. Therefore a differential ρ_* is a linear transformation $\rho_* : \mathfrak{g} \to \mathrm{End}(V)$. The structures of the Lie bracket are induced by left (or right) translations on G and $\mathrm{Aut}(V)$.

Notice $\mathcal{Z} : TE \to \mathfrak{g}$ and $\rho_* : \mathfrak{g} \to \mathrm{End}(V)$. Thus we state the following definition.

Definition 4.3.4. Let $E \times_\rho V$ be an associated bundle to a principal G-bundle $\pi : E \to M$ by a representation $\rho : G \to \mathrm{Aut}(V)$ and $\mathcal{Z}$ be a connection on the principal G-bundle $\pi : E \to M$. Then the *associated connection* on the associated bundle $E \times_\rho V$ is $\rho_* \circ \mathcal{Z}$.

Let us denoted by $\mathcal{V} = E \times_\rho V$ an associated bundle to a principal bundle $\pi : E \to M$ by representation $\rho : G \to \mathrm{Aut}(V)$. We will write a section of an associated bundle $\varphi \in \Gamma(M, \mathcal{V})$ by

$$\varphi = [p, w], \quad \text{for } m \in M \text{ by } \varphi(m) = [p(m), w(m)], \tag{4.1}$$

where $p : U \subset M \to E$ and $w = \nu \circ p : U \subset M \to V$, where ν is G-ekvivariantní V-valued function corresponding to the section φ by the Proposition 4.2.1.

Definition 4.3.5. We will denoted by V^∞ the set of all smooth vectors in V. Let $\mathcal{Z} : TE \to \mathfrak{g}$ be a principal connection and $\rho_* : \mathfrak{g} \to \mathrm{End}(V^\infty)$ be a tangent mapping to representation ρ. The *covariant derivation associated to principal connection* $\mathcal{Z}$ is

$$\nabla_X \varphi = \nabla_X [p, w] = \big[p, X(v) + \rho_*\big(p^* \mathcal{Z}(X)\big) w\big],$$

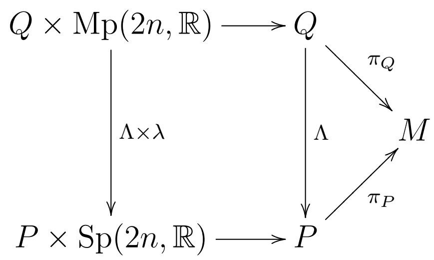
where $X \in \mathfrak{X}(M)$ and $\varphi = [p, w] \in \Gamma(M, \mathcal{V})$ as in (4.1).

4.4 Symplectic spinor bundle

Let us denote by P a bundle of symplectic frames $\pi_P : P \to M$ over a symplectic manifold (M, ω) (see Example 5). Then P is a principal $\mathrm{Sp}(2n, \mathbb{R})$-bundle over the manifold M. The *local symplectic frame* is a local section of $\pi_P : P \to M$.

Let us recall that $\lambda : \mathrm{Mp}(2n, \mathbb{R}) \to \mathrm{Sp}(2n, \mathbb{R})$ is a double covering of the symplectic group, see 3.3.

Definition 4.4.1. Let $\pi_Q : Q \to M$ be a principal $\mathrm{Mp}(2n, \mathbb{R})$-bundle over a symplectic manifold. The *metaplectic structure* on a symplectic manifold (M, ω) is a principal $\mathrm{Mp}(2n, \mathbb{R})$-bundle $\pi_Q : Q \to M$ with a surjective bundle homomorphism $\Lambda : Q \to P$ over the identity on M, i.e the following diagram is commutative



The horizontal arrows in the diagram are respective actions of groups on the relevant principal bundles.

Remark 2. The existence of a metaplectic structure of a given symplectic manifold is not automatic. The topological obstruction to the existence of a metaplectic structure is the same as in Riemannian Spin geometry, as is said in [22, Prop. 3.1.2]. A symplectic manifold (M, ω) admits a metaplectic structure if and only if the second Stiefel-Withney class $w_2(M) \in H^2(M, \mathbb{Z}_2)$ of M vanishes. If this is the case, the isomorphy classes of metaplectic stuctures on (M, ω) are classified by the first cohomology group $H^1(M, \mathbb{Z}_2)$. (For explanation and definitions of used topological terms we refer to [41].)

Let us recall that the space of smooth vectors of the Segal-Shale-Weil representation $\mathfrak{m}$ of $\mathrm{Mp}(2n, \mathbb{R})$ is precisely $\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$. We call the elements of Schwartz space $\mathcal{S}(\mathbb{R}^n)$ symplectic spinors as was mentioned in Section 3.3.

Definition 4.4.2. The *symplectic spinor bundle* is a vector bundle associated to a principal $\mathrm{Mp}(2n, \mathbb{R})$-bundle Q over a manifold M by the representation

$$\mathfrak{m} : \mathrm{Mp}(2n, \mathbb{R}) \to \mathrm{Aut}(\mathcal{S}(\mathbb{R}^n))$$

and we denote it by $\mathbf{S} = Q \times_{\mathfrak{m}} \mathcal{S}(\mathbb{R}^n)$. Smooth sections of symplectic spinor bundle $\varphi \in \Gamma(M, \mathbf{S})$ are called *symplectic spinor fields.*

Remark 3. For simplicity, we refer to symplectic spinor fields often as *symplectic spinors* when it does not cause any confusion.

Instead of $\Gamma(M, \mathbf{S})$ we use $\mathcal{C}^\infty(M, \mathbf{S})$ for symplectic spinor fields when we emphasise the smoothness of them. In the case when the symplectic spinor bundle is trivial, sections $\varphi \in \mathcal{C}^\infty(M, \mathbf{S})$ reduces to mappings $\varphi : M \to \mathcal{S}(\mathbb{R}^n)$ that we denote them by $\mathcal{C}^\infty(M, \mathcal{S}(\mathbb{R}^n))$, for relevant n.

Let us recall, that the symplectic Clifford multiplication is $\mathrm{Mp}(2n, \mathbb{R})$-equivariant, see Lemma 3.3.6. Let φ be a symplectic spinor field, i.e. a section of $Q \times_{\mathfrak{m}} \mathcal{S}(\mathbb{R}^n)$, and X be a vector field on M, i.e. it is a section of the associated bundle $TM = Q \times_\lambda \mathbb{R}^{2n}$. Let us write sections of associated bundle $\mathbf{S}$ as $\varphi = [p, f]$ and $X = [p, v]$ as in (4.1), i.e. for $m \in M$

$$\varphi(m) = [p(m), f(m)], \quad X(m) = [p(m), v(m)],$$

where $p : U \subset M \to P$, $v(m) \in \mathbb{R}^{2n}$ and $f(m) \in \mathcal{S}(\mathbb{R}^n)$.

The symplectic Clifford multiplication is already defined for elements $v(m) \in \mathbb{R}^{2n}$ and $f(m) \in \mathcal{S}(\mathbb{R}^n)$ therefore it is possible to define its "lift" onto a symplectic spinor bundle.

Definition 4.4.3. The *symplectic Clifford multiplication* on a symplectic spinor bundle is $\cdot : \mathfrak{X}(M) \times \Gamma(M, \mathbf{S}) \to \Gamma(M, \mathbf{S})$,

$$[p, v] \cdot [p, f] = [p, v \cdot f], \tag{4.2}$$

where $[p, v] \in \mathfrak{X}(M)$ and $[p, f] \in \Gamma(M, \mathbf{S})$.

Definition 4.4.4. Let $\mathcal{Z}$ be a connection on a principal $\mathrm{Sp}(2n, \mathbb{R})$-bundle $\pi_P : P \to M$ corresponding to a symplectic connection ∇ on a manifold M. Let $\bar{\mathcal{Z}}$ be a "lift" of $\mathcal{Z}$ onto a principal $\mathrm{Mp}(2n, \mathbb{R})$-bundle $\pi_Q : Q \to M$. The *symplectic spinor covariant derivative* is a covariant derivative ∇^S associated to $\bar{\mathcal{Z}}$, i.e.

$$\nabla^S : \Gamma(M, \mathbf{S}) \to \Gamma(M, T^*M \otimes \mathbf{S}).$$

By Proposition 4.2.1, a section φ of the associated vector bundle $\mathbf{S} = Q \times_{\mathfrak{m}} \mathcal{S}(\mathbb{R}^n)$ can be understood as an $\mathrm{Mp}(2n, \mathbb{R})$-equivariant $\mathcal{S}(\mathbb{R}^n)$-valued function on Q. Let us denote by $\hat{\varphi}$ this function, i.e. $\hat{\varphi} : Q \to \mathcal{S}(\mathbb{R}^n)$. For a local symplectic frame $s : U \to P$ we denote by $\bar{s} : U \to Q$ one of a "lifts" of s to Q. Let us set $\varphi_s = \hat{\varphi} \circ \bar{s}$.

Let $[q, \psi]$ denote a relevant element in $\mathbf{S}$ for $q \in Q$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$.

Theorem 4.4.1. Let ∇^S be a symplectic spinor covariant derivative on M. Then

$$\nabla^S_X \varphi = \left[\bar{s}, X(\varphi_s)\right] - \frac{i}{2}\sum_{j=1}^{n} e_j \cdot (\nabla_X e_j) \cdot \varphi - e_j \cdot (\nabla_X e_{j+n}) \cdot \varphi, \tag{4.3}$$

where $X \in \mathfrak{X}(M)$, $\varphi \in \Gamma(M, \mathbf{S})$, $\{e_1, \ldots, e_{2n}\}$ is a local symplectic frame of M and ∇ is a symplectic connection on M.

See [22, Prop. 3.2.6] for a proof, let us only notice that the proof is based on Definition 4.3.5.

Remark 4. A symplectic spinor covariant derivative on $(\mathbb{R}^{2n}, \omega)$ is, by (4.3), equal to the partial derivative because the covariant derivatives $\nabla_{e_j} e_k$ vanish for the coordinate fields $e_j, e_k \in \{\frac{\partial}{\partial x^a}, \frac{\partial}{\partial y^b}\}_{a,b=1}^n$.

Theorem 4.4.2. For every $X, Y \in \Gamma(M, TM)$ and $\varphi \in \Gamma(M, \mathbf{S})$, the symplectic spinor covariant derivative ∇^S satisfies

$$\nabla^S_X (Y \cdot \varphi) = (\nabla_X Y) \cdot \varphi + Y \cdot \nabla^S_X \varphi.$$

We refer an interested reader to the book [22, Ch. 3.2] where relevant properties of symplectic spinor covariant derivative on the symplectic spinor bundle are treated.

5. Symplectic operators

Let (M, ω) be a $2n$-dimensional symplectic manifold with a metaplectic structure and a symplectic connection ∇. The symplectic spinor covariant derivative ∇^S on the symplectic spinor fields is defined to be the covariant derivative associated to $\bar{\mathcal{Z}}$. See Section 4.4.

Let us denote by $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}\}$ a local symplectic frame with respect to ω on (M, ω) and the dual frame by $\{\epsilon^1, \ldots, \epsilon^{2n}\}$. We will be using the symbol ω^{jk}, $j, k = 1, \ldots, 2n$, for elements of the inverse matrix to $\omega_{jk} = \omega(e_j, e_k)$ of the symplectic form ω, i.e.

$$\omega^{jk} = 1 \text{ for } k = n + j, \quad \omega^{jk} = -1 \text{ for } k = j - n, \quad \omega^{jk} = 0 \text{ otherwise.}$$

5.1 Symplectic Dirac operator

The symplectic Dirac operator acting on symplectic spinor fields is defined in a similar way as the Dirac operator on Riemannian manifolds. The definition of the symplectic Dirac operator is given in the book [22].

For purpose of the following definition we will denote by c_s instead by $\cdot$ the symplectic Clifford multiplication on a symplectic spinor bundle defined in (4.2).

Definition 5.1.1. The *symplectic Dirac operator* D_s on a symplectic manifold (M, ω) is a first order differential operator acting on smooth symplectic spinor fields

$$D_s : \Gamma(M, \mathbf{S}) \longrightarrow \Gamma(M, \mathbf{S}).$$

The symplectic Dirac operator is defined as the composition

$$\begin{aligned} &D_s = c_s \circ \omega^{-1} \circ \nabla^S, &(5.1)\\ &D_s : \Gamma(M, \mathbf{S}) \to \Gamma(M, T^*M \otimes \mathbf{S}) \to \Gamma(M, TM \otimes \mathbf{S}) \to \Gamma(M, \mathbf{S}), \end{aligned}$$

where we identify the bundles T^*M and TM by putting a tangent vector field into the first argument of ω. The resulting map is denoted by ω^{-1}.

The symplectic Dirac operator is a symplectically invariant endomorphism of smooth symplectic spinors. Two symplectic Dirac operators are defined in [22], namely the symplectic Dirac operator D_s as in Definition 5.1.1 and the symplectic Dirac operator $\tilde{D}_s$, defined with use of the Riemannian metric g on M instead of symplectic form ω for identifying $\Gamma(M, T^*M \otimes \mathbf{S})$ with $\Gamma(M, TM \otimes \mathbf{S})$, i.e.

$$\tilde{D}_s = c_s \circ g^{-1} \circ \nabla^S.$$

Unfortunately, the operator $\tilde{D}_s$ is not symplectically invariant. We do not use it in the thesis.

Proposition 5.1.1. Let $\{e_j\}_{j=1}^{2n}$ be a local symplectic frame with respect to ω on a $2n$-dimensional symplectic manifold (M, ω). Then the symplectic Dirac operator D_s is locally equal to

$$D_s \varphi = \sum_{j,k=1}^{2n} \omega^{jk} e_j \cdot \nabla^S_{e_k} \varphi, \tag{5.2}$$

where $\varphi \in \Gamma(M, \mathbf{S})$.

Let us consider the real symplectic manifold $(\mathbb{R}^2, \omega)$. Since the first cohomology group $H^1(\mathbb{R}^2, \mathbb{Z}_2) = \{0\}$, its only metaplectic structure is the trivial $\mathrm{Mp}(2,\mathbb{R})$-bundle $Q = \mathbb{R}^2 \times \mathrm{Mp}(2,\mathbb{R})$, see [22, Ex 4.1.4]. Consequently, the symplectic spinor bundle $\mathbf{S} = Q \times_{\mathfrak{m}} \mathcal{S}(\mathbb{R})$ is trivial and their sections are mappings

$$\varphi : \mathbb{R}^2 \to \mathcal{S}(\mathbb{R}).$$

The mapping φ can be considered as a mapping $\varphi : \mathbb{R}^3 \to \mathbb{C}$ such that, for any fixed $(x, y) \in \mathbb{R}^2$ the mapping $\tilde{\varphi}(x, y) : q \mapsto \varphi(x, y, q)$ is a Schwartz function, $q \in \mathbb{R}$, i.e. $\tilde{\varphi}(x, y) \in \mathcal{S}(\mathbb{R})$.

A symplectic spinor covariant derivative on a real symplectic manifold is just the partial derivative, see Remark 4 at the end of Section 4.4. Then according to (5.2), the symplectic Dirac operator takes the form

$$D_s\varphi(x, y, q) = iq\frac{\partial \varphi}{\partial y}(x, y, q) - \frac{\partial^2 \varphi}{\partial x \partial q}(x, y, q). \tag{5.3}$$

For simplicity, we often write ∂_x, ∂_y and ∂_q instead of $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial q}$. Then the symplectic Dirac operator is

$$D_s = iq\partial_y - \partial_x\partial_q. \tag{5.4}$$

5.2 Symplectic spinor valued forms

We are interested in this section in symplectic spinor valued exterior forms with values in the vector space

$$\bigwedge^{\bullet}(\mathbb{R}^{2n})^* \otimes \mathcal{S}(\mathbb{R}^n),$$

where $(\mathbb{R}^{2n})^*$ is the dual vector space to $\mathbb{R}^{2n}$.

The representation of the metaplectic group $\mathrm{Mp}(2n,\mathbb{R})$ on symplectic spinor valued forms

$$\varrho : \mathrm{Mp}(2n, \mathbb{R}) \to \mathrm{Aut}\Big(\bigwedge^{\bullet}(\mathbb{R}^{2n})^* \otimes \mathcal{S}(\mathbb{R}^n)\Big), \tag{5.5}$$

is defined by

$$\varrho(g)(\alpha \otimes \phi) = \big(\lambda(g)^*\big)^{\wedge r}\alpha \otimes \mathfrak{m}(g)\phi, \tag{5.6}$$

where $\lambda : \mathrm{Mp}(2n,\mathbb{R}) \to \mathrm{Sp}(2n,\mathbb{R})$ is the twofold covering of a symplectic group, the symbol ${}^{\wedge r}$ denotes the r-th exterior power, $r = 0, \ldots, 2n$, $\mathfrak{m}$ is the Segal-Shale-Weil representation and $\alpha \in \bigwedge^r(\mathbb{R}^{2n})^*$, $\phi \in \mathcal{S}(\mathbb{R}^n)$. We extend the representation ϱ by linearity to any element.

Definition 5.2.1. The *space of exterior differential forms with values in symplectic spinors* is

$$\Omega^r(M, \mathbf{S}) = \Gamma\Big(M, Q \times_{\varrho} \Big(\bigwedge^{r}(\mathbb{R}^{2n})^* \otimes \mathcal{S}(\mathbb{R}^n)\Big)\Big). \tag{5.7}$$

Proposition 5.2.1. The following decomposition into irreducible $\mathfrak{mp}(2n,\mathbb{R})$-modules holds. For all $j = 0, \ldots, 2n$, $n \geq 1$

$$\bigwedge^{j}(\mathbb{R}^{2n})^* \otimes \mathcal{S}(\mathbb{R}^n)^{\pm} = \bigoplus_{\{k|(j,k)\in I_n\}} E^{jk\pm}, \tag{5.8}$$

where $I_n = \{(j,k)|j = 0,\dots,n, k = 0,\dots,j\} \cup \{(j,k)|j = n+1,\dots,2n, k = 0,\dots,2n-j\}$.

See [30] for a proof and a description of highest weight $\mathfrak{mp}(2n,\mathbb{R})$-modules $E^{ik^\pm}$ in the terms of representation theory.

Modules $E^{jk^\pm}$ are Fréchet spaces. In addition, for any (j,k), $(j,l) \in I_n$, $k \neq l$ we have $E^{jk^\pm} \neq E^{jl^\pm}$ (as $\mathfrak{mp}(2n,\mathbb{R})$-modules) for all combinations of $\pm$ on both sides. Thus for $j = 0,\dots,2n$, the decomposition of the tensor product

$$\bigwedge^j (\mathbb{R}^{2n})^* \otimes (\mathcal{S}(\mathbb{R}^n)^+ \oplus \mathcal{S}(\mathbb{R}^n)^-) = \bigoplus_k (E^{jk^+} \oplus E^{jk^-})$$

is multiplicity-free. It means that $\bigwedge^j(\mathbb{R}^{2n})^* \otimes (\mathcal{S}(\mathbb{R}^n)^+ \oplus \mathcal{S}(\mathbb{R}^n)^-)$ splits into non-isomorphic irreducible submodules $E^{jk^\pm}$. Let us set $E^{jk} = E^{jk^+} \oplus E^{jk^-}$ and consider the associated vector bundle

$$\mathcal{E}^{jk} = Q \times_\varrho E^{jk} \tag{5.9}$$

for $(j,k) \in I_n$. Let us recall that the symplectic spinor bundle for $2n$ dimensional symplectic manifold was defined as the associated bundle $\mathbf{S} = Q \times_\mathfrak{m} \mathcal{S}(\mathbb{R}^n)$.

There exist uniquely defined invariant projections

$$p^{j,k} : \Omega^j(M,\mathbf{S}) \to \Gamma(M,\mathcal{E}^{jk}) \text{ for } (j,k) \in I_n, \tag{5.10}$$

because the decomposition of $\bigwedge^j(\mathbb{R}^{2n})^* \otimes (\mathcal{S}(\mathbb{R}^n)^+ \oplus \mathcal{S}(\mathbb{R}^n)^-)$ is multiplicity-free.

Let us introduce two $\mathrm{Mp}(2n,\mathbb{R})$-equivariant endomorphisms of $\bigwedge^\bullet(\mathbb{R}^{2n})^* \otimes \mathcal{S}(\mathbb{R}^n)$ acting on the decomposition.

Definition 5.2.2. Let $\alpha \otimes \varphi \in \bigwedge^r(\mathbb{R}^{2n})^* \otimes \mathcal{S}(\mathbb{R}^n)$. We set for $r = 0,\dots,2n$

$$\begin{aligned} X &: \bigwedge^r(\mathbb{R}^{2n})^* \otimes \mathcal{S}(\mathbb{R}^n) \to \bigwedge^{r+1}(\mathbb{R}^{2n})^* \otimes \mathcal{S}(\mathbb{R}^n), \\ Y &: \bigwedge^r(\mathbb{R}^{2n})^* \otimes \mathcal{S}(\mathbb{R}^n) \to \bigwedge^{r-1}(\mathbb{R}^{2n})^* \otimes \mathcal{S}(\mathbb{R}^n) \end{aligned} \tag{5.11}$$

by formulas

$$\begin{aligned} X(\alpha \otimes \varphi) &= -\sum_{j=1}^{2n} \epsilon^j \wedge \alpha \otimes e_j \cdot \varphi, \\ Y(\alpha \otimes \varphi) &= \sum_{j,k=1}^{2n} \omega^{jk} \iota_{e_j} \alpha \otimes e_k \cdot \varphi, \end{aligned} \tag{5.12}$$

where $\iota_v\alpha$ denotes the contraction of an exterior form α by a vector $v \in \mathbb{R}^{2n}$.

The operators X and Y are well defined in the sense that they are independent of a choice of a symplectic basis $\{e_j\}_{j=1}^{2n}$. The operators are related to Howe duality on the symplectic spinor valued exterior forms. See [30], for more details about this operators.

Since the operators X, Y are $\mathrm{Mp}(2n,\mathbb{R})$-equivariant with respect to representation ϱ, it is possible to define their lifts to sections of the corresponding associated bundles. Let us denote this lifts by the same symbols. Then we can write an explicit formulas for projections (5.10), see [14, Prop. 5.1.3].

Proposition 5.2.2. Let (M,ω) be a $2n$-dimensional symplectic manifold with a metaplectic structure. Projections $p^{1j} : \Omega^1(M,\mathbf{S}) \to \Gamma(M,\mathcal{E}^{1j})$ for $j = 0,1$ are given by

$$p^{10} = \frac{i}{n}XY$$

$$p^{11} = \mathrm{Id}\,|_{\Omega^1(M,\mathbf{S})} - \frac{i}{n}XY.$$

Definition 5.2.3. Let us choose a local symplectic frame $\{e_j\}_{j=1}^{2n}$ and let us denote its dual coframe by $\{\epsilon^j\}_{j=1}^{2n}$. The *symplectic spinor exterior covariant derivative* d^{∇^S} induced by the symplectic spinor covariant derivative ∇^S is defined by

$$\mathrm{d}^{\nabla^S}(\alpha\otimes\varphi) = \mathrm{d}\alpha\otimes\varphi + (-1)^r\sum_{j=1}^{2n}\epsilon^j\wedge\alpha\otimes\nabla^S_{e_j}\varphi, \tag{5.13}$$

where $\alpha\otimes\varphi\in\Omega^r(M,\mathbf{S})$, $\alpha\in\Omega^r(M)$ and $\varphi\in\Gamma(M,\mathbf{S})$. For the other elements of $\Omega^r(M,\mathbf{S})$, d^{∇^S} is extended by linearity.

5.3 Symplectic twistor operators

We shall define symplectic twistor operators acting on symplectic spinor valued exterior forms. The definition is taken from [29]. Its contact projective analogue was introduced in [24].

Definition 5.3.1. The *j-th symplectic twistor operators* $(T_s)_j$

$$(T_s)_j : \Gamma(M,\mathcal{E}^{jj}) \to \Gamma(M,\mathcal{E}^{j+1,j+1}),$$

$$(T_s)_j = p^{j+1,j+1}\mathrm{d}^{\nabla^S}|_{\Gamma(M,\mathcal{E}^{jj})}$$

for $j = 0,\ldots,n$.

We are interested only in the zeroth symplectic twistor operator $(T_s)_0$ in this thesis. From this reason and for simplicity of notation, we write T_s instead of $(T_s)_0$ and refer to it as to the *symplectic twistor operator*.

Let us present an equivalent definition of the (zeroth) symplectic twistor operator. We will denote the symplectic Clifford multiplication on a symplectic spinor bundle by c_s again. See (4.2).

Definition 5.3.2. The *symplectic twistor operator* T_s on a symplectic manifold (M,ω) is the first order differential operator

$$T_s : \Gamma(M,\mathbf{S}) \to \Gamma(M,\mathcal{T})$$

acting on smooth symplectic spinor fields

$$T_s = P_{\mathrm{Ker}(c_s)}\circ\omega^{-1}\circ\nabla^S, \tag{5.14}$$
$$T_s : \Gamma(M,\mathbf{S}) \to \Gamma(M,T^*M\otimes\mathbf{S}) \to \Gamma(M,TM\otimes\mathbf{S}) \to \Gamma(M,\mathcal{T}),$$

where $\mathcal{T}$ is the space of symplectic twistors, $T^*M\otimes\mathbf{S}\simeq\mathbf{S}\oplus\mathcal{T}$, given by the projection

$$P_{\mathrm{Ker}(c_s)} : \Gamma(M,T^*M\otimes\mathbf{S}) \longrightarrow \Gamma(M,\mathcal{T})$$

on the kernel of the symplectic Clifford multiplication c_s.

The symplectic twistor operator is $\mathrm{Mp}(2n,\mathbb{R})$-invariant. See Lemma 6.4.1.

Let $\varphi \in \Gamma(M,\mathbf{S})$ be a symplectic spinor field, i.e. a section of the symplectic spinor bundle over the manifold M. The symplectic covariant derivative $\nabla^S = d^{\nabla^S}|_{\Gamma(M,\mathbf{S})}$ can be locally written in the form

$$\nabla^S\varphi = \sum_{k=1}^{2n} \epsilon^k \otimes \nabla^S_{e_k}\varphi. \tag{5.15}$$

Proposition 5.3.1. In the local symplectic coframe $\{\epsilon^1\}_{j=1}^{2n}$ dual to the symplectic frame $\{e_j\}_{j=1}^{2n}$, we have on a $2n$-dimensional symplectic manifold, the following formulas for T_s

$$T_s(\varphi) = \sum_{k=1}^{2n} \epsilon^k \otimes \nabla^S_{e_k}\varphi + \frac{i}{n}\sum_{j,k,l=1}^{2n} \epsilon^l \otimes \omega^{kj} e_l \cdot e_j \cdot \nabla^S_{e_k}\varphi \tag{5.16}$$

$$= \Big(1+\frac{1}{n}\Big)\sum_{k=1}^{2n} \epsilon^k \otimes \nabla^S_{e_k}\varphi + \frac{i}{n}\sum_{j,k,l=1}^{2n} \epsilon^l \otimes \omega^{kj} e_j \cdot e_l \cdot \nabla^S_{e_k}\varphi, \tag{5.17}$$

where $\varphi \in \Gamma(M,\mathbf{S})$.

Proof. Using the expression for the covariant derivation (5.15) and expressions (5.12) and (3.7), we obtain

$$\begin{aligned}
T(\varphi) = p^{11}(\nabla^S\varphi) &= \sum_{k=1}^{2n} \epsilon^k \otimes \nabla^S_{e_k}\varphi - \frac{i}{n} XY\Big(\sum_{m=1}^{2n} \epsilon^m \otimes \nabla^S_{e_m}\varphi\Big)\\
&= \sum_{k=1}^{2n} \epsilon^k \otimes \nabla^S_{e_k}\varphi - \frac{i}{n} X\Big(\sum_{j,k=1}^{2n} \omega^{kj} e_j \cdot \nabla^S_{e_k}\varphi\Big)\\
&= \sum_{k=1}^{2n} \epsilon^k \otimes \nabla^S_{e_k}\varphi + \frac{i}{n}\Big(\sum_{j,k,l=1}^{2n} \epsilon^l \otimes \omega^{kj} e_l \cdot e_j \cdot \nabla^S_{e_k}\varphi\Big)\\
&= \sum_{k=1}^{2n} \epsilon^k \otimes \nabla^S_{e_k}\varphi + \frac{i}{n}\Big(\sum_{j,k,l=1}^{2n} \epsilon^l \otimes \omega^{kj} (e_j \cdot e_l \cdot \nabla^S_{e_k}\varphi - i\omega(e_l,e_j)\nabla^S_{e_k}\varphi)\Big)\\
&= \Big(1+\frac{1}{n}\Big)\sum_{k=1}^{2n} \epsilon^k \otimes \nabla^S_{e_k}\varphi + \frac{i}{n}\sum_{j,k,l=1}^{2n} \epsilon^l \otimes \omega^{kj} e_j \cdot e_l \cdot \nabla^S_{e_k}\varphi,
\end{aligned}$$

which proves the formula. □

From the local expression for the symplectic Dirac (5.2) and the symplectic twistor operator (5.16) we get the following.

Lemma 5.3.2.

$$T_s = \sum_{l=1}^{2n} \epsilon^l \otimes \left(\nabla^S_{e_l} - \frac{i}{n} e_l \cdot D_s\right), \tag{5.18}$$

where $\{\epsilon^1\}_{j=1}^{2n}$ is local symplectic coframe dual to the symplectic frame $\{e_j\}_{j=1}^{2n}$.

6. Symplectic twistor operator on $(\mathbb{R}^2, \omega)$

Text of this chapter is based on an article
Symplectic twistor operator and its solution space on $\mathbb{R}^2$,
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Marie Dostálová, Petr Somberg.
Abstract We introduce the symplectic twistor operator T_s in the symplectic spin geometry of real dimension two, as a symplectic analogue of the Dolbeault operator in the complex spin geometry of complex dimension 1. Based on the techniques of the metaplectic Howe duality and the algebraic Weyl algebra, we compute the space of its solutions on the real symplectic manifold $(\mathbb{R}^2, \omega)$.
Key words: Symplectic spin geometry, Metaplectic Howe duality, Symplectic twistor operator, Symplectic Dirac operator.
MSC classification: 53C27, 53D05, 81R25.

6.1 Introduction and motivation

Central problems and questions in differential geometry of Riemannian spin manifolds are usually reflected in analytic and spectral properties of the pair of first order differential operators acting on spinors, the Dirac operator and the twistor operator. In particular, there is a rather subtle relation between geometry and topology of a given manifold and the spectra resp. the solution spaces of these operators. See, e.g., [1], [18] and references therein.

Based on the Segal-Shale-Weil representation, the symplectic version of the Dirac operator D_s was introduced in [28], and some of its basic analytic and spectral properties were studied in [5], [22], [24]. Introducing the metaplectic Howe duality, [9], a representation theoretical characterization of the solution space of symplectic Dirac operator was determined on the real symplectic manifold $(\mathbb{R}^{2n}, \omega)$. However, an explicit analytic description of this space is still missing and this fact has also substantial consequences for the present chapter.

A variant of the first order symplectic twistor operator T_s was introduced in [24] in the framework of contact parabolic geometry, descending to the symplectic twistor operator on symplectic leaves of foliation. Basic properties, including the solution space, of the symplectic twistor operator on $\mathbb{R}^{2n}$ are discussed in Chapter 7. In particular, the case $n = 1$ fits into the framework of Chapter 7 as well, but all the results for $n = 1$ and $n > 1$ follow from intrinsically different reasons. Consequently, there is a substantial difference between the cases $n = 1$ and $n > 1$, and the approach in Chapter 7 based on the procedure of the geometrical prolongation of the symplectic twistor differential equation did not enlighten the reason for this difference. Roughly speaking, the problem behind this is that many first order operators (e.g., the Dirac and twistor operators on spinors) coincide in the case of one complex dimension with the Cauchy-Riemann (Dolbeault) and its conjugate operators.

The aim of the present chapter is to fill this gap and discuss the case of $n = 1$ by different methods, namely, by analytical and combinatorial techniques. A part of the problem of finding the solution space of T_s is the discovery of certain canonical representative solutions of the symplectic Dirac operator D_s and the discovery of certain non-trivial identities in the algebraic Weyl algebra.

The system of partial differential equations representing T_s is overdetermined, acting on the space of functions valued in an infinite dimensional vector space of the Segal-Shale-Weil representation, and the solution space of T_s is (even locally) infinite dimensional. Notice that the techniques of the metaplectic Howe duality are not restricted to $(\mathbb{R}^2, \omega)$, but it is not straightforward for $(\mathbb{R}^{2n}, \omega)$, $n > 1$, to write more explicit formulas for solutions with values in the higher dimensional non-commutative algebraic Weyl algebra.

The structure of the present chapter goes as follows. In the first Section, we review basic properties of the symplectic spin geometry in the real dimension 2, with emphasis on the metaplectic Howe duality. In Section 6.3, we give a general definition of the symplectic twistor operator T_s. The space of polynomial solutions of T_s on $(\mathbb{R}^2, \omega)$ is analysed in Section 6.5, relying on two basic principles. The first one is representation theoretical, coming from the action of the metaplectic Lie algebra on the function space of interest. The second one is then the construction of representative solutions in the particular irreducible subspaces of the function space. As a byproduct of our approach, we construct specific polynomial solutions of the symplectic Dirac operator D_s, which is also new according to our best knowledge.

6.2 Metaplectic Lie algebra $\mathfrak{mp}(2, \mathbb{R})$, symplectic Clifford algebra and class of simple weight modules for $\mathfrak{mp}(2, \mathbb{R})$

In the present section, we collect some basic algebraic and representation theoretical information needed in the analysis of the solution space of the symplectic twistor operator T_s. See, e.g., [5], [9], [19], [22], [24].

Let us consider a 2-dimensional symplectic vector space $(\mathbb{R}^2, \omega = \mathrm{d}x \wedge \mathrm{d}y)$, and a symplectic basis $\{e_1, e_2\}$ with respect to the non-degenerate two form $\omega \in \wedge^2(\mathbb{R}^2)^\star$. The linear action of $\mathfrak{sp}(2, \mathbb{R}) \simeq \mathfrak{sl}(2, \mathbb{R})$ on $\mathbb{R}^2$ induces the action on its tensor representations, and we have $g^\star\omega = \omega$ for all $g \in \mathrm{Sp}(2, \mathbb{R})$. The set of three matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

is a basis of $\mathfrak{sp}(2, \mathbb{R})$.

The metaplectic Lie algebra $\mathfrak{mp}(2, \mathbb{R})$ is the Lie algebra of the two-fold group covering $\lambda : \mathrm{Mp}(2, \mathbb{R}) \to \mathrm{Sp}(2, \mathbb{R})$ of the symplectic Lie group $\mathrm{Sp}(2, \mathbb{R})$. It can be realized by homogeneity two elements in the symplectic Clifford algebra $Cl_s(\mathbb{R}^2, \omega)$, where the isomorphism $\lambda_\star : \mathfrak{mp}(2, \mathbb{R}) \to \mathfrak{sp}(2, \mathbb{R})$ is given by

$$\begin{aligned} &\lambda_\star(e \cdot e) = -2X, \\ &\lambda_\star(f \cdot f) = 2Y, \\ &\lambda_\star(e \cdot f + f \cdot e) = 2H. \end{aligned} \tag{6.1}$$

Definition 6.2.1. The symplectic Clifford algebra $Cl_s(\mathbb{R}^2, \omega)$ is an associative unital algebra over $\mathbb{C}$, realized as a quotient of the tensor algebra $T(e, f)$ by a two-sided ideal $I \subset T(e, f)$, generated by

$$v \cdot w - w \cdot v = -i\omega(v, w)$$

for all $v, w \in \mathbb{R}^2$.

The symplectic Clifford algebra $Cl_s(\mathbb{R}^2, \omega)$ is isomorphic to the Weyl algebra W_2 of complex valued algebraic differential operators on $\mathbb{R}$, and the symplectic Lie algebra $\mathfrak{sp}(2, \mathbb{R})$ can be realized as a subalgebra of W_2. In particular, the Weyl algebra is an associative algebra generated by $\{q, \partial_q\}$, the multiplication operator by q and the differentiation ∂_q. The symplectic Lie algebra $\mathfrak{sp}(2, \mathbb{R})$ has a basis $\{-\frac{i}{2}q^2, -\frac{i}{2}\frac{\partial^2}{\partial q^2}, q\frac{\partial}{\partial q} + \frac{1}{2}\}$.

The symplectic spinor representation is the irreducible Segal-Shale-Weil representation of $Cl_s(\mathbb{R}^2, \omega)$ on $L^2(\mathbb{R}, e^{-\frac{q^2}{2}} \mathrm{d}q_{\mathbb{R}})$, the space of square integrable functions on $(\mathbb{R}, \mathrm{d}\mu = e^{-\frac{q^2}{2}} \mathrm{d}q_{\mathbb{R}})$, where $\mathrm{d}q_{\mathbb{R}}$ is the Lebesgue measure. Its action, the symplectic Clifford multiplication c_s, acts on the subspace of $\mathcal{C}^\infty$(smooth)-vectors given by the Schwartz space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing complex valued functions on $\mathbb{R}$ as its dense subspace. The space $\mathcal{S}(\mathbb{R})$ can be regarded as a smooth Fréchet globalization of the space of $\tilde{K}$-finite vectors in the representation, where $\tilde{K} \subset \mathrm{Mp}(2, \mathbb{R})$ is the maximal compact subgroup which is the double cover of $K = \mathrm{U}(1) \subset \mathrm{Sp}(2, \mathbb{R})$. Though we shall work in the smooth globalization $\mathcal{S}(\mathbb{R})$, our representative vectors constructed in Section 6.5 belong to the underlying Harish-Chandra module of $\tilde{K}$-finite vectors preserved by c_s.

The function spaces associated to Segal-Shale-Weil representation are supported on $\mathbb{R} \subset \mathbb{R}^2$, a maximal isotropic subspace of $(\mathbb{R}^2, \omega)$. The restriction of Segal-Shale-Weil representation to $\mathfrak{mp}(2, \mathbb{R})$ decomposes into two representations realized on the subspace of even resp. odd functions

$$\varrho : \mathfrak{mp}(2, \mathbb{R}) \to \mathrm{End}(\mathcal{S}(\mathbb{R})), \tag{6.2}$$

where the basis vectors act by

$$\begin{aligned} &\varrho(e \cdot e) = iq^2, \\ &\varrho(f \cdot f) = -i\partial_q^2, \\ &\varrho(e \cdot f + f \cdot e) = q\partial_q + \partial_q q. \end{aligned} \tag{6.3}$$

Because it is a complex representation of $\mathfrak{mp}(2, \mathbb{R})$ we may consider complex algebra $\mathfrak{mp}(2, \mathbb{C})$ and isomorphic complex algebra $\mathfrak{sp}(2, \mathbb{C})$.

In this representation $Cl_s(\mathbb{R}^2, \omega)$ acts on $L^2(\mathbb{R}, e^{-\frac{q^2}{2}} \mathrm{d}q_{\mathbb{R}})$ by unbounded operators with the subdomain $\mathcal{S}(\mathbb{R})$. The space of $\tilde{K}$-finite vectors has a basis $\{q^j e^{-\frac{q^2}{2}}\}_{j=0}^\infty$. Its even $\mathfrak{mp}(2, \mathbb{C})$-submodule is generated by $\{q^{2j} e^{-\frac{q^2}{2}}\}_{j=0}^\infty$ and the odd by $\{q^{2j+1} e^{-\frac{q^2}{2}}\}_{j=0}^\infty$. It is an irreducible representation of $\mathfrak{mp}(2, \mathbb{C}) \ltimes \mathfrak{h}(2)$, the semidirect product of $\mathfrak{mp}(2, \mathbb{C})$ and the 3-dimensional Heisenberg Lie algebra spanned by $\{e_1, e_2, \mathrm{Id}\}$. Cf., [16]. In this chapter, we denote the Segal-Shale-Weil representation by $\mathcal{S}$. We have $\mathcal{S} \simeq \mathcal{S}^+ \oplus \mathcal{S}^-$ as $\mathfrak{mp}(2, \mathbb{C})$-module.

Let us denote by $\mathrm{Pol}(\mathbb{R}^2, \mathbb{C})$ the vector space of complex valued polynomials on $\mathbb{R}^2$, and by $\mathrm{Pol}_l(\mathbb{R}^2, \mathbb{C})$ the subspace of homogeneity l polynomials. The complex vector space $\mathrm{Pol}_l(\mathbb{R}^2,\mathbb{C})$ is as an irreducible $\mathfrak{mp}(2,\mathbb{C})$-module isomorphic to $\mathrm{S}^l(\mathbb{C}^2)$, the l-th symmetric power of the complexification of the fundamental vector representation $\mathbb{R}^2$, $l \in \mathbb{N}_0$.

6.3 Segal-Shale-Weil representation and metaplectic Howe duality

Let us review a representation-theoretical result of [2]. We consider the Borel subalgebra of $\mathfrak{sp}(2,\mathbb{C})$ generated by $X' = \frac{1}{2}\left(\begin{smallmatrix} -i & 1 \\ 1 & i \end{smallmatrix}\right)$ and $H' = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$ and Borel subalgebra of $\mathfrak{mp}(2,\mathbb{C})$ generated by elements $\lambda_\star^{-1}(X')$ and $\lambda_\star^{-1}(H')$.

Let ϖ_1 be the fundamental weight of the Lie algebra $\mathfrak{sp}(2,\mathbb{C})$, and let $L(\varpi)$ denote the simple module over universal enveloping algebra $\mathcal{U}(\mathfrak{mp}(2,\mathbb{C}))$ of $\mathfrak{mp}(2,\mathbb{C})$ generated by the highest weight vector of the weight ϖ. Then the Segal-Shale-Weil representation for $\mathfrak{mp}(2,\mathbb{C})$ on $\mathrm{Pol}(\mathbb{R}^2,\mathbb{C}) \otimes \mathcal{S}(\mathbb{R})$ is the highest weight representation $L(-\frac{1}{2}\varpi_1) \oplus L(-\frac{3}{2}\varpi_1)$.

The decomposition of the space of polynomial functions on $\mathbb{R}^2$ valued in the Segal-Shale-Weil representation corresponds to the tensor product of $L(-\frac{1}{2}\varpi_1) \oplus L(-\frac{3}{2}\varpi_1)$ with symmetric powers $\mathrm{S}^l(\mathbb{C}^{2n})$, $l \in \mathbb{N}_0$, of the fundamental vector representation $\mathbb{C}^2$ of $\mathfrak{sp}(2,\mathbb{C})$. Note that all summands in the decomposition are again irreducible representations of $\mathfrak{mp}(2,\mathbb{C})$.

Lemma 6.3.1. ([2]) Let $l \in \mathbb{N}_0$.

1. We have for $L(-\frac{1}{2}\varpi_1)$ and any l

$$\begin{aligned} L(-\frac{1}{2}\varpi_1) \otimes \mathrm{S}^l(\mathbb{C}^2) \ \simeq\ & L(-\frac{1}{2}\varpi_1) \oplus L(\varpi_1 - \frac{1}{2}\varpi_1) \oplus \ldots \\ & \oplus L((l-1)\varpi_1 - \frac{1}{2}\varpi_1) \oplus L(l\varpi_1 - \frac{1}{2}\varpi_1), \end{aligned}$$

2. We have for $L(-\frac{3}{2}\varpi_1)$ and any l

$$\begin{aligned} L(-\frac{3}{2}\varpi_1) \otimes \mathrm{S}^l(\mathbb{C}^2) \ \simeq\ & L(-\frac{3}{2}\varpi_1) \oplus L(\varpi_1 - \frac{3}{2}\varpi_1) \oplus \ldots \\ & \oplus L((l-1)\varpi_1 - \frac{3}{2}\varpi_1) \oplus L(l\varpi_1 - \frac{3}{2}\varpi_1). \end{aligned}$$

Another way of realizing this decomposition is based on the metaplectic Howe duality, [9]. The metaplectic analogue of the classical theorem on the separation of variables allows to decompose the space $\mathrm{Pol}(\mathbb{R}^2,\mathbb{C}) \otimes \mathcal{S}(\mathbb{R})$ of complex polynomials valued in the Segal-Shale-Weil representation under the action of $\mathfrak{mp}(2,\mathbb{R})$ into a direct sum of simple weight $\mathfrak{mp}(2,\mathbb{R})$-modules

$$\mathrm{Pol}(\mathbb{R}^2,\mathbb{C}) \otimes \mathcal{S}(\mathbb{R}) \simeq \bigoplus_{l=0}^{\infty} \bigoplus_{j=0}^{\infty} X_s^j M_l, \tag{6.4}$$

where we use the notation $M_l = M_l^+ \oplus M_l^-$. This decomposition takes the form of an infinite triangle

$$
\begin{array}{cccccc}
P_0 \otimes \mathcal{S} & P_1 \otimes \mathcal{S} & P_2 \otimes \mathcal{S} & P_3 \otimes \mathcal{S} & P_4 \otimes \mathcal{S} & P_5 \otimes \mathcal{S} \\
\| & \| & \| & \| & \| & \| \\
M_0 \longrightarrow & X_s M_0 \longrightarrow & X_s^2 M_0 \longrightarrow & X_s^3 M_0 \longrightarrow & X_s^4 M_0 \longrightarrow & X_s^5 M_0 \\
 & \oplus & \oplus & \oplus & \oplus & \oplus \\
 & M_1 \longrightarrow & X_s M_1 \longrightarrow & X_s^2 M_1 \longrightarrow & X_s^3 M_1 \longrightarrow & X_s^4 M_1 \\
 & & \oplus & \oplus & \oplus & \oplus \\
 & & M_2 \longrightarrow & X_s M_2 \longrightarrow & X_s^2 M_2 \longrightarrow & X_s^3 M_2 \\
 & & & \oplus & \oplus & \oplus \\
 & & & M_3 \longrightarrow & X_s M_3 \longrightarrow & X_s^2 M_3 \\
 & & & & \oplus & \oplus \\
 & & & & M_4 \longrightarrow & X_s M_4 \\
 & & & & & \oplus \\
 & & & & & M_5
\end{array}
\tag{6.5}
$$

Now, let us explain the notation used in the previous scheme. First of all, we use the shorthand notation $P_l = \mathrm{Pol}_l(\mathbb{R}^2, \mathbb{C})$, $l \in \mathbb{N}_0$, $\mathcal{S} = \mathcal{S}(\mathbb{R})$, and all spaces and arrows in the picture have the following meaning. We denote $M_l = \mathrm{Pol}_l(\mathbb{R}^2, \mathbb{C}) \cap \mathrm{Ker}(D_s)$, where we set the three operators ($i \in \mathbb{C}$ is the complex unit)

$$
\begin{aligned}
X_s &= y\partial_q + ixq, \\
D_s &= iq\partial_y - \partial_x\partial_q, \\
E &= x\partial_x + y\partial_y,
\end{aligned}
\tag{6.6}
$$

The operator D_s acts horizontally as X_s but in the opposite direction. They fulfil the $\mathfrak{sl}(2, \mathbb{C})$-commutation relations

$$
\begin{aligned}
[E, D_s] &= -D_s, \\
[E, X_s] &= X_s, \\
[D_s, X_s] &= -i(E + 1).
\end{aligned}
\tag{6.7}
$$

Let $\varphi \in \mathrm{Pol}(\mathbb{R}^2, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R})$, $h \in \mathrm{Mp}(2, \mathbb{R})$ and $\lambda(h) = g \in \mathrm{Sp}(2, \mathbb{R})$. We define the action of $\mathrm{Mp}(2, \mathbb{R})$ to be

$$
\begin{aligned}
&\tilde{\varrho}(h)\varphi(x, y, q) = \varrho(h)\varphi\left(\lambda(g^{-1})\begin{pmatrix} x \\ y \end{pmatrix}, q\right) = \varrho(h)\varphi(dx - by, -cx + ay, q), \\
&g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}).
\end{aligned}
\tag{6.8}
$$

where ϱ acts on the Segal-Shale-Weil representation via (6.2). Passing to the infinitesimal action, we get the following operators representing the basis elements of $\mathfrak{mp}(2, \mathbb{R})$

$$
\begin{aligned}
\frac{d}{dt}\Big|_{t=0} \tilde{\varrho}(\exp(tX))\varphi(x, y, q) &= \frac{d}{dt}\Big|_{t=0} \varrho\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \varphi(x - yt, y, q) \\
&= -\frac{i}{2}q^2 e^{-\frac{i}{2}tq^2}\varphi(x - yt, y, q)\Big|_{t=0} \\
&\quad + e^{-\frac{i}{2}tq^2}\frac{d}{dt}\varphi(x - yt, y, q)\Big|_{t=0} \\
&= \big(-\frac{i}{2}q^2 - y\frac{\partial}{\partial x}\big)\varphi(x, y, q),
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\Big|_{t=0}\tilde{\varrho}(\exp(tH))\varphi(x,y,q) &= \frac{d}{dt}\Big|_{t=0}\varrho\begin{pmatrix} e^t & t \\ 0 & e^{-1}\end{pmatrix}\varphi(xe^{-t},ye^t,q) \\
&= \frac{1}{2}e^{\frac{1}{2}t}\varphi(xe^{-t},ye^t,qe^t)+e^{\frac{1}{2}t}\frac{d}{dt}\varphi(xe^{-t},ye^t,qe^t)\Big|_{t=0} \\
&= \big(\frac{1}{2}-x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+q\frac{\partial}{\partial q}\big)\varphi(x,y,q),
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\varrho}(X) &= -y\frac{\partial}{\partial x}-\frac{i}{2}q^2,\ \tilde{\varrho}(Y) = -x\frac{\partial}{\partial y}-\frac{i}{2}\frac{\partial^2}{\partial q^2}, \\
\tilde{\varrho}(H) &= -x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+q\frac{\partial}{\partial q}+\frac{1}{2}.
\end{aligned} \tag{6.9}
$$

They satisfy commutation rules of $\mathfrak{mp}(2,\mathbb{R})$

$$
\begin{aligned}
[\tilde{\varrho}(X),\tilde{\varrho}(Y)] &= \tilde{\varrho}(H), \\
[\tilde{\varrho}(H),\tilde{\varrho}(X)] &= 2\tilde{\varrho}(X), \\
[\tilde{\varrho}(H),\tilde{\varrho}(Y)] &= -2\tilde{\varrho}(Y).
\end{aligned}
$$

Notice that we have not derived the explicit formula for $\tilde{\varrho}(Y)$, because it easily follows from the Lie algebra structure. The action of the Casimir operator $Cas \in \mathcal{U}(\mathfrak{mp}(2,\mathbb{R}))\otimes Cl_s(\mathbb{R}^2,\omega)$

$$
Cas = \tilde{\varrho}(H)^2+1+2\tilde{\varrho}(X)\tilde{\varrho}(Y)+2\tilde{\varrho}(Y)\tilde{\varrho}(X),
$$

is given by the differential operator

$$
\begin{aligned}
Cas = & x^2\partial_x^2+y^2\partial_y^2+2x\partial_x+4y\partial_y+2xy\partial_x\partial_y+\frac{1}{4} \\
& -2xq\partial_x\partial_q+2yq\partial_y\partial_q+2iy\partial_x\partial_q^2+2ixq^2\partial_y \\
= & E_x(E_x-1)+E_y(E_y-1)+2E_x+4E_y+2E_xE_y+\frac{1}{4} \\
& -2E_xE_q+2E_yE_q+2iy\partial_x\partial_q^2+2ixq^2\partial_y.
\end{aligned} \tag{6.10}
$$

Here we use the notation $\partial_x = \frac{\partial}{\partial x}$, $\partial_x = \frac{\partial}{\partial x}$ and $E_x = x\partial_x$, $E_y = y\partial_y$, $E_q = q\partial_q$ for the Euler homogeneity operators.

Lemma 6.3.2. The operators X_s and D_s commute with the operators $\tilde{\varrho}(X), \tilde{\varrho}(Y)$ and $\tilde{\varrho}(H)$. In other words, they are $\mathfrak{mp}(2,\mathbb{R})$-invariant differential operators on complex polynomials with values in the Segal-Shale-Weil representation.

Proof. We have

$$
[D_s,\tilde{\varrho}(H)] = iq\partial_y[\partial_y,y]+iq\partial_q[q,\partial_q]+\partial_x\partial_q[\partial_x,x]-\partial_x\partial_q[\partial_q,q] = 0. \tag{6.11}
$$

Remaining commutators are computed analogously. □

The action of $\mathfrak{mp}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{C})$ generates the multiplicity free decomposition of the representation and the pair of Lie algebras in the product is called the metaplectic Howe dual pair. The operators X_s, D_s act on the previous picture horizontally and isomorphically identify the neighbouring $\mathfrak{mp}(2,\mathbb{R})$-modules. The modules M_l, $l \in \mathbb{N}$, on the left-most diagonal are termed symplectic monogenics, and are characterized as l-homogeneous solutions of the symplectic Dirac operator D_s. Thus the decomposition is given as a tensor product of the symplectic monogenics multiplied by algebra of polynomial invariants $\mathbb{C}[X_s]$. The operator X_s maps polynomial symplectic spinors valued in the odd part of $\mathcal{S}$ into symplectic spinors valued in the even part of $\mathcal{S}$. This means that M_m^- is valued in $\mathcal{S}^-$, $X_s M_m^-$ is valued in $\mathcal{S}^+$, etc.

6.4 Symplectic twistor operator T_s

We start with an abstract definition of the symplectic twistor operator T_s and then we specialize to the real symplectic manifold $(\mathbb{R}^2, \omega)$.

Definition 6.4.1. Let (M,ω) be a symplectic manifold of dimension $2n$, with symplectic connection ∇. Let ∇^s be the associated symplectic spin covariant derivative and $\omega \in \mathcal{C}^\infty(M, \wedge^2 T^\star M)$ a non-degenerate 2-form such that $\nabla\omega = 0$. We denote by

$$\{e_1, \ldots, e_{2n}\} \equiv \{e_1, \ldots, e_n, f_1, \ldots, f_n\}$$

a local symplectic frame. The symplectic twistor operator T_s on M is the first order differential operator T_s acting on smooth symplectic spinors $\mathbf{S}$

$$\begin{aligned} &\nabla^s : \mathcal{C}^\infty(M,\mathbf{S}) \longrightarrow T^\star M \otimes \mathcal{C}^\infty(M,\mathbf{S}), \\ &T_s = P_{\mathrm{Ker}(c_s)} \circ \omega^{-1} \circ \nabla^s : \mathcal{C}^\infty(M,\mathbf{S}) \longrightarrow \mathcal{C}^\infty(M,\mathcal{T}), \end{aligned} \tag{6.12}$$

where $\mathcal{T}$ is the space of symplectic twistors, $T^\star M \otimes \mathbf{S} \simeq \mathbf{S} \oplus \mathcal{T}$, given by algebraic projection

$$P_{\mathrm{Ker}(c_s)} : T^\star M \otimes \mathcal{C}^\infty(M,\mathbf{S}) \longrightarrow \mathcal{C}^\infty(M,\mathcal{T})$$

on the kernel of the symplectic Clifford multiplication c_s. In the local symplectic coframe $\{\epsilon^1\}_{j=1}^{2n}$ dual to the symplectic frame $\{e_j\}_{j=1}^{2n}$, we have the local formula for T_s

$$T_s = \left(1 + \frac{1}{n}\right) \sum_{k=1}^{2n} \epsilon^k \otimes \nabla^s_{e_k} + \frac{i}{n} \sum_{j,k,l=1}^{2n} \epsilon^l \otimes \omega^{kj} e_j \cdot e_l \cdot \nabla^s_{e_k}, \tag{6.13}$$

where $\cdot$ is the shorthand notation for the symplectic Clifford multiplication and $i \in \mathbb{C}$ is the imaginary unit. We use the convention $\omega^{kj} = 1$ for $j = k+n$ and $k = 1, \ldots, n$, $\omega^{kj} = -1$ for $k = n+1, \ldots, 2n$ and $j = k-n$, and $\omega^{kj} = 0$ otherwise.

The symplectic Dirac operator D_s, defined as the image of the symplectic Clifford multiplication c_s, has the explicit form (6.6).

Lemma 6.4.1. The symplectic twistor operator T_s is $\mathrm{Mp}(2n,\mathbb{R})$-invariant.

Proof. The property of invariance is a direct consequence of the equivariance of symplectic covariant derivative and the invariance of algebraic projection $P_{\mathrm{Ker}(c_s)}$, and amounts to show that

$$T_s(\tilde{\varrho}(g)\varphi) = \lambda(g) \otimes \tilde{\varrho}(g)(T_s\varphi) \tag{6.14}$$

for any $g \in \mathrm{Mp}(2n, \mathbb{R})$ and $\varphi \in \mathcal{C}^\infty(M, \mathbf{S})$. Using the local formula (6.13) for T_s in a local chart $(x_1, \ldots, x_{2n})$, both sides of (6.14) are equal

$$\Big(1 + \frac{1}{n}\Big) \sum_{k=1}^{2n} \epsilon^k \otimes \varrho(g) \frac{\partial}{\partial x_k} \big[\varphi\big(\lambda(g)^{-1}x\big)\big]$$
$$+ \frac{i}{n} \sum_{j,k,l=1}^{2n} \epsilon^l \otimes \omega^{kj} e_j \cdot e_l \cdot \Big[\varrho(g) \frac{\partial}{\partial x_k} \big[\varphi\big(\lambda(g)^{-1}x\big)\big]\Big]$$

and the proof follows. □

In the case $M = (\mathbb{R}^{2n}, \omega)$, the symplectic twistor operator is

$$T_s = \Big(1 + \frac{1}{n}\Big) \sum_{k=1}^{2n} \epsilon^k \otimes \frac{\partial}{\partial x_k} + \frac{i}{n} \sum_{j,k,l=1}^{2n} \epsilon^l \otimes \omega^{kj} e_j \cdot e_l \cdot \frac{\partial}{\partial x_k}. \tag{6.15}$$

Lemma 6.4.2. In the case of the real symplectic manifold $(\mathbb{R}^2, \omega)$ with coordinates x, y and $\omega = \mathrm{d}x \wedge \mathrm{d}y$, a symplectic frame $\{e_1, e_2\}$ and its dual coframe $\{\epsilon^1, \epsilon^2\}$, the symplectic twistor operator $T_s : \mathcal{C}^\infty(\mathbb{R}^2, \mathbf{S}) \to \mathcal{C}^\infty(\mathbb{R}^2, \mathcal{T})$ acts on a smooth symplectic spinor $\varphi(x, y, q) \in C^\infty(\mathbb{R}^2, \mathbf{S})$ by

$$T_s(\varphi) = \epsilon^1 \otimes \Big(\frac{\partial \varphi}{\partial x} - q\frac{\partial^2 \varphi}{\partial q \partial x} + iq^2 \frac{\partial \varphi}{\partial y}\Big) + \epsilon^2 \otimes \Big(2\frac{\partial \varphi}{\partial y} + i\frac{\partial^3 \varphi}{\partial q^2 \partial x} + q\frac{\partial^2 \varphi}{\partial q \partial y}\Big). \tag{6.16}$$

The last display follows from (6.15) by direct substitution for the symplectic Clifford multiplication. The next Lemma simplifies the condition on a symplectic spinor to be in the kernel of T_s.

Lemma 6.4.3. A smooth symplectic spinor $\varphi(x, y, q) \in C^\infty(\mathbb{R}^2, \mathbf{S})$ is in the kernel of T_s if and only if it fulfils the partial differential equation

$$\Big(\frac{\partial}{\partial x} - q\frac{\partial^2}{\partial q \partial x} + iq^2 \frac{\partial}{\partial y}\Big)\varphi = 0. \tag{6.17}$$

Proof. The claim is a consequence of Lemma 6.4.2, because the covectors ϵ^1, ϵ^2 are linearly independent and the differential operators in (6.16) (the two components of T_s by ϵ^1 and ϵ^2) have the same solution space (i.e., φ solving the first one implies that φ solves the second one). This implies the equivalence statement in the Lemma. □

Notice that $\tilde{\varrho}(X), \tilde{\varrho}(Y)$ and $\tilde{\varrho}(H)$ preserve the solution space of the symplectic twistor equation (6.17), i.e., if the symplectic spinor φ solves (6.17) then $\tilde{\varrho}(X)\varphi, \tilde{\varrho}(Y)\varphi$ and $\tilde{\varrho}(H)\varphi$ solve (6.17) . This is a consequence of $\mathrm{Mp}(2, \mathbb{R})$-invariance of the symplectic twistor operator T_s on $\mathbb{R}^2$ (in fact, the same observation is true in any dimension.) By abuse of notation, we use T_s in Section 6.5 to denote the operator (6.17) and call it the symplectic twistor operator - this terminology is justified by the reduction in Lemma 6.4.3. In the chapter, we work with polynomial (in x, y or $z, \bar{z}$) smooth symplectic spinors, i.e. with elements of $\mathrm{Pol}(\mathbb{R}^2, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R})$.

6.5 Polynomial solution space of symplectic twistor operator T_s on $(\mathbb{R}^2, \omega)$

Let us consider the complex vector space of symplectic spinor valued polynomials $\mathrm{Pol}(\mathbb{R}^2, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R})$, $\mathcal{S}(\mathbb{R}) \simeq \mathcal{S}(\mathbb{R})^- \oplus \mathcal{S}(\mathbb{R})^+$, together with its decomposition on irreducible subspaces with respect to the natural action of $\mathfrak{mp}(2, \mathbb{R})$. It follows from the $\mathfrak{mp}(2, \mathbb{R})$-invariance of the symplectic twistor operator that it is sufficient to characterize its behaviour on any non-zero vector in an irreducible $\mathfrak{mp}(2, \mathbb{R})$-submodule. Further, the action of the symplectic twistor operator preserves the subspace of homogeneous symplectic spinors. This is what we are going to accomplish in the present section. Note that the meaning of the natural number $n \in \mathbb{N}$ used in previous sections to denote the dimension of the underlying real symplectic manifold is different from its use in the present section.

The main technical difficulty consists of finding suitable representative smooth vectors in each irreducible $\mathfrak{mp}(2, \mathbb{R})$-subspace. We shall find a general characterizing condition for a polynomial (in the real variables x, y) valued in the Schwartz space $\mathcal{S}(\mathbb{R})$ (in the variable q) as a formal power series, and the representative vectors are always conveniently chosen as polynomials in q weighted by the exponential $e^{-\frac{q^2}{2}}$. In other words, the constructed vectors are $\tilde{K}$-finite vectors in $\mathcal{S}(\mathbb{R})$. These representative vectors are then evaluated on the symplectic twistor operator T_s.

First of all, the constant symplectic spinors belong to the solution space of T_s. We have

Lemma 6.5.1.

$$T_s(X_s e^{-\frac{q^2}{2}}) = T_s(ie^{-\frac{q^2}{2}} q(x + iy)) = 0, \tag{6.18}$$

$$T_s(X_s q e^{-\frac{q^2}{2}}) = T_s(e^{-\frac{q^2}{2}} (iq^2(x + iy) + y)) = 0. \tag{6.19}$$

The next Lemma is preparatory for further considerations.

Lemma 6.5.2. We have for any $n \in \mathbb{N}_0$, $(X_s)^n \in \mathrm{End}(\mathrm{Pol}(\mathbb{R}^2, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R}))$, the following identity

$$(X_s)^n = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{k=0}^{n-2j} A^n_{jk} y^{n-j-k} (ix)^{j+k} q^k \partial_q^{n-2j-k}. \tag{6.20}$$

Here $\lfloor \frac{n}{2} \rfloor$ is the floor function applied to $\frac{n}{2}$, and the coefficients $A^n_{jk} \in \mathbb{C}$ fulfil the 4-term recurrent relation

$$A^n_{jk} = A^{(n-1)}_{jk} + A^{(n-1)}_{j(k-1)} + (k + 1) A^{(n-1)}_{(j-1)(k+1)}. \tag{6.21}$$

We use the normalization $A^0_{00} = 1$, and $A^n_{jk} \neq 0$ only for $n \in \mathbb{N}_0$, $j = 0, \ldots, \lfloor \frac{n}{2} \rfloor$, and $k = 0, \ldots, n - 2j$.

Proof. The proof is by induction on $n \in \mathbb{N}_0$. The claim is trivial for $n = 0$, and for $n = 1$ we have

$$(X_s)^1 = A^1_{00} y \partial_q + A^1_{01} ixq,$$

where $A^1_{00} = A^0_{00} = 1$ and $A^1_{01} = A^0_{00} = 1$.

We assume that the formula holds for $n-1$ and aim to prove it for n

$$
\begin{aligned}
&(ixq + y\partial_q)\Big(\sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{k=0}^{n-1-2j} A^{(n-1)}_{jk} y^{n-1-j-k}(ix)^{j+k} q^k \partial_q^{n-1-2j-k}\Big) \\
&= \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{k=0}^{n-1-2j} A^{(n-1)}_{jk} \Big(y^{n-1-j-k}(ix)^{j+k+1} q^{k+1} \partial_q^{n-1-2j-k} \\
&\quad + y^{n-j-k}(ix)^{j+k} q^k \partial_q^{n-2j-k} + k y^{n-j-k}(ix)^{j+k} q^{k-1} \partial_q^{n-1-2j-k}\Big) \\
&= \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{k=0}^{n-2j} A^{(n-1)}_{j(k-1)} y^{n-j-k}(ix)^{j+k} q^k \partial_q^{n-2j-k} \\
&\quad + \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{k=0}^{n-2j} A^{(n-1)}_{jk} y^{n-j-k}(ix)^{j+k} q^k \partial_q^{n-2j-k} \\
&\quad + \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor+1} \sum_{k=0}^{n-2j} (k+1) A^{(n-1)}_{(j-1)(k+1)} y^{n-j-k}(ix)^{j+k} q^k \partial_q^{n-2j-k},
\end{aligned}
$$

where we shifted the indexes in the first sum $k \mapsto k-1$, in the third sum by $k \mapsto k+1$ and $j \mapsto j-1$ and added zero elements in the summations. Altogether we get

$$
\begin{aligned}
&\sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{k=0}^{n-2j} \Big(A^{(n-1)}_{jk} + A^{(n-1)}_{j(k-1)} + (k+1) A^{(n-1)}_{(j-1)(k+1)} \Big) y^{n-j-k}(ix)^{j+k} q^k \partial_q^{n-2j-k} \\
&+ \sum_{k=0}^{n-2\lfloor \frac{n-1}{2} \rfloor -2} (k+1) A^{(n-1)}_{\lfloor \frac{n-1}{2} \rfloor (k+1)} y^{n-\lfloor \frac{n-1}{2} \rfloor -1-k} (ix)^{\lfloor \frac{n-1}{2} \rfloor +1+k} q^k \partial_q^{n-2\lfloor \frac{n-1}{2} \rfloor -2-k}.
\end{aligned}
$$

Now we apply the induction argument to the first term. The second term is non zero only for even n, when the previous expression equals to

$$
\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{k=0}^{n-2j} A^n_{jk} y^{n-j-k}(ix)^{j+k} q^k \partial_q^{n-2j-k},
$$

which completes the required statement. □

Remark 5. Notice that for $j = 0$, the solution of recurrent relation in (6.21) corresponds to the binomial coefficients. It follows from $A^{(n-1)}_{(-1)(k+1)} = 0$,

$$
A^n_{0k} = A^{(n-1)}_{0k} + A^{(n-1)}_{0(k-1)},
$$

and therefore, $A^n_{0k} = \binom{n}{k}$.

Lemma 6.5.3. We have $A^n_{1(n-2)} = \frac{n(n-1)}{2} = \binom{n}{n-2}$.

Proof. We use the relation $A^n_{1(n-2)} = A^{(n-1)}_{1(n-2)} + A^{(n-1)}_{1(n-3)} + (n-1)A^{(n-1)}_{0(n-1)}$, where $A^{n-1}_{1(n-2)} = 0$ (because it is out of the range for the index k in the equation (6.21).) The proof goes by induction in n: we start with $A^2_{10} = A^1_{01} = 1$, and claim $A^n_{1(n-2)} = \frac{n(n-1)}{2}$. The induction step gives $A^{(n+1)}_{1(n-1)} = A^n_{1(n-2)} + nA^n_{0n} = \frac{n^2-n}{2} + n = \frac{n^2+n}{2}$. □

A direct consequence of the Baker-Campbell-Hausdorff formula or its dual, Zassenhaus formula, for three operators A, B, C fulfilling the commutation relations $[A, B] = C$ and $[A, C] = [B, C] = 0$ gives

$$(A+B)^n = \sum_{\substack{l \le n \\ l \equiv n \mod 2}} \left(\sum_{r=0}^{l} \binom{l}{r} A^r B^{l-r} \right) \left(-\frac{C}{2} \right)^{\frac{n-l}{2}} \frac{n!}{l! \left(\frac{n-l}{2}\right)!}. \tag{6.22}$$

Thus we get the solution of the recursion relation (6.21).

Lemma 6.5.4. For $X_s = ixq + y\partial_q$ with $[ixq, y\partial_q] = -ixy$, we have

$$(ixq + y\partial_q)^n = \sum_{\substack{l \le n \\ l \equiv n \mod 2}} \left(\sum_{r=0}^{k} \frac{n!}{r!(l-r)! \left(\frac{n-l}{2}\right)! 2^{\frac{n-l}{2}}} (ix)^{r+\frac{n-l}{2}} y^{l-r+\frac{n-l}{2}} q^r \partial_q^{l-r} \right) \tag{6.23}$$

for any $n \in \mathbb{N}_0$, and the comparison with Lemma 6.5.2 yields the solution of the recursion relation (6.21):

$$A^n_{jk} = \frac{n!}{k!(n-2j-k)!j!2^j}, \tag{6.24}$$

where the index l in (6.23) corresponds to $n - 2j$ in (6.20), and the index r in (6.23) corresponds to k in (6.20).

Let us remark that the composition $T_s \circ (X_s)^n$ for $n = 2, 3$, acting on $e^{-\frac{q^2}{2}}$ and $qe^{-\frac{q^2}{2}}$, is non-vanishing. This means that some irreducible $\mathfrak{mp}(2, \mathbb{R})$-components in the decomposition (6.5) are not in the kernel of T_s

$$\begin{aligned}
T_s(X_s^2 e^{-\frac{q^2}{2}}) &= e^{-\frac{q^2}{2}}(q^2x + iy + iq^2y) \neq 0, \\
T_s(X_s^2 q e^{-\frac{q^2}{2}}) &= e^{-\frac{q^2}{2}}(q^3x + iq^3y) \neq 0, \\
T_s(X_s^3 e^{-\frac{q^2}{2}}) &= e^{-\frac{q^2}{2}}(3iq^3x^2 - 6q^3xy - 3iq^3y^2) \neq 0, \\
T_s(X_s^3 q e^{-\frac{q^2}{2}}) &= e^{-\frac{q^2}{2}}(3iq^4x^2 + 6q^2xy - 6q^4xy + 3iy^2 + 6iq^2y^2 \\
&\qquad - 3iq^4y^2) \neq 0,
\end{aligned} \tag{6.25}$$

Lemma 6.5.5. Let $n \in \mathbb{N}_0$. Then

$$\begin{aligned}
T_s \circ (X_s)^n = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{k=0}^{n-2j} A^n_{jk} \Big(& i(j+k) y^{n-j-k} (ix)^{j+k-1} q^k \partial_q^{n-2j-k} \\
&+ y^{n-j-k}(ix)^{j+k} q^k \partial_x \partial_q^{n-2j-k} - i(j+k) y^{n-j-k} (ix)^{j+k-1} q^{k+1} \partial_q^{n-2j-k+1} \\
&- y^{n-j-k} (ix)^{j+k} q^{k+1} \partial_x \partial_q^{n-2j-k+1} - ik(j+k) y^{n-j-k} (ix)^{j+k-1} q^k \partial_q^{n-2j-k} \\
&- k y^{n-j-k} (ix)^{j+k} q^k \partial_x \partial_q^{n-2j-k} + i(n-j-k) y^{n-j-k-1} (ix)^{j+k} q^{k+2} \partial_q^{n-2j-k} \\
&+ i y^{n-j-k} (ix)^{j+k} q^{k+2} \partial_y \partial_q^{n-2j-k} \Big).
\end{aligned} \tag{6.26}$$

In particular, $T_s((X_s)^n e^{-\frac{q^2}{2}}) \neq 0$ and $T_s((X_s)^n q e^{-\frac{q^2}{2}}) \neq 0$ for all $n > 1$.

Proof. The proof is based on the identity in Lemma 6.5.2. The non-triviality of the composition is detected by the coefficient in the monomial $x^{n-1}q^n e^{-\frac{q^2}{2}}$ in an expansion of $T_s((X_s)^n e^{-\frac{q^2}{2}})$. It follows from the identity (6.26) that this coefficient is

$$\begin{aligned} & i^n(A^n_{0n}n - A^n_{0n}n^2 + A^n_{1(n-2)})x^{n-1}q^n e^{-\frac{q^2}{2}} = \\ & = i^n \left(\binom{n}{n}(n-n^2) + \binom{n}{n-2} \right) x^{n-1}q^n e^{-\frac{q^2}{2}} \\ & = -i^n \frac{n(n-1)}{2} x^{n-1}q^n e^{-\frac{q^2}{2}}, \end{aligned} \tag{6.27}$$

which is non-zero for all $n > 1$.

As for the action on the vector $qe^{-\frac{q^2}{2}}$, the situation is analogous. The coefficient of the monomial $x^{n-1}q^{n+1}e^{-\frac{q^2}{2}}$ in $T_s((X_s)^n q e^{-\frac{q^2}{2}})$ is $-i^n \frac{n(n-1)}{2}$, which is again non-zero for all $n > 1$. The proof is complete. □

In the next part we focus for a while on symplectic spinors given by iterative action of X_s on $\mathcal{S}^+$, and complete the task of finding all subspaces of polynomial solutions of T_s (expressed in the real variables x, y).

Lemma 6.5.6. The vectors $e^{-\frac{q^2}{2}}(x+iy)^m \in \mathrm{Pol}_m(\mathbb{R}^2, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R})^+$, $m \in \mathbb{N}_0$, are in the kernel of D_s, but not in the kernel of the symplectic twistor operator T_s.

Proof. We get by direct computation,

$$\begin{aligned} D_s\big(e^{-\frac{q^2}{2}}(x+iy)^m\big) =& iq\partial_y e^{-\frac{q^2}{2}}(x+iy)^m - \partial_x\partial_q e^{-\frac{q^2}{2}}(x+iy)^m \\ =& e^{-\frac{q^2}{2}}(-mq(x+iy)^{m-1} + mq(x+iy)^{m-1}) = 0, \\ T_s\big(e^{-\frac{q^2}{2}}(x+iy)^m\big) =& \partial_x e^{-\frac{q^2}{2}}(x+iy)^m - q\partial_x\partial_q e^{-\frac{q^2}{2}}(x+iy)^m \\ & + iq^2\partial_y e^{-\frac{q^2}{2}}(x+iy)^m = e^{-\frac{q^2}{2}} m(x+iy)^{m-1} \neq 0 \end{aligned}$$

for any natural number $m > 0$. □

Lemma 6.5.7. Let $m \in \mathbb{N}_0$. Then the vectors $X_s e^{-\frac{q^2}{2}}(x+iy)^m$ in $\mathrm{Pol}_{m+1}(\mathbb{R}^2, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R})^+$ are in the kernel of the symplectic twistor operator T_s.

Proof. We have

$$\begin{aligned} & T_s(X_s e^{-\frac{q^2}{2}}(x+iy)^m) = T_s(iqe^{-\frac{q^2}{2}}(x+iy)^{m+1}) \\ & = i(m+1)e^{-\frac{q^2}{2}}(q-q+q^2-q^2)(x+iy)^m = 0. \end{aligned}$$

□

Remark 6. The non-trivial elements in $\mathrm{Ker}(T_s)$ are

$$qe^{-\frac{q^2}{2}}(x+iy)^k,\ k \in \mathbb{N}_0. \tag{6.28}$$

The next Lemma completes the information on the behaviour of T_s for remaining $\mathfrak{mp}(2,\mathbb{R})$-modules coming from the action of X_s on $\mathcal{S}^+$.

Lemma 6.5.8. For all natural numbers $n > 1$ and all $m \in \mathbb{N}_0$, we have

$$T_s((X_s)^n e^{-\frac{q^2}{2}}(x+iy)^m) \neq 0. \tag{6.29}$$

Proof. We focus on the coefficient of the monomial $x^{n-1+m}q^n e^{-\frac{q^2}{2}}$ in the expanded form of $T_s((X_s)^n e^{-\frac{q^2}{2}}(x+iy)^m)$. It follows from (6.26) that the contribution to this coefficient is

$$\begin{aligned} i^n(A^n_{0n}n - &A^n_{0n}n^2 + A^n_{1(n-2)} + A^n_{0n}m - A^n_{0n}mn)x^{n-1+m}q^n e^{-\frac{q^2}{2}} \\ &= i^n\left(\binom{n}{n}(n-n^2+m-mn) + \binom{n}{n-2}\right)x^{n-1+m}q^n e^{-\frac{q^2}{2}} \\ &= -i^n\frac{(n+2m)(n-1)}{2}x^{n-1+m}q^n e^{-\frac{q^2}{2}}, \end{aligned} \tag{6.30}$$

which is non-zero for all natural numbers $n > 1$ and all $m \in \mathbb{N}_0$. ☐

Let us summarize the previous lemmas in the final Theorem.

Theorem 6.5.9. The solution space of the symplectic twistor operator T_s acting on $\mathrm{Pol}(\mathbb{R}^2,\mathbb{C}) \otimes \mathcal{S}(\mathbb{R})^+$ consists of the set of $\mathfrak{mp}(2,\mathbb{R})$-modules in the boxes, realized in the decomposition of $\mathrm{Pol}(\mathbb{R}^2,\mathbb{C}) \otimes \mathcal{S}(\mathbb{R})^+$ on $\mathfrak{mp}(2,\mathbb{R})$ irreducible subspaces

$$\begin{array}{cccccccc}
\boxed{M_0^+} \to & \boxed{X_sM_0^+} \to & X_s^2M_0^+ \to & X_s^3M_0^+ \to & X_s^4M_0^+ \to & X_s^5M_0^+ & \dots \\
{\scriptstyle e^{-\frac{q^2}{2}}} & \oplus & \oplus & \oplus & \oplus & \oplus & \\
& M_1^+ \longrightarrow & \boxed{X_sM_1^+} \to & X_s^2M_1^+ \to & X_s^3M_1^+ \to & X_s^4M_1^+ & \dots \\
& {\scriptstyle e^{-\frac{q^2}{2}}(x+iy)} & \oplus & \oplus & \oplus & \oplus & \\
& & M_2^+ \longrightarrow & \boxed{X_sM_2^+} \to & X_s^2M_2^+ \to & X_s^3M_2^+ & \dots \\
& & {\scriptstyle e^{-\frac{q^2}{2}}(x+iy)^2} & \oplus & \oplus & \oplus & \\
& & & M_3^+ \longrightarrow & \boxed{X_sM_3^+} \to & X_s^2M_3^+ & \dots \\
& & & {\scriptstyle e^{-\frac{q^2}{2}}(x+iy)^3} & \oplus & \oplus & \\
& & & & M_4^+ \longrightarrow & \boxed{X_sM_4^+} & \dots \\
& & & & {\scriptstyle e^{-\frac{q^2}{2}}(x+iy)^4} & \oplus & \\
& & & & & M_5^+ & \dots
\end{array} \tag{6.31}$$

Notice that non-zero representative vectors in the solution space of D_s are pictured under the spaces of symplectic monogenics.

This completes the picture in the case of $\mathcal{S}^+$. As we shall see, the representative solutions of D_s for arbitrary homogeneity are far more complicated for $\mathcal{S}^-$ than for $\mathcal{S}^+$, which were chosen to be the powers of $z = x + iy$. A rather convenient way to simplify the presentation is to pass from the real coordinates x, y to the complex coordinates $z, \overline{z}$ for the standard complex structure on $\mathbb{R}^2$, where $\partial_x = (\partial_z + \partial_{\bar{z}})$ and $\partial_y = i(\partial_z - \partial_{\bar{z}})$.

Lemma 6.5.10. The operators X_s, D_s and T_s are in the complex coordinates $z, \bar{z}$ given by

$$
\begin{aligned}
X_s &= \frac{i}{2}\big((q-\partial_q)z + (q+\partial_q)\bar{z}\big),\\
D_s &= -\big((q+\partial_q)\partial_z + (-q+\partial_q)\partial_{\bar{z}}\big),\\
T_s &= \big((1-q\partial_q-q^2)\partial_z + (1-q\partial_q+q^2)\partial_{\bar{z}}\big).
\end{aligned}
\tag{6.32}
$$

In the rest of the chapter we suppress the overall constants in X_s, D_s, T_s. The reason is that both the metaplectic Howe duality and the solution space of D_s, T_s are independent of the normalization of X_s, D_s, T_s. In other words, the representative solutions differ by a non-zero multiple, a property which has no effect on our results. An element of the solution space of D_s is called symplectic monogenic.

We start with the characterization of elements in the solution space of D_s, both for $\mathcal{S}^+$ and $\mathcal{S}^-$.

Theorem 6.5.11. 1. The symplectic spinor of the homogeneity $m \in \mathbb{N}_0$ in the variables z, $\bar{z}$,

$$\varphi = e^{-\frac{q^2}{2}} q\big(A^m(q)z^m + A^{m-1}(q)z^{m-1}\bar{z} + \ldots + A^1(q)z\bar{z}^{m-1} + A^0(q)\bar{z}^m\big), \tag{6.33}$$

with coefficients in the formal power series in q,

$$A^r(q) = a_0^r + a_2^r q^2 + a_4^r q^4 + \ldots,\ a_k^r \in \mathbb{C},\ r = 0, \ldots, m,\ k \in 2\mathbb{N}_0$$

is in the kernel of D_s provided the coefficients a_k^r satisfy the system of recurrence relations

$$
\begin{aligned}
0 &= m(k+1)a_k^m + (k+1)a_k^{m-1} - 2a_{k-2}^{m-1},\\
0 &= (m-1)(k+1)a_k^{m-1} + 2(k+1)a_k^{m-2} - 4a_{k-2}^{m-2},\\
&\ \ldots\\
0 &= 2(k+1)a_k^2 + (m-1)(k+1)a_k^1 - 2(m-1)a_{k-2}^1,\\
0 &= (k+1)a_k^1 + m(k+1)a_k^0 - 2ma_{k-2}^0,
\end{aligned}
\tag{6.34}
$$

equivalent to

$$(m-p)(k+1)a_k^{m-p} + (p+1)(k+1)a_k^{m-1-p} - 2(p+1)a_{k-2}^{m-1-p} = 0, \tag{6.35}$$

for all $p = 0, 1, \ldots, m-1$.

2. The symplectic spinor of the homogeneity $m \in \mathbb{N}_0$ in the variables z, $\bar{z}$,

$$\varphi = e^{-\frac{q^2}{2}} \big(A^m(q)z^m + A^{m-1}(q)z^{m-1}\bar{z} + \ldots + A^1(q)z\bar{z}^{m-1} + A^0(q)\bar{z}^m\big), \tag{6.36}$$

with coefficients in the formal power series in q,

$$A^r(q) = a_0^r + a_2^r q^2 + a_4^r q^4 + \ldots,\ a_k^r \in \mathbb{C},\ r = 0, \ldots, m,\ k \in 2\mathbb{N}_0$$

is in the kernel of D_s provided the coefficients a_k^r satisfy the system of recurrence relations

$$\begin{aligned}
0 &= mka_k^m + ka_k^{m-1} - 2a_{k-2}^{m-1},\\
0 &= (m-1)ka_k^{m-1} + 2ka_k^{m-2} - 4a_{k-2}^{m-2},\\
&\dots\\
0 &= 2ka_k^2 + (m-1)ka_k^1 - 2(m-1)a_{k-2}^1,\\
0 &= ka_k^1 + mka_k^0 - 2ma_{k-2}^0,
\end{aligned} \tag{6.37}$$

equivalent to

$$(m-p)ka_k^{m-p} + (p+1)ka_k^{m-1-p} - 2(p+1)a_{k-2}^{m-1-p} = 0, \tag{6.38}$$

for all $p = 0, 1, \dots, m-1$.

Proof. Because

$$(q+\partial_q)e^{-\frac{q^2}{2}}qA^r(q) = e^{-\frac{q^2}{2}}[q^2 + 1 - q^2 + q\partial_q]A^r(q),$$

$$(-q+\partial_q)e^{-\frac{q^2}{2}}qA^r(q) = e^{-\frac{q^2}{2}}[-q^2 + 1 - q^2 + q\partial_q]A^r(q),$$

the action of D_s on the vector $e^{-\frac{q^2}{2}}qA^r(q)$ is

$$\begin{aligned}
&D_s\Big(e^{-\frac{q^2}{2}}q\big(A^m(q)z^m + A^{m-1}(q)z^{m-1}\bar{z} + \dots + A^1(q)z\bar{z}^{m-1} + A^0(q)\bar{z}^m\big)\Big)\\
&= e^{-\frac{q^2}{2}}\Big(z^{m-1}\big(m[1+q\partial_q]A^m(q) + [1+q\partial_q - 2q^2]A^{m-1}(q)\big)\\
&\qquad z^{m-2}\bar{z}\big((m-1)[1+q\partial_q]A^{m-1}(q) + 2[1+q\partial_q - 2q^2]A^{m-2}(q)\big)\\
&\qquad \vdots\\
&\qquad z\bar{z}^{m-1}\big(2[1+q\partial_q]A^2(q) + (m-1)[1+q\partial_q - 2q^2]A^1(q)\big)\\
&\qquad \bar{z}^m\big([1+q\partial_q]A^1(q) + m[1+q\partial_q - 2q^2]A^0(q)\big)\Big).
\end{aligned} \tag{6.39}$$

The action of $[1+q\partial_q]$ on $A^r(q)$ yields $\sum_{k\in 2\mathbb{N}}(k+1)a_k^r q^k$, and the action of $[1+q\partial_q - 2q^2]$ on $A^r(q)$ gives $\sum_{k\in 2\mathbb{N}}((k+1)a_k^r - 2a_{k-2}^r)q^k$, for all $r = 0, \dots, m$.

As for the second part, we have

$$\begin{aligned}
(q+\partial_q)e^{-\frac{q^2}{2}}A^r(q) &= e^{-\frac{q^2}{2}}[\partial_q]A^r(q),\\
(-q+\partial_q)e^{-\frac{q^2}{2}}A^r(q) &= e^{-\frac{q^2}{2}}[-2q+\partial_q]A^r(q),
\end{aligned}$$

and the rest of the proof is analogous to the first part. The proof is complete. □

Remark 7. We observe that the choice of the constant $A^0(q) = a_0^0 \neq 0$, i.e. $a_k^0 \neq 0$ only for $k = 0$, leads to the solution (polynomial in q) of the recurrence relation for all coefficients in the symplectic spinor (6.33)

$$\begin{aligned}
A^0(q) =& a_0^0,\\
A^1(q) =& \left(-1+\frac{2}{3}q^2\right)\binom{m}{1}a_0^0,\\
& \dots\\
A^r(q) =& \left((-1)^r+\ldots+\frac{2^r}{(2r+1)!!}q^{2r}\right)\binom{m}{r}a_0^0,\\
& \dots\\
A^m(q) =& \left((-1)^m+\ldots+\frac{2^m}{(2m+1)!!}q^{2m}\right)\binom{m}{m}a_0^0,
\end{aligned}$$

where $(2m+1)!! = (2m+1)\cdot(2m-1)\cdots 3\cdot 1$. In this way, we get simple representative vectors in the kernel of D_s, valued in $\mathcal{S}^-$ for each homogeneity m. We have for $m=1,2,3$

$$\begin{aligned}
& e^{-\frac{q^2}{2}}q\Big(\Big(-1+\frac{2}{3}q^2\Big)z+\bar{z}\Big)a_0^0,\\
& e^{-\frac{q^2}{2}}\Big(q\Big(1-\frac{4}{3}q^2+\frac{4}{15}q^4\Big)z^2+\Big(-2+\frac{4}{3}q^2\Big)z\bar{z}+\bar{z}^2\Big)a_0^0,\\
& e^{-\frac{q^2}{2}}\Big(q\Big(-1+2q^2-\frac{12}{15}q^4+\frac{8}{105}q^6\Big)z^3+\Big(3-4q^2+\frac{4}{5}q^4\Big)z^2\bar{z}\\
& \quad +(-3+2q^2)z\bar{z}^2+\bar{z}^3\Big)a_0^0.
\end{aligned} \tag{6.40}$$

The same formulas expressed in the real variables x, y

$$\begin{aligned}
& \frac{2}{3}e^{-\frac{q^2}{2}}\Big(q^3(x+iy)-3iqy\Big)a_0^0,\\
& \frac{4}{15}e^{-\frac{q^2}{2}}\Big(q^5(x+iy)^2+10q^3y(-ix+y)-15qy^2\Big)a_0^0,\\
& \frac{8}{105}e^{-\frac{q^2}{2}}\Big(q^7(x+iy)^3-21iq^5(x+iy)^2y-105q^3(x+iy)y^2+105iqy^3\Big)a_0^0.
\end{aligned} \tag{6.41}$$

Another observation is that for a chosen homogeneity m in $z, \bar{z}$, the highest exponent of q is at least $2m+1$ and our solution realizes this minimum. The representative symplectic monogenics valued in $\mathcal{S}^+$ were already given for each homogeneity in Lemma 6.5.6.

In the following Theorem, we characterize the solution space for T_s separately in the even case (including both even powers of X_s acting on $\mathcal{S}^+$ and odd powers of X_s acting on $\mathcal{S}^-$) and the odd case (including both odd powers of X_s acting on $\mathcal{S}^+$ and even powers of X_s acting on $\mathcal{S}^-$.)

Theorem 6.5.12. 1. The symplectic spinor of the homogeneity $m \in \mathbb{N}_0$ in the variables z, $\bar{z}$,

$$\varphi = e^{-\frac{q^2}{2}}q\big(A^m(q)z^m+A^{m-1}(q)z^{m-1}\bar{z}+\ldots+A^1(q)z\bar{z}^{m-1}+A^0(q)\bar{z}^m\big) \tag{6.42}$$

with coefficients in the formal power series in q,

$$A^r = a_0^r + a_2^r q^2 + a_4^r q^4 + \dots,\ a_k^r \in \mathbb{C},\ r = 0, \dots, m,\ k \in 2\mathbb{N}_0,$$

is in the kernel of the symplectic twistor operator T_s provided the coefficients a_k^r satisfy the recurrence relations

$$\begin{aligned} 0 &= mka_k^m + ka_k^{m-1} - 2a_{k-2}^{m-1}, \\ 0 &= (m-1)ka_k^{m-1} + 2ka_k^{m-2} - 4a_{k-2}^{m-2}, \\ &\dots \\ 0 &= 2ka_k^2 + (m-1)ka_k^1 - 2(m-1)a_{k-2}^1, \\ 0 &= ka_k^1 + mka_k^0 - 2ma_{k-2}^0, \end{aligned} \tag{6.43}$$

equivalent to

$$(m-p)ka_k^{m-p} + (p+1)ka_k^{m-1-p} - 2(p+1)a_{k-2}^{m-1-p} = 0, \tag{6.44}$$

for all $p = 0, 1, \dots, m-1$.

2. The symplectic spinor of the homogeneity $m \in \mathbb{N}_0$ in the variables z, $\bar{z}$,

$$\varphi = e^{-\frac{q^2}{2}}\big(A^m(q)z^m + A^{m-1}(q)z^{m-1}\bar{z} + \dots + A^1(q)z\bar{z}^{m-1} + A^0(q)\bar{z}^m\big) \tag{6.45}$$

with coefficients in the formal power series in q,

$$A^r = a_0^r + a_2^r q^2 + a_4^r q^4 + \dots,\ a_k^r \in \mathbb{C},\ r = 0, \dots, m,\ k \in 2\mathbb{N}_0,$$

is in the kernel of the symplectic twistor operator T_s provided the coefficients a_k^r satisfy the recurrence relations

$$\begin{aligned} 0 &= m(k-1)a_k^m + (k-1)a_k^{m-1} - 2a_{k-2}^{m-1}, \\ 0 &= (m-1)(k-1)a_k^{m-1} + 2(k-1)a_k^{m-2} - 4a_{k-2}^{m-2}, \\ &\dots \\ 0 &= 2(k-1)a_k^2 + (m-1)(k-1)a_k^1 - 2(m-1)a_{k-2}^1, \\ 0 &= (k-1)a_k^1 + m(k-1)a_k^0 - 2ma_{k-2}^0, \end{aligned} \tag{6.46}$$

equivalent to

$$(m-p)(k-1)a_k^{m-p} + (p+1)(k-1)a_k^{m-1-p} - 2(p+1)a_{k-2}^{m-1-p} = 0, \tag{6.47}$$

for all $p = 0, 1, \dots, m-1$.

Proof. Concerning the first part, we have

$$\begin{aligned} &T_s\Big(e^{-\frac{q^2}{2}}q\big(A^m(q)z^m + A^{m-1}(q)z^{m-1}\bar{z} + \dots + A^1(q)z\bar{z}^{m-1} + A^0(q)\bar{z}^m\big)\Big) \\ &= e^{-\frac{q^2}{2}}q^2\Big(z^{m-1}\left(m[-\partial_q]A^m(q) + [2q-\partial_q]A^{m-1}(q)\right) \\ &\qquad + z^{m-2}\bar{z}\left((m-1)[-\partial_q]A^{m-1}(q) + 2[2q-\partial_q]A^{m-2}(q)\right) \\ &\qquad \dots \\ &\qquad + \bar{z}^m\left([-\partial_q]A^1(q) + m[2q-\partial_q]A^0(q)\right)\Big) = 0, \end{aligned}$$

where

$$[-\partial_q]A^r(q) = \; -2a_2^r q - 4a_4^r q^3 - 6a_6^r q^5 - \ldots,$$
$$[2q-\partial_q]A^r(q) = (2a_0^r - 2a_2^r)q + (2a_2^r - 4a_4^r)q^3 + \ldots,$$

etc. Then the coefficients of $A^r(q) = a_0^r + a_2^r q^2 + a_4^r q^4 + \ldots$, $r = 0, \ldots, m$ satisfy the recurrence relations

$$(m-p)ka_k^{m-p} + (p+1)ka_k^{m-1-p} - 2(p+1)a_{k-2}^{m-1-p} = 0,\ p = 0, \ldots, m-1.$$

As for the second part, we get

$$\begin{aligned}(1-q\partial_q - q^2)e^{-\frac{q^2}{2}}A^r(q) &= e^{-\frac{q^2}{2}}[1-q\partial_q]A^r(q),\\ (1-q\partial_q + q^2)e^{-\frac{q^2}{2}}A^r(q) &= e^{-\frac{q^2}{2}}[1+2q^2-q\partial_q]A^r(q).\end{aligned} \tag{6.48}$$

The annihilation condition for the symplectic twistor operator T_s acting on (6.45) is equivalent to

$$\begin{aligned}&T_s\Big(e^{-\frac{q^2}{2}}\big(A^m(q)z^m + A^{m-1}(q)z^{m-1}\bar{z} + \ldots + A^1(q)z\bar{z}^{m-1} + A^0(q)\bar{z}^m\big)\Big)\\ &= e^{-\frac{q^2}{2}}\Big(z^{m-1}\big(m[1-q\partial_q]A^m(q) + [1+2q^2-q\partial_q]A^{m-1}(q)\big)\\ &\qquad z^{m-2}\bar{z}\big((m-1)[1-q\partial_q]A^{m-1}(q) + 2[1+2q^2-q\partial_q]A^{m-2}(q)\big)\\ &\qquad \vdots\\ &\qquad z\bar{z}^{m-1}\big(2[1-q\partial_q]A^2(q) + (m-1)[1+2q^2-q\partial_q]A^1(q)\big)\\ &\qquad \bar{z}^m\big([1-q\partial_q]A^1(q) + m[1+2q^2-q\partial_q]A^0(q)\big)\Big),\end{aligned} \tag{6.49}$$

and this completes the proof of the Theorem. □

Remark 8. The explicit solution vectors for the symplectic twistor operator T_s are, for the choice of $A^0(q) = a_0^0 \neq 0$, given in homogeneities $m = 1, 2, 3$ by

$$\begin{aligned}&e^{-\frac{q^2}{2}}\big(\left(-1+2q^2\right)z + \bar{z}\big)a_0^0,\\ &e^{-\frac{q^2}{2}}\bigg(\Big(1-4q^2+\frac{4}{3}q^4\Big)z^2 + \left(-2+4q^2\right)z\bar{z} + \bar{z}^2\bigg)a_0^0,\\ &e^{-\frac{q^2}{2}}\bigg(\Big(-1+6q^2-4q^4+\frac{8}{15}q^6\Big)z^3 + \left(3-12q^2+4q^4\right)z^2\bar{z}\\ &\qquad + \left(-3+6q^2\right)z\bar{z}^2 + \bar{z}^3\bigg)a_0^0.\end{aligned}$$

The same solutions expressed in the variables x, y are

$$\begin{aligned}&2e^{-\frac{q^2}{2}}\left(q^2(x+iy) - iy\right)a_0^0,\\ &\frac{4}{3}e^{-\frac{q^2}{2}}\left(q^4(x+iy)^2 + 6q^2y(-ix+y) - 3y^2\right)a_0^0,\\ &\frac{8}{15}e^{-\frac{q^2}{2}}\left(q^6(x+iy)^3 - 15iq^4(x+iy)^2y - 45q^2(x+iy)y^2 + 15iy^3\right)a_0^0.\end{aligned} \tag{6.50}$$

Theorem 6.5.13. Let $\varphi = \varphi(z, \bar{z}, q) \in \mathrm{Pol}(\mathbb{R}^2, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R})^-$ be a polynomial symplectic spinor in the solution space of the symplectic Dirac operator D_s, i.e. the symplectic spinor φ satisfying the recurrence relations in the first part of Theorem (6.5.11). Then $X_s(\varphi)$ is in kernel of the symplectic twistor operator, $T_s\big(X_s(\varphi)\big) = 0$.

Proof. Let us consider the polynomial symplectic spinor of homogeneity m,

$$\varphi = e^{-\frac{q^2}{2}} q\big(A^m(q)z^m + A^{m-1}(q)z^{m-1}\bar{z} + \ldots + A^1(q)z\bar{z}^{m-1} + A^0(q)\bar{z}^m\big),$$

where $A^r(q) = a_0^r + a_2^r q^2 + a_4^r q^4 + \ldots$, $r = 0, \ldots, m$ satisfies the recursive relations (6.35). The functions $A^l(q)qe^{-\frac{q^2}{2}}$, $l = 0, \ldots, m$ are Schwartz functions. We use the notational simplification $\varphi(z, \bar{z}, q) = e^{-\frac{q^2}{2}} qW$, $W = W(z, \bar{z}, q)$. Then

$$X_s(e^{-\frac{q^2}{2}} qW) = e^{-\frac{q^2}{2}} \left([2q^2 - 1 - q\partial_q]zW + [1 + q\partial_q]\bar{z}W\right),$$

which can be rewritten as

$$X_s(e^{-\frac{q^2}{2}} qW) = e^{-\frac{q^2}{2}} \big(B^{m+1}(q)z^{m+1} + B^m(q)z^m\bar{z} + \ldots + B^0(q)\bar{z}^{m+1}\big),$$

where $B^r(q) = b_0^r + b_2^r q^2 + b_4^r q^4 + \ldots$, $r = 0, \ldots, m+1$, and the coefficients of this formal power series satisfy

$$b_k^m = 2a_{k-2}^{m-1} + (k+1)(a_k^m - a_k^{m-1}). \tag{6.51}$$

We show that $B^r(q)$ satisfy the recurrence relations (6.47) for $p = 0, 1, \ldots, m$ in Theorem (6.5.12). It follows from (6.51) that

$$\begin{aligned}
&(m+1-p)(k-1)\big(2a_{k-2}^{m-p} + (k+1)(a_k^{m-p+1} - a_k^{m-p})\big)\\
&+ (p+1)(k-1)\big(2a_{k-2}^{m-p-1} + (k+1)(a_k^{m-p} - a_k^{m-p-1})\big)\\
&- 2(p+1)\big(2a_{k-4}^{m-p-1} + (k-1)(a_{k-2}^{m-p} - a_{k-2}^{m-p-1})\big)\\
&= 2\big((m-p)(k-1)a_{k-2}^{m-p} + (p+1)(k-1)a_{k-2}^{m-p-1} - 2(p+1)a_{k-4}^{m-p-1}\big)\\
&+ (k-1)\big((m-p+1)(k+1)a_k^{m-p+1} + p(k+1)a_k^{m-p} - 2pa_{k-2}^{m-p}\big)\\
&- (k-1)\big((m-p)(k-1)a_k^{m-p} + (p+1)(k-1)a_k^{m-p-1} - 2(p+1)a_{k-2}^{m-p-1}\big)\\
&+ 2(k-1)a_{k-2}^{m-p} - (k-1)(k+1)a_k^{m-p} + (k-1)(k+1)a_k^{m-p} - 2(k-1)a_{k-2}^{m-p}\\
&= 0,
\end{aligned} \tag{6.52}$$

where we used for the last equality the relation (6.35) to verify that each of the three rows in the last but one expression equals to zero. The proof is complete. □

Theorem 6.5.14. Let $\varphi = \varphi(z, \bar{z}, q) \in \mathrm{Pol}_m(\mathbb{R}^2, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R})^-$, $m \in \mathbb{N}_0$, be a symplectic spinor polynomial in the solution space of the symplectic Dirac operator D_s. Then φ is not in the kernel of the symplectic twistor operator T_s if and only if $m > 0$.

Proof. By our assumption, the symplectic spinor φ satisfies the recurrence relation in Theorem 6.5.11. Recall the recurrence relations for symplectic spinors valued in $\mathcal{S}^-$, which are in the solution space of $\mathrm{Ker}(T_s)$, (6.44)

$$(m-p)ka_k^{m-p} + (p+1)ka_k^{m-1-p} - 2(p+1)a_{k-2}^{m-1-p} = 0,\ p = 0, \ldots, m-1.$$

By Theorem 6.5.11, the coefficients a_k^r satisfy the relations (6.35)

$$(m-p)(k+1)a_k^{m-p} + (p+1)(k+1)a_k^{m-1-p} + 2(p+1)a_{k-2}^{m-1-p} = 0.$$

The comparison of the last two relations leads to

$$(m-p)a_k^{m-p} + (p+1)a_k^{m-1-p} = 0 \tag{6.53}$$

for all k, p, and these are just the coefficients by $q^{k+1}z^{m-1-p}\bar{z}^p$ in $T_s(\varphi)$. We choose the symplectic monogenic φ as in Remark 7. For $k = 2, p = 0$, the coefficient in $T_s(\varphi)$ by $q^3\bar{z}^{m-1}$ is $(a_2^1 + ma_2^0)$. Our choice for φ to be a solution for D_s gives $a_2^1 = \frac{2m}{3}a_0^0$ and $a_2^0 = 0$, therefore the coefficient in (6.53) will not be equal to zero and consequently will not be in $\mathrm{Ker}(T_s)$ for $m > 0$, $m \in \mathbb{N}$. By $\mathfrak{mp}(2,\mathbb{R})$-invariance, the whole metaplectic module does not belong to the kernel of T_s, which finishes the proof. □

Theorem 6.5.15. Let $m \in \mathbb{N}_0, k \in 2\mathbb{N}_0$.

1. The recurrence relations for the coefficients a_k^r of an even (even homogeneity in q) symplectic spinor φ,

   $$\varphi = e^{-\frac{q^2}{2}}\big(A^m(q)z^m + A^{m-1}(q)z^{m-1}\bar{z} + \ldots + A^1(q)z\bar{z}^{m-1} + A^0(q)\bar{z}^m\big),$$

   $A^r(q) = a_0^r + a_2^r q^2 + a_4^r q^4 + \ldots$, $r = 0, \ldots, m$, which is in the kernel of the square of the symplectic Dirac operator D_s^2, are

   $$\begin{aligned}&(m-p)(m-p-1)(k+2)(k+1)a_{k+2}^{m-p}+\\&(m-1-p)(p+1)\big(2(k+2)(k+1)a_{k+2}^{m-1-p} - 2(2k+1)a_k^{m-1-p}\big)+\\&(p+1)(p+2)\big((k+2)(k+1)a_{k+2}^{m-2-p} - 2(2k+1)a_k^{m-2-p} + 4a_{k-2}^{m-2-p}\big)\\&= 0\end{aligned} \tag{6.54}$$

   for $p = 0, \ldots, m-2$.

2. The recurrence relations for the coefficients a_k^r of an odd (odd homogeneity in q) symplectic spinor φ,

   $$\varphi = e^{-\frac{q^2}{2}}q\big(A^m(q)z^m + A^{m-1}(q)z^{m-1}\bar{z} + \ldots + A^1(q)z\bar{z}^{m-1} + A^0(q)\bar{z}^m\big),$$

   $A^r(q) = a_0^r + a_2^r q^2 + a_4^r q^4 + \ldots$, $r = 0, \ldots, m$, which is in the kernel of the square of the symplectic Dirac operator D_s^2, are

   $$\begin{aligned}&(m-p)(m-p-1)(k+2)(k+3)a_{k+2}^{m-p}+\\&(m-1-p)(p+1)\big(2(k+2)(k+3)a_{k+2}^{m-1-p} - 2(2k+3)a_k^{m-1-p}\big)+\\&(p+1)(p+2)\big((k+2)(k+3)a_{k+2}^{m-2-p} - 2(2k+3)a_k^{m-2-p} + 4a_{k-2}^{m-2-p}\big)\\&= 0.\end{aligned} \tag{6.55}$$

   for $p = 0, \ldots, m-2$.

Proof. The second power of the symplectic Dirac operator D_s is equal to

$$D_s^2 = (q^2 + 2q\partial_q + 1 + \partial_q^2)\partial_z^2 + 2(-q^2 + \partial_q^2)\partial_z\partial_{\bar{z}} + (q^2 - 2q\partial_q - 1 + \partial_q^2)\partial_{\bar{z}}^2. \tag{6.56}$$

In the even case, the action of D_s^2 results in

$$\begin{aligned} &D_s^2\Big(e^{-\frac{q^2}{2}}\big(A^m(q)z^m + A^{m-1}(q)z^{m-1}\bar{z} + \ldots + A^0(q)\bar{z}^m\big)\Big) \\ &= e^{-\frac{q^2}{2}}\Big(z^{m-2}\big(m(m-1)[\partial_q^2]A^m(q) + (m-1)[2\partial_q^2 - 4q\partial_q - 2]A^{m-1}(q) \\ &\qquad + 2[\partial_q^2 - 4q\partial_q - 2 + 4q^2]A^{m-2}(q)\big) + \ldots + \\ &\qquad \bar{z}^{m-2}\big(2[\partial_q^2]A^2(q) + (m-1)[2\partial_q^2 - 4q\partial_q - 2]A^1(q) + \\ &\qquad + m(m-1)[\partial_q^2 - 4q\partial_q - 2 + 4q^2]A^0(q)\big)\Big), \end{aligned} \tag{6.57}$$

where

$$\begin{aligned} [\partial_q^2]A^r(q) &= 2a_2^r + 12a_4^r q^2 + \ldots \\ [2\partial_q^2 - 4q\partial_q - 2]A^r(q) &= 4a_2^r - 2a_0^r + (24a_4^r - 8a_2^r - 2a_2^r)q^2 + \ldots \\ [\partial_q^2 - 4q\partial_q - 2 + 4q^2]A^r(q) &= 2a_2^r - 2a_0^r + (12a_4^r - 8a_2^r - 2a_2^r + 4a_0^r)q^2 + \ldots \end{aligned}$$

The odd homogeneity case is analogous. Denoting $\varphi = e^{-\frac{q^2}{2}}qW$, where $W = A^m(q)z^m + \ldots + A^0(q)\bar{z}^m$, we get

$$\begin{aligned} \partial_z^2(q^2 + 2q\partial_q + 1 + \partial_q^2)e^{-\frac{q^2}{2}}qW &= \partial_z^2 e^{-\frac{q^2}{2}}[2\partial_q + q\partial_q^2]W, \\ 2\partial_z\partial_{\bar{z}}(-q^2 + \partial_q^2)e^{-\frac{q^2}{2}}qW &= 2\partial_z\partial_{\bar{z}}e^{-\frac{q^2}{2}}[q\partial_q^2 - 2q^2\partial_q + 2\partial_q - 3q]W, \\ \partial_{\bar{z}}^2(q^2 - 2q\partial_q - 1 + \partial_q^2)e^{-\frac{q^2}{2}}qW &= \partial_{\bar{z}}^2 e^{-\frac{q^2}{2}}[q\partial_q^2 - 4q^2\partial_q + 2\partial_q + 4q^3 - 6q]W, \end{aligned}$$

and the proof follows.

The irreducible $\mathfrak{mp}(2,\mathbb{R})$-submodules in the kernel of D_s^2 were put into boxes on the scheme of the $\mathfrak{mp}(2,\mathbb{R})$-decomposition of $\mathrm{Pol}(\mathbb{R}^2,\mathbb{C}) \otimes \mathcal{S}(\mathbb{R})$

$$\begin{array}{cccccccccccc} \boxed{M_0} & \to & \boxed{X_sM_0} & \to & X_s^2M_0 & \to & X_s^3M_0 & \to & X_s^4M_0 & \to & X_s^5M_0 \\ & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus \\ & & \boxed{M_1} & \to & \boxed{X_sM_1} & \to & X_s^2M_1 & \to & X_s^3M_1 & \to & X_s^4M_1 \\ & & & & \oplus & & \oplus & & \oplus & & \oplus \\ & & & & \boxed{M_2} & \to & \boxed{X_sM_2} & \to & X_s^2M_2 & \to & X_s^3M_2 \\ & & & & & & \oplus & & \oplus & & \oplus \\ & & & & & & \boxed{M_3} & \to & \boxed{X_sM_3} & \to & X_s^2M_3 \\ & & & & & & & & \oplus & & \oplus \\ & & & & & & & & \boxed{M_4} & \to & \boxed{X_sM_4} \\ & & & & & & & & & & \oplus \\ & & & & & & & & & & \boxed{M_5} \end{array} \tag{6.58}$$

□

Theorem 6.5.16. The solution space of the symplectic twistor operator T_s is a subspace of the space of solutions of the square of the symplectic Dirac operator D_s^2. In particular, the recurrence relations for D_s^2 specialized to even resp. odd symplectic spinors from Theorem 6.5.15 are solved by (6.47) resp. (6.44).

Proof. Let us start with even symplectic spinors. It is straightforward to rewrite the recurrence relations in Theorem 6.5.15,

$$\begin{aligned}&(m-p)(m-p-1)(k+2)(k+1)a_{k+2}^{m-p}+\\&(m-1-p)(p+1)\big(2(k+2)(k+1)a_{k+2}^{m-1-p}-2(2k+1)a_k^{m-1-p}\big)+\\&(p+1)(p+2)\big((k+2)(k+1)a_{k+2}^{m-2-p}-2(2k+1)a_k^{m-2-p}+4a_{k-2}^{m-2-p}\big)=0,\end{aligned}$$

into

$$\begin{aligned}&(m-1-p)(k+2)\big((m-p)(k+1)a_{k+2}^{m-p}+(p+1)(k+1)a_{k+2}^{m-1-p}-2(p+1)a_k^{m-1-p}\big)\\&+(p+1)(k+2)\big((m-1-p)(k+1)a_{k+2}^{m-1-p}+(p+2)(k+1)a_{k+2}^{m-2-p}-2(p+2)a_k^{m-2-p}\big)\\&-2(p+1)\big((m-1-p)(k-1)a_k^{m-1-p}+(p+2)(k-1)a_k^{m-2-p}-2(p+2)a_{k-2}^{m-2-p}\big)=0.\end{aligned}$$

Because each of the last three rows corresponds to a recurrence relation (6.47), the claim follows.

In the odd case, the recurrence relations

$$\begin{aligned}&(m-p)(m-p-1)(k+2)(k+3)a_{k+2}^{m-p}+\\&(m-1-p)(p+1)\big(2(k+2)(k+3)a_{k+2}^{m-1-p}-2(2k+3)a_k^{m-1-p}\big)+\\&(p+1)(p+2)\big((k+2)(k+3)a_{k+2}^{m-2-p}-2(2k+3)a_k^{m-2-p}+4a_{k-2}^{m-2-p}\big)=0,\end{aligned}$$

can be rewritten as

$$\begin{aligned}&(m-1-p)(k+3)\big((m-p)(k+2)a_{k+2}^{m-p}+(p+1)(k+2)a_{k+2}^{m-1-p}-2(p+1)a_k^{m-1-p}\big)\\&+(p+1)(k+3)\big((m-1-p)(k+2)a_{k+2}^{m-1-p}+(p+2)(k+2)a_{k+2}^{m-2-p}-2(p+2)a_k^{m-2-p}\big)\\&-2(p+1)\big((m-1-p)ka_k^{m-1-p}+(p+2)ka_k^{m-2-p}-2(p+2)a_{k-2}^{m-2-p}\big)=0,\end{aligned}$$

and each of the last three rows corresponds to the recurrence relation (6.44). □

Theorem 6.5.17. The solution space of the symplectic twistor operator T_s, acting on $\mathrm{Pol}(\mathbb{R}^2,\mathbb{C})\otimes\mathcal{S}(\mathbb{R})$, consists of the set of $\mathfrak{mp}(2,\mathbb{R})$-modules pictured in the squares realized in the decomposition of $\mathrm{Pol}(\mathbb{R}^2,\mathbb{C})\otimes\mathcal{S}(\mathbb{R})$ on $\mathfrak{mp}(2,\mathbb{R})$ irreducible subspaces, (6.5)

1. $\mathrm{Pol}(\mathbb{R}^2,\mathbb{C})\otimes\mathcal{S}(\mathbb{R})^-$:

$$\begin{array}{ccccccccc}\boxed{M_0^-} & \longrightarrow & \boxed{X_sM_0^-} & \longrightarrow & X_s^2M_0^- & \longrightarrow & X_s^3M_0^- & \longrightarrow & \dots\\ qe^{-\frac{q^2}{2}} & & \oplus & & \oplus & & \oplus & & \\ & & M_1^- & \longrightarrow & \boxed{X_sM_1^-} & \longrightarrow & X_s^2M_1^- & \longrightarrow & \dots\\ & & e^{-\frac{q^2}{2}}(q^3(x+iy)-3iqy) & & \oplus & & \oplus & & \\ & & & & M_2^- & \longrightarrow & \boxed{X_sM_2^-} & \longrightarrow & \dots\\ & & & & e^{-\frac{q^2}{2}}(q^5(x+iy)^2+10q^3y(-ix+y)-15qy^2) & & \oplus & & \\ & & & & & & M_3^- & \longrightarrow & \dots\end{array} \tag{6.59}$$

2. $\mathrm{Pol}(\mathbb{R}^2,\mathbb{C})\otimes\mathcal{S}(\mathbb{R})^+$:

$$\begin{array}{ccccccccc}\boxed{M_0^+} & \longrightarrow & \boxed{X_sM_0^+} & \longrightarrow & X_s^2M_0^+ & \longrightarrow & X_s^3M_0^+ & \longrightarrow & \dots\\ e^{-\frac{q^2}{2}} & & \oplus & & \oplus & & \oplus & & \\ & & M_1^+ & \longrightarrow & \boxed{X_sM_1^+} & \longrightarrow & X_s^2M_1^+ & \longrightarrow & \dots\\ & & e^{-\frac{q^2}{2}}(x+iy) & & \oplus & & \oplus & & \\ & & & & M_2^+ & \longrightarrow & \boxed{X_sM_2^+} & \longrightarrow & \dots\\ & & & & e^{-\frac{q^2}{2}}(x+iy)^2 & & \oplus & & \\ & & & & & & M_3^+ & \longrightarrow & \dots\end{array} \tag{6.60}$$

Notice that the representative vectors in the solution space of D_s are pictured under the spaces of symplectic monogenics. In the case of $\mathcal{S}^+$, we exploit the symplectic monogenics constructed in Theorem 6.5.9.

Proof. It follows from the metaplectic Howe duality, [9], that Theorem 6.5.16 characterizes the $\mathfrak{mp}(2,\mathbb{R})$-submodule of $\mathrm{Pol}(\mathbb{R}^2,\mathbb{C}) \otimes \mathcal{S}(\mathbb{R})$ contained in the solution space of T_s. Then Theorem 6.5.9, Theorem 6.5.14 and Theorem 6.5.15 characterize the space of solutions as the image of the space of symplectic monogenics by X_s, in addition to the space of constant symplectic spinors. The proof is complete. □

In previous sections, we discussed the space of polynomial solutions. A natural question is an extension of the function space from polynomials to the class of analytic, smooth, hyperfunction, generalized, etc., function spaces. For example, one can consider convergent power series constructed from the polynomial solutions. We shall not attempt to discuss this question in a greater generality, but observe the existence of a wider class of solutions.

Let us consider the function element $z^n f(q)$ for $f \in \mathcal{S}(\mathbb{R})$, $n \in \mathbb{N}_0$. The substitution into (6.17) implies that it belongs to the solution space of T_s provided $f(q)$ solves the ordinary differential equation

$$(1-q^2)f(q) = q\frac{\partial}{\partial q}f(q). \tag{6.61}$$

This equation has a unique solution $f(q) = qe^{-\frac{q^2}{2}}$ in $\mathcal{S}(\mathbb{R})$, and so $z^n qe^{-\frac{q^2}{2}}$ are in the kernel of the symplectic twistor operator for all $n \in \mathbb{N}_0$.

A generalization of this result is contained in the following lemma.

Lemma 6.5.18. Let $h(z)$ be arbitrary holomorphic function on $\mathbb{C}$. Then the complex analytic symplectic spinor

$$h(z)qe^{-\frac{q^2}{2}} \tag{6.62}$$

is in the kernel of the symplectic twistor operator T_s.

Consequently, the space of holomorphic functions on $\mathbb{C}$ is embedded into the space of smooth solutions of the symplectic twistor operator T_s.

Notice that an admissible continuous representation spaces of a reductive Lie group G can be conveniently described in terms of a globalization of the underlying Harish-Chandra $(\mathfrak{g}, K)$-module, where $\mathfrak{g}$ resp. K are the Lie algebra resp. maximal compact subgroup of G. In this way, one has continuous representation of G on the space of analytic, smooth, Frechet, hyperfunction, generalized, etc., functions. It is still natural to ask for a characterization of solutions of both T_s and D_s on the space of such functions.

7. Symplectic twistor operator on $(\mathbb{R}^{2n}, \omega)$

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Symplectic twistor operator on $\mathbb{R}^{2n}$ and the Segal-Shale-Weil representation,

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Abstract The aim of our article is the study of solution space of the symplectic twistor operator T_s in symplectic spin geometry on the real symplectic manifold $(\mathbb{R}^{2n}, \omega)$, which is the symplectic analogue of the twistor operator in (pseudo-) Riemannian spin geometry. In particular, we observe a substantial difference between the case $n = 1$ of real dimension 2 and the case of $\mathbb{R}^{2n}$, $n > 1$. For $n > 1$, the solution space of T_s is isomorphic to the Segal-Shale-Weil representation.

Key words: Symplectic twistor operator, Symplectic Dirac operator, Metaplectic Howe duality.

MSC classification: 53C27, 53D05, 81R25.

7.1 Introduction and motivation

In the case when the second Stiefel-Whitney class of an orientable Riemannian manifold is trivial, there is a double cover of the frame bundle and consequently there is an associated vector bundle for the spinor representation of the spin structure group. There are two basic first order invariant differential operators acting on spinor valued fields, namely the Dirac operator and the twistor operator. Their spectral properties are reflected in the geometric properties of the underlying manifold. In Riemannian geometry, the twistor equation appeared as an integrability condition for the canonical almost complex structure on the twistor space, and it plays a prominent role in conformal differential geometry due to its larger symmetry group. In physics, its solution space defines infinitesimal isometries in Riemannian supergeometry. For an exposition with panorama of examples, cf. [1], [18] and references therein.

The symplectic version of Dirac operator D_s was introduced in [28], and its differential geometric properties were studied in [5], [22], [24]. The metaplectic Howe duality for D_s, introduced in [9], allows to characterize the space of solutions for the symplectic Dirac operator D_s on the real symplectic manifold $(\mathbb{R}^{2n}, \omega)$.

The aim of the present chapter is to study the symplectic twistor operator T_s in context of the the metaplectic Howe duality, and consequently to determine its solution space on the real symplectic manifold $(\mathbb{R}^{2n}, \omega)$. The operators D_s, T_s were considered from a different perspective in [24], [31], [32]. From an analytic point of view, T_s is represented by an overdetermined system of partial differential equations and acts on the space of polynomials valued in the vector space of the Segal-Shale-Weil representation. From the point of view of representation theory, T_s is $\mathfrak{mp}(2n, \mathbb{R})$-invariant and the initial problem is solved by understanding of the interaction of T_s with the generators D_s, X_s of the Howe dual Lie algebra $\mathfrak{sl}(2)$.

As we shall see, as for T_s there is a substantial difference between the situation for $n = 1$ and $n > 1$. Namely, there is in $\mathrm{Ker}(T_s)$ an infinite number of irreducible $\mathfrak{mp}(2n, \mathbb{R})$-modules with different infinitesimal character for $n = 1$, while for $n > 1$ the kernel contains just the Segal-Shale-Weil representation, a result of independent interest. This is the reason why we decided to treat the case $n = 1$ in a separate chapter (see Chapter 6) using different, more combinatorial approach, which will be useful in complete understanding of the full infinite dimensional symmetry group of our operator.

The structure of chapter goes as follows. In the first section, we review the subject of symplectic spin geometry and metaplectic Howe duality. In the second section, we start with the definition of the symplectic twistor operator T_s and compute the space of polynomial solutions of T_s on $(\mathbb{R}^{2n}, \omega)$. These results follow from a careful study of algebraic and differential properties of T_s. In the last third section, we give a collection of unsolved problems related to the topic of the present chapter.

7.2 Metaplectic Lie algebra $\mathfrak{mp}(2n, \mathbb{R})$, symplectic Clifford algebra and class of simple weight modules for $\mathfrak{mp}(2n, \mathbb{R})$

In the present section, we recall several algebraic and representation theoretic results used in the next section for the analysis of the solution space of the symplectic twistor operator T_s. See, e.g., [5], [9], [19], [22], [24].

Let us consider $2n$-dimensional symplectic vector space $(\mathbb{R}^{2n}, \omega = \sum_{j=1}^{n} \epsilon^j \wedge \epsilon^{n+j})$, $n \in \mathbb{N}$, and a symplectic basis $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2n}\}$ with respect to the non-degenerate two form $\omega \in \wedge^2(\mathbb{R}^{2n})^\star$. Let $E_{k,j}$ be the $2n \times 2n$ matrix with 1 on the intersection of the k-th row and the j-th column and zero otherwise. The set of matrices

$$X_{jk} = E_{j,k} - E_{n+k,n+j}, \quad Y_{jk} = E_{j,n+k} + E_{k,n+j}, \quad Z_{jk} = E_{n+j,k} + E_{n+k,j},$$

for $j, k = 1, \dots, n$ is a basis of $\mathfrak{sp}(2n, \mathbb{R})$. This basis can be realized by first order differential operators

$$X_{jk} = x_j \partial_{x_k} - x_{n+j} \partial_{x_{n+j}}, \quad Y_{jk} = x_j \partial_{x_{n+k}} + x_k \partial_{x_{n+j}}, \quad Z_{jk} = x_{n+j} \partial_{x_k} + x_{n+k} \partial_{x_j}.$$

The metaplectic Lie algebra $\mathfrak{mp}(2n, \mathbb{R})$ is the Lie algebra of the two-fold group covering $\lambda : \mathrm{Mp}(2n, \mathbb{R}) \to \mathrm{Sp}(2n, \mathbb{R})$ of the symplectic Lie group $\mathrm{Sp}(2n, \mathbb{R})$. It can be realized by homogeneity two elements in the symplectic Clifford algebra $Cl_s(\mathbb{R}^{2n}, \omega)$, where the isomorphism

$$\lambda_\star : \mathfrak{mp}(2n, \mathbb{R}) \to \mathfrak{sp}(2n, \mathbb{R})$$

is given by

$$\begin{aligned} &\lambda_\star(e_k \cdot e_j) = -Y_{kj}, \\ &\lambda_\star(e_{n+k} \cdot e_{n+j}) = Z_{kj}, \\ &\lambda_\star(e_k \cdot e_{n+j} + e_{n+j} \cdot e_k) = 2X_{kj}, \end{aligned} \tag{7.1}$$

for $j, k = 1, \ldots, n$.

Definition 7.2.1. The symplectic Clifford algebra $Cl_s(\mathbb{R}^{2n}, \omega)$ is an associative unital algebra over $\mathbb{C}$, given by the quotient of the tensor algebra $T(e_1, \ldots, e_{2n})$ by a two-sided ideal $I \subset T(e_1, \ldots, e_{2n})$ generated by

$$v \cdot w - w \cdot v = -i\omega(v, w)$$

for all $v, w \in \mathbb{R}^{2n}$, where $i \in \mathbb{C}$ is the complex unit.

The symplectic Clifford algebra $Cl_s(\mathbb{R}^{2n}, \omega)$ is isomorphic to the Weyl algebra W_{2n} of complex valued algebraic differential operators on $\mathbb{R}^n$, and the symplectic Lie algebra $\mathfrak{sp}(2n, \mathbb{R})$ can be realized as a subalgebra of W_{2n}. In particular, the Weyl algebra is an associative algebra generated by $\{q_1, \ldots, q_n, \partial_{q_1}, \ldots, \partial_{q_n}\}$, the multiplication operator by q_j and differentiation ∂_{q_j}, for $j = 1, \ldots, n$ The symplectic Lie algebra $\mathfrak{sp}(2n, \mathbb{R})$ has a basis $\{-\frac{i}{2}q_j^2, -\frac{i}{2}\frac{\partial^2}{\partial q_j^2}, q_j\frac{\partial}{\partial q_j}+\frac{1}{2}\}$, $j = 1, \ldots, n$.

The symplectic spinor representation is an irreducible Segal-Shale-Weil representation of $Cl_s(\mathbb{R}^{2n}, \omega)$ on $L^2(\mathbb{R}^n, e^{-\frac{1}{2}\sum_{j=1}^n q_j^2} dq_{\mathbb{R}^n})$, the space of square integrable functions on $(\mathbb{R}^n, e^{-\frac{1}{2}\sum_{j=1}^n q_j^2} dq_{\mathbb{R}^n})$, where $dq_{\mathbb{R}^n}$ is the Lebesgue measure. Its action, the symplectic Clifford multiplication c_s, acts on the subspace of $\mathcal{C}^\infty$(smooth)-vectors given by the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing complex valued functions on $\mathbb{R}^n$ as a dense subspace. The space $\mathcal{S}(\mathbb{R}^n)$ can be regarded as a smooth (Fréchet) globalization of the space of $\tilde{K}$-finite vectors in the representation, where $\tilde{K} \subset \mathrm{Mp}(2n, \mathbb{R})$ is the maximal compact subgroup given by the double cover of $K = \mathrm{U}(n) \subset \mathrm{Sp}(2n, \mathbb{R})$. Though we shall work in the smooth globalization $\mathcal{S}(\mathbb{R}^n)$, the representative vectors are usually chosen to belong to the underlying Harish-Chandra module of $\tilde{K}$-finite vectors preserved by c_s.

The function spaces associated to the Segal-Shale-Weil representation are supported on $\mathbb{R}^n \subset \mathbb{R}^{2n}$, a maximal isotropic subspace of $(\mathbb{R}^{2n}, \omega)$. In its restriction to $\mathfrak{mp}(2n, \mathbb{R})$, $\mathcal{S}(\mathbb{R}^2)$ decomposes into two unitary representations realized on the subspace of even resp. odd functions

$$\varrho : \mathfrak{mp}(2n, \mathbb{R}) \to \mathrm{End}(\mathcal{S}(\mathbb{R}^2)), \tag{7.2}$$

where the basis vectors act by

$$\begin{aligned}
&\varrho(e_j \cdot e_k) = iq_j q_k, \\
&\varrho(e_{n+j} \cdot e_{n+k}) = -i\partial_{q_j}\partial_{q_k}, \\
&\varrho(e_j \cdot e_{n+j} + e_{n+j} \cdot e_j) = q_j\partial_{q_j} + \partial_{q_j} q_j.
\end{aligned} \tag{7.3}$$

for all $j, k = 1, \ldots, n$. Because it is a complex representation of $\mathfrak{mp}(2, \mathbb{R})$ we may consider complex algebra $\mathfrak{mp}(2, \mathbb{C})$ and isomorphic complex algebra $\mathfrak{sp}(2, \mathbb{C})$.

In this representation, the symplectic Clifford algebra $Cl_s(\mathbb{R}^{2n}, \omega)$ acts on $L^2(\mathbb{R}^n,\ e^{-\frac{1}{2}\sum_{j=1}^n q_j^2} dq_{\mathbb{R}^n})$ by unbounded operators with domain $\mathcal{S}(\mathbb{R}^n)$. The space of $\tilde{K}$-finite vectors consists of even resp. odd homogeneity $\mathfrak{mp}(2n, \mathbb{C})$-submodule

$$\{\mathrm{Pol}_{even}(q_1, \ldots, q_n)e^{-\frac{1}{2}\sum_{j=1}^n q_j^2}\}, \quad \{\mathrm{Pol}_{odd}(q_1, \ldots, q_n)e^{-\frac{1}{2}\sum_{j=1}^n q_j^2}\}.$$

It is also an irreducible representation of $\mathfrak{mp}(2n, \mathbb{C}) \ltimes \mathfrak{h}(n)$, the semidirect product of $\mathfrak{mp}(2n, \mathbb{C})$ and $(2n+1)$-dimensional Heisenberg Lie algebra $\mathfrak{h}(n)$ spanned by $\{e_1, \ldots, e_{2n}, \mathrm{Id}\}$. Cf. [16].

Let us denote by $\mathrm{Pol}(\mathbb{R}^{2n}, \mathbb{C})$ the vector space of complex valued polynomials on $\mathbb{R}^{2n}$, and by $\mathrm{Pol}_l(\mathbb{R}^{2n}, \mathbb{C})$ the subspace of homogeneity l polynomials. The complex vector space $\mathrm{Pol}_l(\mathbb{R}^{2n}, \mathbb{C})$ is as an irreducible $\mathfrak{mp}(2n, \mathbb{C})$-module isomorphic to $\mathrm{S}^l(\mathbb{C}^{2n})$, the l-th symmetric power of the complexification of the fundamental vector representation $\mathbb{R}^{2n}$, $l \in \mathbb{N}_0$.

7.3 Segal-Shale-Weil representation and metaplectic Howe duality

Let us review a representation-theoretical result of [2], formulated in the opposite convention of highest weight metaplectic modules. Let us consider Borel subalgebra of $\mathfrak{sp}(2, \mathbb{C})$ generated by $X' = \frac{1}{2}\left(\begin{smallmatrix} -i & 1 \\ 1 & i \end{smallmatrix}\right)$ and $H' = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$ and Borel subalgebra of $\mathfrak{mp}(2, \mathbb{C})$ generated by elements $\lambda_\star^{-1}(X')$ and $\lambda_\star^{-1}(H')$. Let $\varpi_1, \dots, \varpi_n$ be the fundamental weights of the Lie algebra $\mathfrak{sp}(2n, \mathbb{C})$, and let $L(\varpi)$ denote the simple module over the universal enveloping algebra $\mathcal{U}(\mathfrak{mp}(2n, \mathbb{C}))$ of $\mathfrak{mp}(2n, \mathbb{C})$ generated by the highest weight vector of the weight ϖ.

Algebraically, the decomposition of the space of polynomial functions on $\mathbb{R}^{2n}$ valued in the Segal-Shale-Weil representation corresponds to the tensor product of $L(-\frac{1}{2}\varpi_n)$ resp. $L(\varpi_{n-1} - \frac{3}{2}\varpi_n)$ with symmetric powers $\mathrm{S}^k(\mathbb{C}^{2n})$ of the fundamental vector representation $\mathbb{C}^{2n}$ of $\mathfrak{sp}(2n, \mathbb{C})$, $k \in \mathbb{N}_0$. The following result is well known.

Proposition 7.3.1. ([2]) We have for $L(-\frac{1}{2}\varpi_n)$

1. In the even case $k = 2l$ ($2l + 1$ terms on the right-hand side)

$$
\begin{aligned}
&L(-\frac{1}{2}\varpi_n) \otimes \mathrm{S}^k(\mathbb{C}^{2n}) \simeq L(-\frac{1}{2}\varpi_n) \oplus L(\varpi_1 + \varpi_{n-1} - \frac{3}{2}\varpi_n) \\
&\oplus L(2\varpi_1 - \frac{1}{2}\varpi_n) \oplus L(3\varpi_1 + \varpi_{n-1} - \frac{3}{2}\varpi_n) \oplus \dots \\
&\oplus L((2l-1)\varpi_1 + \varpi_{n-1} - \frac{3}{2}\varpi_n) \oplus L(2l\varpi_1 - \frac{1}{2}\varpi_n),
\end{aligned}
$$

2. In the odd case $k = 2l + 1$ ($2l + 2$ terms on the right-hand side)

$$
\begin{aligned}
&L(-\frac{1}{2}\varpi_n) \otimes \mathrm{S}^k(\mathbb{C}^{2n}) \simeq L(\varpi_{n-1} - \frac{3}{2}\varpi_n) \oplus L(\varpi_1 - \frac{1}{2}\varpi_n) \\
&\oplus L(2\varpi_1 + \varpi_{n-1} - \frac{3}{2}\varpi_n) \oplus L(3\varpi_1 - \frac{1}{2}\varpi_n) \oplus \dots \\
&\oplus L(2l\varpi_1 + \varpi_{n-1} - \frac{3}{2}\varpi_n) \oplus L((2l+1)\varpi_1 - \frac{1}{2}\varpi_n),
\end{aligned}
$$

We have for $L(\varpi_{n-1} - \frac{3}{2}\varpi_n)$

1. In the even case $k = 2l$ ($2l + 1$ terms on the right-hand side)

$$
\begin{aligned}
&L(\varpi_{n-1} - \frac{3}{2}\varpi_n) \otimes \mathrm{S}^k(\mathbb{C}^{2n}) \simeq L(\varpi_{n-1} - \frac{3}{2}\varpi_n) \oplus L(\varpi_1 - \frac{1}{2}\varpi_n) \\
&\oplus L(2\varpi_1 + \varpi_{n-1} - \frac{3}{2}\varpi_n) \oplus \dots \oplus L((2l-1)\varpi_1 - \frac{1}{2}\varpi_n) \\
&\oplus L(2l\varpi_1 + \varpi_{n-1} - \frac{3}{2}\varpi_n),
\end{aligned}
$$

2. In the odd case $k = 2l+1$ ($2l+2$ terms on the right-hand side)

$$L(\varpi_{n-1} - \frac{3}{2}\varpi_n) \otimes \mathrm{S}^k(\mathbb{C}^{2n}) \simeq L(-\frac{1}{2}\varpi_n) \oplus L(\varpi_1 + \varpi_{n-1} - \frac{3}{2}\varpi_n) \oplus \dots$$
$$\oplus L(2l\varpi_1 - \frac{1}{2}\varpi_n) \oplus L((2l+1)\varpi_1 + \varpi_{n-1} - \frac{3}{2}\varpi_n).$$

A more geometrical reformulation of this statement is realized in the algebraic (polynomial) Weyl algebra and termed metaplectic Howe duality, [9]. The metaplectic analogue of the classical theorem on the separation of variables allows to decompose the space $\mathrm{Pol}(\mathbb{R}^{2n}, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R}^n)$ of complex polynomials valued in the Segal-Shale-Weil representation under the action of $\mathfrak{mp}(2n, \mathbb{R})$ into a direct sum of simple $\mathfrak{mp}(2n, \mathbb{R})$-modules

$$\mathrm{Pol}(\mathbb{R}^{2n}, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R}^n) \simeq \bigoplus_{l=0}^{\infty} \bigoplus_{j=0}^{\infty} X_s^j M_l, \tag{7.4}$$

where we use the notation $M_l = M_l^+ \oplus M_l^-$. This decomposition takes the form of an infinite triangle

$$\begin{array}{ccccccccccc}
P_0 \otimes \mathcal{S} & & P_1 \otimes \mathcal{S} & & P_2 \otimes \mathcal{S} & & P_3 \otimes \mathcal{S} & & P_4 \otimes \mathcal{S} & & P_5 \otimes \mathcal{S} \\
\| & & \| & & \| & & \| & & \| & & \| \\
M_0 & \longrightarrow & X_s M_0 & \longrightarrow & X_s^2 M_0 & \longrightarrow & X_s^3 M_0 & \longrightarrow & X_s^4 M_0 & \longrightarrow & X_s^5 M_0 \\
 & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus \\
 & & M_1 & \longrightarrow & X_s M_1 & \longrightarrow & X_s^2 M_1 & \longrightarrow & X_s^3 M_1 & \longrightarrow & X_s^4 M_1 \\
 & & & & \oplus & & \oplus & & \oplus & & \oplus \\
 & & & & M_2 & \longrightarrow & X_s M_2 & \longrightarrow & X_s^2 M_2 & \longrightarrow & X_s^3 M_2 \\
 & & & & & & \oplus & & \oplus & & \oplus \\
 & & & & & & M_3 & \longrightarrow & X_s M_3 & \longrightarrow & X_s^2 M_3 \\
 & & & & & & & & \oplus & & \oplus \\
 & & & & & & & & M_4 & \longrightarrow & X_s M_4 \\
 & & & & & & & & & & \oplus \\
 & & & & & & & & & & M_5
\end{array} \tag{7.5}$$

Now, let us explain the notation used on the previous picture. First of all, we used the shorthand notation $P_l = \mathrm{Pol}_l(\mathbb{R}^{2n}, \mathbb{C})$, $l \in \mathbb{N}_0$, $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$, and the spaces and arrows in the picture have the following meaning. We denote $M_l = \mathrm{Pol}_l(\mathbb{R}^2, \mathbb{C}) \cap \mathrm{Ker}(D_s)$, where we set the three operators

$$X_s = \sum_{j=1}^{n} (x_{n+j}\partial_{q_j} + i x_j q_j),$$
$$D_s = \sum_{j=1}^{n} (i q_j \partial_{x_{n+j}} - \partial_{x_j}\partial_{q_j}),$$
$$E = \sum_{j=1}^{2n} x_j \partial_{x_j}. \tag{7.6}$$

The operator D_s and X_s acts on the previous picture horizontally but in the opposite direction. They fulfil the $\mathfrak{sl}(2,\mathbb{C})$-commutation relations

$$\begin{aligned}
[E+n, D_s] &= -D_s, \\
[E+n, X_s] &= X_s, \\
[X_s, D_s] &= i(E+n).
\end{aligned} \tag{7.7}$$

For the purposes of the present chapter, we do not need the proper normalization of the generators D_s, X_s, E making the isomorphism with standard commutation relations in $\mathfrak{sl}(2,\mathbb{C})$ explicit.

The elements of $\mathrm{Pol}(\mathbb{R}^{2n},\mathbb{C})\otimes\mathcal{S}(\mathbb{R}^n)$ are called polynomial symplectic spinors. Let $\varphi \equiv \varphi(x_1,\ldots,x_{2n},q_1,\ldots,q_n) \in \mathrm{Pol}(\mathbb{R}^{2n},\mathbb{C})\otimes\mathcal{S}(\mathbb{R}^n)$, $h \in \mathrm{Mp}(2n,\mathbb{R})$ and $\lambda(h) = g \in \mathrm{Sp}(2n,\mathbb{R})$ for the double covering $\lambda : \mathrm{Mp}(2n,\mathbb{R}) \to \mathrm{Sp}(2n,\mathbb{R})$. We define the action of $\mathrm{Mp}(2n,\mathbb{R})$ on $\mathrm{Pol}(\mathbb{R}^{2n},\mathbb{C})\otimes\mathcal{S}(\mathbb{R}^n)$ by

$$\tilde{\varrho}(h)\varphi(x_1,\ldots,x_{2n},q_1,\ldots,q_n) = \varrho(h)\varphi(\lambda(g^{-1})(x_1,\ldots,x_{2n})^T,q_1,\ldots,q_n), \tag{7.8}$$

with ϱ acting on the Segal-Shale-Weil representation via (7.2). Passing to the infinitesimal action, we get the operators, which represent the basis elements of $\mathfrak{mp}(2n,\mathbb{R})$. For example, we have for $j = 1,\ldots,n$

$$\begin{aligned}
\tilde{\varrho}(X_{jj})\varphi &= \frac{d}{dt}\Big|_{t=0}\tilde{\varrho}(\exp(tX_{jj}))\varphi(x_1,\ldots,x_{2n},q_1,\ldots,q_n) \\
&= \frac{d}{dt}\Big|_{t=0} e^{\frac{t}{2}}\varphi(x_1,\ldots,x_j e^{-t},\ldots,x_{n+j}e^t\ldots,x_{2n},q_1,\ldots,q_j e^t,\ldots,q_n) \\
&= (\frac{1}{2} - x_j\frac{\partial}{\partial x_j} + x_{n+j}\frac{\partial}{\partial x_{n+j}} + q_j\frac{\partial}{\partial q_j})\varphi(x_1,\ldots,x_{2n},q_1,\ldots,q_n).
\end{aligned}$$

These operators satisfy the commutation relations of the Lie algebra $\mathfrak{mp}(2n,\mathbb{R})$, and preserve the homogeneity in $x_1,\ldots,x_{2n}$. The operators X_s and D_s commute with operators $\tilde{\varrho}(X_{jk}), \tilde{\varrho}(Y_{jk})$ and $\tilde{\varrho}(Z_{jk})$, $j,k = 1,\ldots,n$, hence they are $\mathfrak{mp}(2n,\mathbb{R})$-intertwining differential operators.

The action of $\mathfrak{mp}(2n,\mathbb{R})\times\mathfrak{sl}(2,\mathbb{C})$ generates the multiplicity free decomposition of $\mathrm{Pol}(\mathbb{R}^{2n},\mathbb{C})\otimes\mathcal{S}(\mathbb{R}^n)$ and the pair of Lie algebras in the product is called the metaplectic Howe dual pair. The operators X_s, D_s acting on the previous picture horizontally identify the two neighbouring $\mathfrak{mp}(2n,\mathbb{R})$-modules isomorphically. The modules M_l, $l \in \mathbb{N}_0$ on the most left diagonal of our picture are termed symplectic monogenics, and are characterized as l-homogeneous solutions of the symplectic Dirac operator D_s. Thus the decomposition is given, as a vector space, by tensor product of the symplectic monogenics multiplied by the polynomial algebra $\mathbb{C}[X_s]$ of invariants.

7.4 Symplectic twistor operator T_s and its solution space on $(\mathbb{R}^{2n},\omega)$

We start with an abstract definition of the symplectic twistor operator T_s. Let (M,ω) be a $2n$-dimensional symplectic manifold, $\pi : P \to M$ a principal fibre $\mathrm{Sp}(2n,\mathbb{R})$-bundle of symplectic frames on M. A metaplectic structure on (M,ω) is a principal fibre $\mathrm{Mp}(2n,\mathbb{R})$-bundle $Q \to M$ together with bundle morphism

$Q \to P$, equivariant with respect to the double covering $\mathrm{Mp}(2n, \mathbb{R}) \to \mathrm{Sp}(2n, \mathbb{R})$. The manifold (M, ω) with a metaplectic structure is usually called symplectic spin manifold. The symplectic manifold M admits a metaplectic structure if and only if the second Stiefel-Whitney class $w_2(M)$ is trivial, and the equivalence classes of metaplectic structures are classified by $H^1(M, \mathbb{Z}_2)$. These is a unique metaplectic structure on $(\mathbb{R}^{2n}, \omega)$.

Definition 7.4.1. Let (M, ∇, ω) be a symplectic spin manifold of dimension $2n$, ∇^s the associated symplectic spin covariant derivative and $\omega \in \mathcal{C}^\infty(M, \wedge^2 T^\star M)$ a non-degenerate 2-form such that $\nabla\omega = 0$. We denote by $\{e_1, \ldots, e_{2n}\}$ a local symplectic frame. The symplectic twistor operator T_s on M is the first order differential operator T_s acting on smooth symplectic spinors $\mathbf{S}$

$$\begin{aligned} &\nabla^s : \mathcal{C}^\infty(M, \mathbf{S}) \longrightarrow T^\star M \otimes \mathcal{C}^\infty(M, \mathbf{S}), \\ &T_s := P_{\mathrm{Ker}(c_s)} \circ \omega^{-1} \circ \nabla^s : \mathcal{C}^\infty(M, \mathbf{S}) \longrightarrow \mathcal{C}^\infty(M, \mathcal{T}), \end{aligned} \tag{7.9}$$

where $\mathcal{T}$ is the space of symplectic twistors, $T^\star M \otimes \mathbf{S} \simeq \mathbf{S} \oplus \mathcal{T}$, given by algebraic projection

$$P_{\mathrm{Ker}(c_s)} : T^\star M \otimes \mathcal{C}^\infty(M, \mathbf{S}) \longrightarrow \mathcal{C}^\infty(M, \mathcal{T})$$

on the kernel of the symplectic Clifford multiplication c_s. In the local symplectic coframe $\{\epsilon^1\}_{j=1}^{2n}$ dual to the symplectic frame $\{e_j\}_{j=1}^{2n}$ with respect to ω, we have the local formula for T_s

$$T_s = \sum_{k=1}^{2n} \epsilon^k \otimes \nabla^s_{e_k} + \frac{i}{n} \sum_{j,k,l=1}^{2n} \epsilon^l \otimes \omega^{kj} e_l \cdot e_j \cdot \nabla^s_{e_k}, \tag{7.10}$$

where $\cdot$ is the shorthand notation for the symplectic Clifford multiplication and $i \in \mathbb{C}$ is the imaginary unit. We use the convention $\omega^{kj} = 1$ for $j = k + n$ and $k = 1, \ldots, n$, $\omega^{kj} = -1$ for $k = n+1, \ldots, 2n$ and $j = k-n$, and $\omega^{kj} = 0$ otherwise.

The symplectic Dirac operator D_s is defined as the image of the symplectic Clifford multiplication c_s, and a symplectic spinor in the kernel of D_s is called symplectic monogenic.

Lemma 7.4.1. The symplectic twistor operator T_s is $\mathfrak{mp}(2n, \mathbb{R})$-invariant.

Proof. The property of invariance is a direct consequence of the equivariance of symplectic covariant derivative and invariance of algebraic projection $P_{\mathrm{Ker}(c_s)}$, and amounts to verify

$$T_s(\tilde{\varrho}(g)\varphi) = \lambda(g) \otimes \tilde{\varrho}(g)(T_s\varphi) \tag{7.11}$$

for any $g \in \mathfrak{mp}(2n, \mathbb{R})$ and $\varphi \in \mathcal{C}^\infty(M, \mathbf{S})$. Using the local formula (7.10) for T_s in a local chart $(x_1, \ldots, x_{2n})$, both sides of (7.11) are equal to

$$\begin{aligned} &\sum_{k=1}^{2n} \epsilon^k \otimes \varrho(g) \frac{\partial}{\partial x_k} \big[\varphi\big(\lambda(g)^{-1}x\big)\big] \\ &+ \frac{i}{n} \sum_{j,k,l=1}^{2n} \epsilon^l \otimes \omega^{kj} e_l \cdot e_j \cdot \Big[\varrho(g) \frac{\partial}{\partial x_k} \big[\varphi\big(\lambda(g)^{-1}x\big)\big]\Big] \end{aligned}$$

and the proof follows. □

In the case $M = (\mathbb{R}^{2n}, \omega)$, the symplectic Dirac and the symplectic twistor operators are given by

$$D_s = \sum_{j,k=1}^{2n} \omega^{kj} e_k \cdot \frac{\partial}{\partial x_j}, \tag{7.12}$$

$$T_s = \sum_{l=1}^{2n} \epsilon^l \otimes \frac{\partial}{\partial x_l} + \frac{i}{n} \sum_{j,k,l=1}^{2n} \epsilon^l \otimes \omega^{kj} e_l \cdot e_j \cdot \frac{\partial}{\partial x_k} = \sum_{l=1}^{2n} \epsilon^l \otimes \Big(\frac{\partial}{\partial x_l} - \frac{i}{n} e_l \cdot D_s \Big), \tag{7.13}$$

and we restrict their action to the space of polynomial symplectic spinors.

Lemma 7.4.2. Let $\varphi \in \mathrm{Pol}(\mathbb{R}^{2n}, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R}^n)$ be a symplectic spinor in the solution space of the symplectic twistor operator T_s. Then φ is in the kernel of the square of the symplectic Dirac operator D_s^2.

Proof. Let φ be a polynomial symplectic spinor in $\mathrm{Ker}(T_s)$,

$$T_s \varphi = \sum_{l=1}^{2n} \epsilon^l \otimes \Big(\frac{\partial}{\partial x_l} - \frac{i}{n} e_l \cdot D_s \Big) \varphi = 0, \tag{7.14}$$

i.e.

$$\Big(\frac{\partial}{\partial x_l} - \frac{i}{n} e_l \cdot D_s \Big) \varphi = 0, \quad l = 1, \dots, 2n. \tag{7.15}$$

We apply to the last equation partial differentiation operator $\frac{\partial}{\partial x_m}$, multiply it by the skew symmetric form ω^{ml} and sum over $m = 1, \dots, 2n$:

$$\sum_{l,m=1}^{2n} \Big(\omega^{ml} \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_l} - \frac{i}{n} \omega^{ml} e_l \cdot \frac{\partial}{\partial x_m} D_s \Big) \varphi = 0. \tag{7.16}$$

The first part is zero because of the skew-symmetry of ω and the symmetry in m, l, and the second part is (a non-zero multiple of) the square of the symplectic Dirac operator D_s^2. Hence

$$\sum_{l,m=1}^{2n} \frac{i}{n} \omega^{ml} e_l \cdot \frac{\partial}{\partial x_m} D_s \varphi = -\frac{i}{n} D_s^2 \varphi = 0 \tag{7.17}$$

and the proof is complete. □

Lemma 7.4.3. Let $n \in \mathbb{N}$ and $\varphi \in \mathrm{Pol}(\mathbb{R}^{2n}, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R}^n)$ be a symplectic spinor fulfilling

$$\varphi \in \mathrm{Ker}(T_s) \cap \mathrm{Ker}(D_s). \tag{7.18}$$

Then φ is a constant (i.e., independent of $x_1, \dots, x_{2n}$) symplectic monogenic spinor. This is described by the following picture:

- $\mathrm{Pol}(\mathbb{R}^{2n}, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R}^n)^-$:

$$\begin{array}{ccccccccc}
\boxed{M_0^-} & \longrightarrow & X_s M_0^- & \longrightarrow & X_s^2 M_0^- & \longrightarrow & X_s^3 M_0^- & \longrightarrow & \dots \\
 & & \oplus & & \oplus & & \oplus & & \\
 & & M_1^- & \longrightarrow & X_s M_1^- & \longrightarrow & X_s^2 M_1^- & \longrightarrow & \dots \\
 & & & & \oplus & & \oplus & & \\
 & & & & M_2^- & \longrightarrow & X_s M_2^- & \longrightarrow & \dots \\
 & & & & & & \oplus & & \\
 & & & & & & M_3^- & \longrightarrow & \dots
\end{array} \tag{7.19}$$

- $\mathrm{Pol}(\mathbb{R}^{2n},\mathbb{C})\otimes\mathcal{S}(\mathbb{R}^n)^+$:

$$\begin{array}{ccccccccc}
\boxed{M_0^+} & \longrightarrow & X_sM_0^+ & \longrightarrow & X_s^2M_0^+ & \longrightarrow & X_s^3M_0^+ & \longrightarrow & \dots \\
 & & \oplus & & \oplus & & \oplus & & \\
 & & M_1^+ & \longrightarrow & X_sM_1^+ & \longrightarrow & X_s^2M_1^+ & \longrightarrow & \dots \\
 & & & & \oplus & & \oplus & & \\
 & & & & M_2^+ & \longrightarrow & X_sM_2^+ & \longrightarrow & \dots \\
 & & & & & & \oplus & & \\
 & & & & & & M_3^+ & \longrightarrow & \dots
\end{array} \tag{7.20}$$

Proof. Let $\varphi\in\mathrm{Pol}(\mathbb{R}^{2n},\mathbb{C})\otimes\mathcal{S}(\mathbb{R}^n)$ be a solution of the symplectic twistor operator, see (7.15),

$$\Big(\frac{\partial}{\partial x_l}-\frac{i}{n}e_l\cdot D_s\Big)\varphi=0,\quad l=1,\dots,2n,$$

and at the same time $\varphi\in\mathrm{Ker}(D_s)$. This implies

$$\frac{\partial}{\partial x_l}\varphi=0,\quad l=1,\dots,2n,$$

so φ is a constant symplectic spinor. The proof is complete. □

Lemma 7.4.4. Let $\varphi\in\mathrm{Pol}(\mathbb{R}^{2n},\mathbb{C})\otimes\mathcal{S}(\mathbb{R}^n)$ be a symplectic monogenic spinor of homogeneity $h\in\mathbb{N}_0$, i.e. $D_s(\varphi)=0$. Then the symplectic spinor $X_s(\varphi)$ has the following property

1. If $n=1$, then $X_s(\varphi)$ is in the kernel of T_s for any homogeneity $h\in\mathbb{N}_0$. This is described by the following picture

   - $\mathrm{Pol}(\mathbb{R}^{2n},\mathbb{C})\otimes\mathcal{S}(\mathbb{R}^n)^-$:

$$\begin{array}{ccccccccc}
M_0^- & \longrightarrow & \boxed{X_sM_0^-} & \longrightarrow & X_s^2M_0^- & \longrightarrow & X_s^3M_0^- & \longrightarrow & \dots \\
 & & \oplus & & \oplus & & \oplus & & \\
 & & M_1^- & \longrightarrow & \boxed{X_sM_1^-} & \longrightarrow & X_s^2M_1^- & \longrightarrow & \dots \\
 & & & & \oplus & & \oplus & & \\
 & & & & M_2^- & \longrightarrow & \boxed{X_sM_2^-} & \longrightarrow & \dots \\
 & & & & & & \oplus & & \\
 & & & & & & M_3^- & \longrightarrow & \dots
\end{array} \tag{7.21}$$

   - $\mathrm{Pol}(\mathbb{R}^{2n},\mathbb{C})\otimes\mathcal{S}(\mathbb{R}^n)^+$:

$$\begin{array}{ccccccccc}
M_0^+ & \longrightarrow & \boxed{X_sM_0^+} & \longrightarrow & X_s^2M_0^+ & \longrightarrow & X_s^3M_0^+ & \longrightarrow & \dots \\
 & & \oplus & & \oplus & & \oplus & & \\
 & & M_1^+ & \longrightarrow & \boxed{X_sM_1^+} & \longrightarrow & X_s^2M_1^+ & \longrightarrow & \dots \\
 & & & & \oplus & & \oplus & & \\
 & & & & M_2^+ & \longrightarrow & \boxed{X_sM_2^+} & \longrightarrow & \dots \\
 & & & & & & \oplus & & \\
 & & & & & & M_3^+ & \longrightarrow & \dots
\end{array} \tag{7.22}$$

2. If $n > 1$, then $X_s(\varphi)$ is in the kernel of T_s if and only if the homogeneity of φ is equal to $h = 0$. This is described by the following picture

   - $\mathrm{Pol}(\mathbb{R}^{2n}, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R}^n)^-$:

$$\begin{array}{ccccccccc} M_0^- & \longrightarrow & \boxed{X_s M_0^-} & \longrightarrow & X_s^2 M_0^- & \longrightarrow & X_s^3 M_0^- & \longrightarrow & \dots \\ & & \oplus & & \oplus & & \oplus & & \\ & & M_1^- & \longrightarrow & X_s M_1^- & \longrightarrow & X_s^2 M_1^- & \longrightarrow & \dots \\ & & & & \oplus & & \oplus & & \\ & & & & M_2^- & \longrightarrow & X_s M_2^- & \longrightarrow & \dots \\ & & & & & & \oplus & & \\ & & & & & & M_3^- & \longrightarrow & \dots \end{array} \tag{7.23}$$

   - $\mathrm{Pol}(\mathbb{R}^{2n}, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R}^n)^+$:

$$\begin{array}{ccccccccc} M_0^+ & \longrightarrow & \boxed{X_s M_0^+} & \longrightarrow & X_s^2 M_0^+ & \longrightarrow & X_s^3 M_0^+ & \longrightarrow & \dots \\ & & \oplus & & \oplus & & \oplus & & \\ & & M_1^+ & \longrightarrow & X_s M_1^+ & \longrightarrow & X_s^2 M_1^+ & \longrightarrow & \dots \\ & & & & \oplus & & \oplus & & \\ & & & & M_2^+ & \longrightarrow & X_s M_2^+ & \longrightarrow & \dots \\ & & & & & & \oplus & & \\ & & & & & & M_3^+ & \longrightarrow & \dots \end{array} \tag{7.24}$$

Proof. Let φ be a non-zero symplectic spinor in the kernel of D_s. The question is when the system of partial differential equations acting on φ,

$$(\partial_{x_k} - \frac{i}{n} e_k \cdot D_s) X_s \varphi = 0, \tag{7.25}$$

holds for all $k = 1, ..., 2n$. In other words, we ask when $X_s(\varphi)$ is in the kernel of the symplectic twistor operator. Let us multiply the k-th equation of the system by x_k and sum over all k,

$$(E - \frac{i}{n} X_s D_s) X_s \varphi = 0. \tag{7.26}$$

We use the $\mathfrak{sl}(2)$-commutation relations for X_s, D_s and for E, X_s, see (7.7), and the fact that φ is in the kernel of D_s. This gives

$$(E X_s - \frac{1}{n} X_s E - X_s) \varphi = 0. \tag{7.27}$$

Assuming that φ is of homogeneity h, $E\varphi = h\varphi$, the last equation reduces to

$$(h + 1 - \frac{h}{n} - 1) X_s s = h(1 - \frac{1}{n}) X_s \varphi = 0. \tag{7.28}$$

Observe that $(1 - \frac{1}{n}) \neq 0$ for $n > 1$, and X_s is an $\mathfrak{mp}(2n, \mathbb{R})$-intertwining map acting injectively on $\mathrm{Pol}(\mathbb{R}^{2n}, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R}^n)$ as a result of the metaplectic Howe duality (i.e., φ being non-zero implies $X_s(\varphi)$ is non-zero.) Because φ is assumed to be non-zero, the last display implies that either

1. $h = 0$ and $n \in \mathbb{N}$ is arbitrary, or
2. $n = 1$ and h is arbitrary.

A straightforward check for $n > 1$ and the homogeneity $h = 0$ gives

$$(\partial_{x_k} - ie_k\cdot D_s)X_s\varphi = \big(e_k\cdot + X_s\partial_{x_k} - \frac{i}{n}e_k\cdot E - e_k\big)\varphi = 0. \tag{7.29}$$

In the case $n = 1$ and arbitrary homogeneity, we have

$$\begin{aligned}(\partial_{x_1} - ie_1 D_s)X_s\varphi =& \big(e_1\cdot + e_1\cdot x_1\partial_{x_1} + e_2\cdot x_2\partial_{x_1} - e_1\cdot x_1\partial_{x_1} - e_1\cdot x_2\partial_{x_2} - e_1\cdot\big)\varphi = \\ =& \big(x_2(e_2\cdot \partial_{x_1} - e_1\cdot \partial_{x_2})\big)\varphi = -x_2 D_s\varphi = 0.\end{aligned} \tag{7.30}$$

For the second component $(\partial_{x_2} - ie_2\cdot D_s)$ of the symplectic twistor operator, the computation is analogous to the first one in (7.30). This completes the proof. □

Let us summarize our results in the final theorem.

Theorem 7.4.5. The solution space of the symplectic twistor operator T_s on the real symplectic manifold $(\mathbb{R}^{2n}, \omega)$ is given by $\mathfrak{mp}(2n, \mathbb{R})$-modules in the boxes on the following pictures

- In the case $n = 1$, we have for $\mathrm{Pol}(\mathbb{R}^{2n}, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R}^n)^{\pm}$:

$$\begin{array}{ccccccccc}
\boxed{M_0^\pm} & \longrightarrow & \boxed{X_s M_0^\pm} & \longrightarrow & X_s^2 M_0^\pm & \longrightarrow & X_s^3 M_0^\pm & \longrightarrow & \dots \\
 & & \oplus & & \oplus & & \oplus & & \\
 & & M_1^\pm & \longrightarrow & \boxed{X_s M_1^\pm} & \longrightarrow & X_s^2 M_1^\pm & \longrightarrow & \dots \\
 & & & & \oplus & & \oplus & & \\
 & & & & M_2^\pm & \longrightarrow & \boxed{X_s M_2^\pm} & \longrightarrow & \dots \\
 & & & & & & \oplus & & \\
 & & & & & & M_3^\pm & \longrightarrow & \dots
\end{array} \tag{7.31}$$

- In the case $n > 1$, we have for $\mathrm{Pol}(\mathbb{R}^{2n}, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R}^n)^{\pm}$:

$$\begin{array}{ccccccccc}
\boxed{M_0^\pm} & \longrightarrow & \boxed{X_s M_0^\pm} & \longrightarrow & X_s^2 M_0^\pm & \longrightarrow & X_s^3 M_0^\pm & \longrightarrow & \dots \\
 & & \oplus & & \oplus & & \oplus & & \\
 & & M_1^\pm & \longrightarrow & X_s M_1^\pm & \longrightarrow & X_s^2 M_1^\pm & \longrightarrow & \dots \\
 & & & & \oplus & & \oplus & & \\
 & & & & M_2^\pm & \longrightarrow & X_s M_2^\pm & \longrightarrow & \dots \\
 & & & & & & \oplus & & \\
 & & & & & & M_3^\pm & \longrightarrow & \dots
\end{array} \tag{7.32}$$

An interested reader can easily verify the previous result for $n > 1$ by taking a simple solution φ of D_s of homogeneity at least one (it is sufficient to generate such a simple solution from dimension $n = 1$ case) and check that $X_s(\varphi) \notin \mathrm{Ker}(T_s)$.

Example 6. In the case $n = 2$ and the homogeneity 2, the symplectic spinor

$$\varphi = e^{-\frac{q_1^2+q_2^2}{2}}(-ix_1x_2 + x_1x_4 + x_2x_3 + ix_3x_4) \tag{7.33}$$

is a solution of D_s. However, $X_s\varphi$ is not a solution of the symplectic twistor operator T_s because, for example, the first and the second components of $T_sX_s(\varphi)$ are nonzero

$$(T_s s)^1 = \epsilon^1 \otimes e^{-\frac{q_1^2+q_2^2}{2}} q_2(x_2+ix_4)^2 \neq 0,$$
$$(T_s s)^2 = \epsilon^2 \otimes e^{-\frac{q_1^2+q_2^2}{2}} q_1(x_1+ix_3)^2 \neq 0.$$

It is much harder to verify the result $X_s s \in \mathrm{Ker}(T_s)$ for all polynomial symplectic spinors φ, $\varphi \in \mathrm{Ker}(D_s)$, in the case $n=1$, and we refer to Chapter 6 for a non-trivial combinatorial proof of this assertion.

We would like to emphasize that the kernel of our solution space realizes (for $n>1$) the Segal-Shale-Weil representation, a prominent $\mathrm{Sp}(2n,\mathbb{R})$-module with far-reaching impact on harmonic analysis.

7.5 Comments and open problems

First of all, notice that in the case of (both even and odd) orthogonal algebras and the spinor representation as an orthogonal analogue of the Segal-Shale-Weil representation, the solution space of the twistor operator for orthogonal Lie algebras on $\mathbb{R}^n$ is given by two copies of the spinor representation, in complete analogy with the symplectic case, see [1] for $n \geq 3$. As for $n=2$, we were not able to find the required result in the available literature, although we believe it is known to specialists. Here one half of the Dirac operator is the Dolbeault operator and the twistor operator is its complex conjugate, while the opposite half of the Dirac and twistor operators are their complex conjugates, respectively. The solution spaces for both halves of the twistor operator on $\mathbb{R}^2$ are the complex linear spans of polynomials $\{z^j\}_{j\in\mathbb{N}_0}$ and $\{\bar{z}^j\}_{j\in\mathbb{N}_0}$, respectively, intersecting non-trivially in the constant polynomials. This is an orthogonal analogue of our results in symplectic category, and indicates an infinite-dimensional symmetry group acting on the solutions spaces of both symplectic Dirac and symplectic twistor operators in the real dimension 2.

Another observation is related to the proof of Lemma 7.4.2 and its structure on curved symplectic manifolds. Let us consider a $2n$-dimensional metaplectic manifold (M,∇,ω), with ∇^s the metaplectic covariant derivative. Then a differential consequence of the symplectic twistor equation on M is

$$\sum_{l,m=1}^{2n} \big(\omega^{ml}(\nabla^s_m,\nabla^s_l) - \frac{i}{n}D_s^2\big)\varphi = 0, \tag{7.34}$$

where the first term (skewing of the composition of metaplectic covariant derivatives) gives the action of the symplectic curvature of the symplectic connection ∇^s on the space of sections of a metaplectic bundle on M. This equation should be thought of as a symplectic analogue of the equation

$$D^2 s = \frac{1}{4}\frac{n}{n-1}Rs, \quad n \geq 3 \tag{7.35}$$

in Riemannian spin geometry with φ a twistor spinor, D the Dirac operator and R the scalar curvature of the Riemannian structure, cf., [1]. The prolongation of the symplectic twistor equation then constructs a linear connection and covariant derivative on the Segal-Shale-Weil representation, in such a way that the covariantly constant sections correspond to symplectic twistor spinors.

8. Symmetries of symplectic Dirac operator

We shall start the present section with a short reminder of the notion of symmetry operators for the classical Dirac operator associated to a quadratic form, see [15] and then pass to the case of our interest: the symplectic Dirac operator.

The Clifford algebra associated to a vector space equipped with a quadratic form B is determined by the relations $e_j e_k + e_k e_j = -2B(e_j, e_k)$, while the symplectic Clifford algebra on $(\mathbb{R}^{2n}, \omega)$ is given by relations $e_j \cdot e_k - e_k \cdot e_j = -\mathrm{i}\omega(e_j, e_k)$ with ω canonical symplectic form, $\mathrm{i} \in \mathbb{C}$. In the orthogonal case, the Dirac operator on $\mathbb{R}^m$ is $D = \sum_{j=1}^{m} e_j \partial_{x_j}$ and its solutions are termed the spherical monogenics. The module of polynomial spherical monogenics of homogeneity h is denoted by $M_h = \big(\mathrm{Pol}_h(\mathbb{R}^m, \mathbb{C}) \otimes \mathbb{S}\big) \cap \mathrm{Ker}(D)$, where $\mathbb{S}$ is the spinor space. In particular, each of the modules M_h, $h \in \mathbb{N}_0$, is an irreducible representation of the Lie algebra $\mathfrak{so}(m)$ acting by the differential operators

$$K_{jk} = x_j \partial_{x_k} - x_k \partial_{x_j} - \frac{1}{2} e_j e_k, \quad j \neq k, \quad j, k = 1, \dots, m.$$

See, e.g., [11]. Moreover, the space $M = \bigoplus_h M_h$ is a representation of the conformal Lie algebra $\mathfrak{so}(m+1, 1, \mathbb{R})$, which is the linear span of K_{jk}, $2E + m - 1$, ∂_{x_j} and $\tilde{T}_j$, $j, k = 1, \dots, m$; here the operators $\tilde{T}_j : M_h \to M_{h+1}$ act by

$$\tilde{T}_j = X e_j + x_j (m + 2E) - |X|^2 \partial_{x_j}. \tag{8.1}$$

Operators $\tilde{T}_j$ can be found for example in [26, Lem. 5.1].

Now let us turn our attention to the real symplectic manifold $(\mathbb{R}^{2n}, \omega)$ with coordinates $x_1, \dots, x_n$, $y_1, \dots, y_n$, and coordinate vector fields $\partial_{x_1}, \dots, \partial_{x_n}, \partial_{y_1}, \dots, \partial_{y_n}$ or equivalently symplectic frame $e_1, \dots, e_{2n}$ fulfilling

$$\omega(e_j, e_{n+j}) = 1, \quad \omega(e_{n+j}, e_j) = -1, \quad j = 1, \dots, n \tag{8.2}$$

and zero otherwise.

Let us remind realizations of $\mathfrak{sp}(2n, \mathbb{R})$ (1.3) on $\mathrm{Pol}(\mathbb{R}^{2n}, \mathbb{C})$ and (1.4) on $\mathcal{S}(\mathbb{R}^n)$. The representation of $\mathfrak{sp}(2n, \mathbb{R})$ on the space of polynomial symplectic spinors $\mathrm{Pol}(\mathbb{R}^{2n}, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R}^n)$ is

$$\begin{aligned}
X_{jk} &= x_j \partial_{x_k} - y_k \partial_{y_j} + q_j \partial_{q_k} + \frac{1}{2} \delta_{j,k}, \\
Y_{jj} &= x_j \partial_{y_j} - \frac{i}{2} q_j^2, \\
Y_{jk} &= x_j \partial_{y_k} + x_k \partial_{y_j} - i q_j q_k \text{ for } j \neq k, \\
Z_{jj} &= y_j \partial_{x_j} - \frac{i}{2} \partial_{q_j}^2, \\
Z_{jk} &= y_j \partial_{x_k} + y_k \partial_{x_j} - i \partial_{q_j} \partial_{q_k} \text{ for } j \neq k,
\end{aligned} \tag{8.3}$$

The three symplectic invariant operators with values in $\mathrm{End}(\mathcal{S}(\mathbb{R}^n))$,

$$X_s = \sum_{j=1}^{n} (y_j \partial_{q_j} + i x_j q_j),$$
$$D_s = \sum_{j=1}^{n} (i q_j \partial_{y_j} - \partial_{x_j} \partial_{q_j}),$$
$$E = \sum_{j=1}^{n} (x_j \partial_{x_j} + y_j \partial_{y_j}), \tag{8.4}$$

are $\mathfrak{sp}(2n, \mathbb{C})$-equivariant and generate the representation of $\mathfrak{sl}(2, \mathbb{C})$ on the space $\mathrm{Pol}(\mathbb{R}^{2n}, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R}^n)$. Their commutation relations are

$$[E + n, D_s] = -D_s,$$
$$[E + n, X_s] = X_s,$$
$$[X_s, D_s] = i(E + n). \tag{8.5}$$

First of all, we find differential operators increasing the homogeneity of polynomial solutions of the symplectic Dirac operator by one. We construct them as a composition of the multiplication by x_l, y_l, $l = 1, \dots, n$, and projection on the kernel of the symplectic Dirac operator D_s.

In the first step we check that D_s^3 acts trivially on $x_l m$, $y_l m$ for $m \in M_h^s$ and coordinate functions x_l, y_l on $\mathbb{R}^{2n}$. For $j = 1, \dots, n$, we have

$$D_s^2(x_j m) = D_s(-\partial_{q_j} m) = -\sum_{k=1}^{2n} (\mathrm{i} q_k \partial_{y_k} - \partial_{q_k} \partial_{x_k}) \partial_{q_j} m = \mathrm{i} \partial_{y_j} m,$$
$$D_s^2(y_j m) = D_s(\mathrm{i} q_j m) = \sum_{k=1}^{2n} (\mathrm{i} q_k \partial_{y_k} - \partial_{q_k} \partial_{x_k}) \mathrm{i} q_j m = -\mathrm{i} \partial_{x_j} m, \tag{8.6}$$

and so $x_l m$, $y_l m$ are in the kernel of D_s^3 for all $l = 1, \dots, n$. Denoting the identity endomorphism Id, the corresponding projector of $x_l m$, $y_l m$ on the homogeneity $h + 1$ subspace of $\mathrm{Ker}(D_s)$ is

$$P_{h+1}^s = \mathrm{Id} + c X_s D_s + d X_s^2 D_s^2$$

for some constants c, d depending on h and n. The relations (6.7) imply that on the spaces of homogeneous symplectic monogenics, the following relations hold

$$\begin{aligned} P_{h+1}^s m_{h+1} &= m_{h+1}, \\ P_{h+1}^s X_s m_h &= X_s m_h + c X_s D_s X_s m_h = \big(1 - ic(h+n)\big) X_s m_h, \\ P_{h+1}^s X_s^2 m_{h-1} &= X_s^2 m_{h-1} + c X_s D_s (X_s^2 m_{h-1}) + d X_s^2 D_s^2 (X_s^2 m_{h-1}) \\ &= X_s^2 m_{h-1} - ic X_s^2 (h - 1 + n) m_{h-1} - ic X_s (h+n) X_s m_{h-1} \\ &\quad - d(2h + 2n - 1)(h + n - 1) X_s^2 m_{h-1}. \end{aligned} \tag{8.7}$$

The second and the third expressions in (8.7) are zero if

$$c = \frac{1}{\mathrm{i}(h+n)}, \quad d = \frac{-1}{(h+n)(2h+2n-1)}.$$

Hence, the projector is

$$P^s_{h+1} = \mathrm{Id} + \frac{1}{i(h+n)} X_s D_s - \frac{1}{(h+n)(2h+2n-1)} X_s^2 D_s^2. \tag{8.8}$$

The actions of the operators $S_l = P^s_{h+1} x_l$, $l = 1, \dots, n$ and $S_{n+l} = P^s_{h+1} y_l$, $l = 1, \dots, n$, on $m \in M^s_h$ are given by

$$\begin{aligned} S_j m &= x_j m - c X_s \partial_{q_j} m + di X_s^2 \partial_{y_j} m, \\ S_{n+j} m &= y_j m + c X_s i q_j m - di X_s^2 \partial_{x_j} m. \end{aligned}$$

Thus we can define for $j = 1, \dots, n$ the collection of differential operators

$$\begin{aligned} Z_j &= -i(h+n)(2h+2n-1) S_{n+j}, \\ Z_{n+j} &= i(h+n)(2h+2n-1) S_j. \end{aligned} \tag{8.9}$$

Lemma 8.0.1. Let $j = 1, \dots, n$. The $\mathfrak{mp}(2n, \mathbb{R})$-equivariant first order differential operators

$$\begin{aligned} Z_j &= X_s^2 \partial_{x_j} - i y_j (E+n)(2E+2n-1) - i X_s q_j (2E+2n-1), \\ Z_{n+j} &= X_s^2 \partial_{y_j} + i x_j (E+n)(2E+2n-1) - X_s \partial_{q_j} (2E+2n-1) \end{aligned} \tag{8.10}$$

preserve the solution space of the symplectic Dirac operator on $(\mathbb{R}^{2n}, \omega)$. The operators Z_l, $l = 1, \dots, 2n$, increase by one the homogeneity in the base variables $x_1, \dots, x_n$, $y_1, \dots, y_n$

$$\begin{aligned} &Z_l : \mathrm{Ker}(D_s) \to \mathrm{Ker}(D_s), \\ &Z_l : M_h \to M_{h+1}, \quad l = 1, \dots, 2n, \end{aligned} \tag{8.11}$$

where M_h is the irreducible $\mathfrak{mp}(2n, \mathbb{R})$-module of symplectic polynomial spinors of homogeneity h in $\mathrm{Ker}(D_s)$.

Proof. The result is a consequence of

$$\begin{aligned} [D_s, X_s^2 \partial_{x_j}] =& - i X_s \partial_{x_j} (2E+2n-1), \qquad (8.12) \\ [D_s, \omega_{jk} \delta^{k,l} x_l (E+n)(2E+2n-1)] =& - e_j (E+n)(2E+2n-1) \\ &+ \omega_{jk} \delta^{k,l} x_l (4E+4n+1) D_s, \\ [D_s, X_s e_j (2E+2n-1)] =& - i e_j (E+n)(2E+2n-1) \\ &- i X_s \partial_{x_j} (2E+2n-1) + 2 X_s e_j D_s, \end{aligned}$$

because the linear combination

$$A X_s^2 \partial_{x_j} + B \omega_{jk} \delta^{k,l} x_l (E+n)(2E+2n-1) + C X_s e_j (2E+2n-1)$$

for $A, B, C \in \mathbb{C}$ and all $j = 1, \dots, 2n$ commutes with D_s provided $A = 1, B = i$ and $C = -1$. To shorten our notation we use $\omega_{jk} = \omega(e_j, e_k)$, see (8.2). □

The differential operators Z_j, Z_{n+j}, $j = 1, \dots, n$ are of third order, and are of second order in the base variables x_j, y_j (due to their quadratic dependence on the homogeneity operator E.)

Lemma 8.0.2. The $\mathfrak{mp}(2n, \mathbb{R})$-equivariant first order differential operators

$$\partial_{x_j}, \partial_{y_j}, \quad j = 1, \ldots, n \tag{8.13}$$

preserve the solution space of the symplectic Dirac operator on $(\mathbb{R}^{2n}, \omega)$ and decrease by one the homogeneity in the base variables $x_1, \ldots, x_n$, $y_1, \ldots, y_n$

$$\begin{aligned} &\partial_{x_j}, \partial_{y_j} : \mathrm{Ker}(D_s) \to \mathrm{Ker}(D_s), \\ &\partial_{x_j}, \partial_{y_j} : M_h \to M_{h-1}, \quad j = 1, \ldots, n, \end{aligned} \tag{8.14}$$

where M_h is the irreducible $\mathfrak{mp}(2n, \mathbb{R})$-module of symplectic polynomial spinors of homogeneity h in $\mathrm{Ker}(D_s)$.

Proof. The proof follows from $[\partial_{x_j}, D_s] = 0$ and $[\partial_{y_j}, D_s] = 0$ for $j = 1, \ldots, n$. □

8.1 First order symmetries of D_s on $(\mathbb{R}^2, \omega)$

The aim of the present section is to compute all first order differential operators which are symmetries of the symplectic Dirac operator. Here we restrict to $n = 2$. The case of general even dimension is notationally tedious.

We start with $(\mathbb{R}^2, \omega)$ and denote the coordinates by $x = x_1, y = y_1$, the coordinate vector fields by ∂_x, ∂_y and the symplectic frame by e_1, e_2 with the action on a symplectic spinor $\varphi \in \mathrm{Pol}(\mathbb{R}^2, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R})$ given by

$$e_1 \cdot \varphi = \mathrm{i}q\varphi, \quad e_2 \cdot \varphi = \partial_q \varphi.$$

Following (8.3), the basis elements of $\mathfrak{mp}(2, \mathbb{R}) (\simeq \mathfrak{sp}(2, \mathbb{R}) \simeq \mathfrak{sl}(2))$ act as

$$\begin{aligned} \tilde{X} &= -y\partial_x - \frac{\mathrm{i}}{2} q^2, \\ \tilde{Y} &= -x\partial_y - \frac{\mathrm{i}}{2} \partial_q^2, \\ \tilde{H} &= -x\partial_x + y\partial_y + q\partial_q + \frac{1}{2}, \end{aligned} \tag{8.15}$$

and satisfy the commutation relations of the Lie algebra $\mathfrak{mp}(2, \mathbb{R})$

$$\begin{aligned} [\tilde{X}, \tilde{Y}] &= \tilde{H}, \\ [\tilde{H}, \tilde{X}] &= 2\tilde{X}, \\ [\tilde{H}, \tilde{Y}] &= -2\tilde{Y}. \end{aligned}$$

Notice that these operators preserve homogeneity in the variables x, y. The three $\mathfrak{mp}(2, \mathbb{R})$-invariant operators

$$\begin{aligned} X_s &= y\partial_q + \mathrm{i}xq, \\ D_s &= \mathrm{i}q\partial_y - \partial_x\partial_q, \\ E &= x\partial_x + y\partial_y \end{aligned} \tag{8.16}$$

form a Lie algebra isomorphic to $\mathfrak{sl}(2)$. The operators X_s, D_s and E commute with $\tilde{X}, \tilde{Y}$ and $\tilde{H}$, i.e., they are $\mathfrak{mp}(2, \mathbb{R})$-intertwining differential operators on complex polynomials valued in the Segal-Shale-Weil representation.

Lemma 8.1.1. The commuting operators

$$Z_1 = -X_s^2\partial_x + \mathrm{i}y(E+1)(2E+1) + X_s\mathrm{i}q(2E+1),$$
$$Z_2 = -X_s^2\partial_y - \mathrm{i}x(E+1)(2E+1) + X_s\partial_q(2E+1) \tag{8.17}$$

preserve the solution space of the symplectic Dirac operator D_s and increase the homogeneity in the variables x, y by one, $Z_j : M_h \to M_{h+1}$, $j = 1, 2$.

The commuting operators

$$\partial_x, \partial_y \tag{8.18}$$

preserve the solution space of the symplectic Dirac operator D_s and decrease the homogeneity in the variables x, y by one.

The commutator $[Z_1, Z_2]$ is zero, and

$$[\partial_x, Z_1] = -2\mathrm{i}\tilde{X}(2E+1),$$
$$[\partial_y, Z_1] = 2X_sD_s + \mathrm{i}\tilde{H}(2E+1) + \mathrm{i}(2E+1)(2E+1) + \frac{\mathrm{i}}{2},$$
$$[\partial_x, Z_2] = -2X_sD_s + \mathrm{i}\tilde{H}(2E+1) - \mathrm{i}(2E+1)(2E+1) - \frac{\mathrm{i}}{2},$$
$$[\partial_y, Z_2] = 2\mathrm{i}\tilde{Y}(2E+1). \tag{8.19}$$

Moreover, we have

$$\begin{array}{ll} [Z_1, \tilde{H}] = -Z_1, & [Z_2, \tilde{H}] = Z_2, \\ [Z_1, \tilde{X}] = 0, & [Z_2, \tilde{X}] = -Z_1, \\ [Z_1, \tilde{Y}] = Z_2, & [Z_2, \tilde{Y}] = 0, \\ [Z_1, E] = -Z_1, & [Z_2, E] = -Z_2, \end{array} \tag{8.20}$$

as well as

$$\begin{array}{ll} [\partial_x, \tilde{H}] = -\partial_x & [\partial_y, \tilde{H}] = \partial_y, \\ [\partial_x, \tilde{X}] = 0 & [\partial_y, \tilde{X}] = -\partial_x, \\ [\partial_x, \tilde{Y}] = -\partial_y & [\partial_y, \tilde{Y}] = 0, \\ [\partial_x, E] = \partial_x, & [\partial_y, E] = \partial_y. \end{array} \tag{8.21}$$

Remark 9. The commutator of commutators $[\partial_x, Z_1] = -2\mathrm{i}\tilde{X}(2E+1)$ and $[\partial_y, Z_2] = 2\mathrm{i}\tilde{Y}(2E+1)$ gives

$$[-2\mathrm{i}\tilde{X}(2E+1), 2\mathrm{i}\tilde{Y}(2E+1)] = 4\tilde{H}(2E+1)(2E+1).$$

We can compute the commutator of this commutator. For example, we have $[\partial_x, Z_1] = -2\mathrm{i}\tilde{X}(2E+1)$, resulting into the third power of $(2E+1)$. In general, we can produce an arbitrarily high power of $(2E+1)$ in iterated commutators, hence the linear span of the operators $\tilde{H}, \tilde{X}, \tilde{Y}, \partial_x, \partial_y, Z_1, Z_2$ and E is not closed under the commutator bracket.

Let us briefly mention the key concept of (generalized) differential symmetries for the symplectic Dirac operator, see [15] and references therein. A differential operator A is a symmetry of D_s if there exists another differential operator B such that

$$D_sA = BD_s. \tag{8.22}$$

Consequently, symmetry operators preserve the solution space of the symplectic Dirac operator.

Theorem 8.1.2. The first order symmetries of the symplectic Dirac operator D_s on $\mathbb{R}^2$ are given by the linear span of differential operators ∂_x, ∂_y, $\tilde{H}$, $\tilde{X}$, E and $y\tilde{H} - 2x\tilde{X} + yE + \frac{3}{2}y$.

Proof. Let us consider a general first order differential operator in the variables x, y, q

$$A = F_0(x,y,q)\partial_x + F_1(x,y,q)\partial_y + F_2(x,y,q)\partial_q + F_3(x,y,q),$$

where $F_j, j = 0,1,2,3$, are convenient functions of x, y and q. Then $D_s A = AD_s + [D_s, A]$, so that (8.22) implies $[D_s, A] = B' D_s$ for a differential operator B'. The computation of commutators gives

$$\begin{aligned}&\big(iq[\partial_y, F_0(x,y,q)] - \partial_q[\partial_x, F_0(x,y,q)] - [\partial_q, F_2(x,y,q)]\partial_q - [\partial_q, F_3(x,y,q)]\big)\partial_x\\ &+ \big(iq[\partial_y, F_1(x,y,q)] - \partial_q[\partial_x, F_1(x,y,q)] + F_2(x,y,q)[iq, \partial_q]\big)\partial_y\\ &- [\partial_q, F_0(x,y,q)]\partial_x^2 - [\partial_q, F_1(x,y,q)]\partial_x\partial_y\\ &+ iq[\partial_y, F_2(x,y,q)]\partial_q - \partial_q[\partial_x, F_2(x,y,q)]\partial_q + iq[\partial_y, F_3(x,y,q)] - \partial_q[\partial_x, F_3(x,y,q)]\\ &= B'(iq\partial_y - \partial_x\partial_q).\end{aligned}$$

The commutator $[\partial_q, F_0(x,y,q)]$ by ∂_x^2 does not depend on ∂_q, and thus equals to zero. Hence $F_0(x,y,q)$ is independent of the variable q, $F_0 \equiv F_0(x,y)$. Then the commutator $[\partial_q, F_1(x,y,q)]$ by $\partial_x\partial_y$ has to be zero as well, i.e., $F_1(x,y)$ is independent of q. Moreover, the commutator $[\partial_x, F_1(x,y)]$ in $\partial_q[\partial_x, F_1(x,y)]\partial_y$ has to be zero, i.e., $F_1 \equiv F_1(y)$.

We can separate the last equation into three equalities

$$\begin{aligned}&\big(iq[\partial_y, F_0(x,y)] - [\partial_q, F_3(x,y,q)] - \big([\partial_x, F_0(x,y)] + [\partial_q, F_2(x,y,q)]\big)\partial_q\big)\partial_x\\ &= -B'\partial_q\partial_x, \qquad (8.23)\\ &\big(iq[\partial_y, F_1(y)] - iF_2(x,y,q)\big)\partial_y = B' iq\partial_y, \qquad (8.24)\\ &iq[\partial_y, F_2(x,y,q)]\partial_q - \partial_q[\partial_x, F_2(x,y,q)]\partial_q + iq[\partial_y, F_3(x,y,q)]\\ &- \partial_q[\partial_x, F_3(x,y,q)] = 0. \qquad (8.25)\end{aligned}$$

The equation (8.23) yields $iq[\partial_y, F_0(x,y)] - [\partial_q, F_3(x,y,q)] = 0$. We set

$$F_3(x,y,q) = F_3'(x,y)\frac{i}{2}q^2 + F_3''(x,y), \tag{8.26}$$

and therefore

$$[\partial_y, F_0(x,y)] = F_3'(x,y). \tag{8.27}$$

The second equality (8.24) implies

$$F_2(x,y,q) = F_2'(x,y)q. \tag{8.28}$$

Then $[\partial_q, F_2(x,y,q)] = F_2'(x,y)$, and equations (8.24) and (8.23) give

$$\begin{aligned}&[\partial_y, F_1(y)] - F_2'(x,y) = [\partial_x, F_0(x,y)] + F_2'(x,y),\\ &[\partial_y, F_1(y)] = [\partial_x, F_0(x,y)] + 2F_2'(x,y). \qquad (8.29)\end{aligned}$$

The equation (8.25) can be rewritten with the use of (8.26) and (8.28) as

$$[\partial_y, F_2'(x,y)]iq^2\partial_q - [\partial_x, F_2'(x,y)](\partial_q + q\partial_q^2) - [\partial_y, F_3'(x,y)]\frac{1}{2}q^3$$
$$+[\partial_y, F_3''(x,y)]iq - [\partial_x, F_3'(x,y)](iq + \frac{1}{2}iq^2\partial_q) - [\partial_x, F_3''(x,y)]\partial_q = 0.$$

Because there is only one commutator by $q\partial_q^2$ and q^3, we have $F_2' \equiv F_2'(y)$, $F_3' \equiv F_3'(x)$. Then the commutators by ∂_q have to be zero and F_3'' is independent of x, $F_3'' \equiv F_3''(y)$. The commutators by $iq^2\partial_q$ and iq give the relations

$$[\partial_y, F_2'(y)] - \frac{1}{2}[\partial_x, F_3'(x)] = 0, \qquad [\partial_y, F_3''(y)] - [\partial_x, F_3'(x)] = 0. \tag{8.30}$$

The solution of (8.30) is $F_2'(y) = \frac{1}{2}\alpha y + \gamma$, $F_3'(x) = \alpha x + \beta$ and $F_3'' = \alpha y + \gamma$. The substitution of this solution into (8.27) yields $F_0(x,y) = \alpha xy + \beta y + F_0'(x)$. Substituting into (8.29), we get $F_0'(x) = \eta x + \zeta$ and $F_1(y) = \alpha y^2 + (2\gamma + \eta)y + \kappa$. Taken altogether, the functions $F_j, j = 0,1,2,3$ are

$$F_0 = \alpha xy + \eta x + \beta y + \zeta, \qquad F_2 = \frac{1}{2}\alpha yq + \gamma q,$$
$$F_1 = \alpha y^2 + (2\gamma + \eta)y + \kappa, \qquad F_3 = (\alpha x + \beta)\,\frac{i}{2}q^2 + \alpha y + \delta,$$

where $\alpha, \beta, \gamma, \delta, \eta, \zeta, \kappa \in \mathbb{C}$ are arbitrary constants. The constant β corresponds to the operator $\tilde{X}$, ζ and κ correspond to ∂_x and ∂_y. A combination of η, γ and δ corresponds to a combination of E, $\tilde{H}$ and the identity operator. Finally, α corresponds to the operator $y\tilde{H} - 2x\tilde{X} + yE + \frac{3}{2}y$. □

We notice that $\tilde{Y}$ is a second order differential operator, but it is first order in the base variables x, y. The operators Z_1, Z_2 are symmetries of D_s but they are third order differential operators, second order in the base variables x, y.

8.2 First order symmetries of D_s in holomorphic variables on $(\mathbb{R}^2, \omega)$

We use the complex coordinates $z = x + iy, \overline{z} = x - iy$, for the standard complex structure on $\mathbb{R}^2$, where $\partial_x = \partial_z + \partial_{\bar{z}}$ and $\partial_y = i(\partial_z - \partial_{\bar{z}})$. In the complex coordinates $z, \bar{z}$ we have

$$\begin{aligned} X_s &= \frac{i}{2}\big((q - \partial_q)z + (q + \partial_q)\overline{z}\big), \\ D_s &= -(q + \partial_q)\partial_z + (q - \partial_q)\partial_{\bar{z}}, \\ E &= z\partial_z + \bar{z}\partial_{\bar{z}} \end{aligned} \tag{8.31}$$

and

$$\begin{aligned} Z_1 &= 2X_s^2\partial_z + \bar{z}(E+1)(2E+1) + iX_s(\partial_q - q)(2E+1), \\ Z_2 &= 2X_s^2\partial_{\bar{z}} - z(E+1)(2E+1) - iX_s(\partial_q + q)(2E+1), \end{aligned} \tag{8.32}$$

where $Z_1 = \bar{Z}_1 + i\bar{Z}_2$ and $Z_2 = \bar{Z}_1 - i\bar{Z}_2$. Cf., $\bar{Z}_1$ and $\bar{Z}_2$ in Lemma 8.1.1 rewritten in the variables $z, \bar{z}$.

The commutator $[Z_1, Z_2] = 0$, and the commutators with (anti-)holomorphic coordinate vector fields are

$$
\begin{aligned}
&[Z_1, \partial_z] = 2iX_t(2E+1),\\
&[Z_1, \partial_{\bar z}] = 2iX_sD_s - H_t(2E+1) - (2E+1)(2E+1) - \frac{1}{2},\\
&[Z_2, \partial_z] = -2iX_sD_s - H_t(2E+1) + (2E+1)(2E+1) + \frac{1}{2},\\
&[Z_2, \partial_{\bar z}] = 2iY_t(2E+1).
\end{aligned} \tag{8.33}
$$

Moreover, we introduce

$$
\begin{aligned}
H_t &= i\bar{\bar{X}} - i\bar{\bar{Y}},\\
X_t &= -\frac{1}{2}\big(\bar{\bar{X}} + \bar{\bar{Y}} + i\bar{\bar{H}}\big),\\
Y_t &= -\frac{1}{2}\big(\bar{\bar{X}} + \bar{\bar{Y}} - i\bar{\bar{H}}\big),
\end{aligned}
$$

where $\bar{\bar{H}}, \bar{\bar{X}}$ and $\bar{\bar{Y}}$ denote the operators (8.15) in variables $z, \bar z$

$$
\begin{aligned}
H_t &= \bar z\partial_{\bar z} - z\partial_z + \frac{1}{2}(q^2 - \partial_q^2),\\
X_t &= i\bar z\partial_z + \frac{i}{4}(q - \partial_q)^2,\\
Y_t &= -iz\partial_{\bar z} + \frac{i}{4}(q + \partial_q)^2.
\end{aligned} \tag{8.34}
$$

The operators H_t, X_t and Y_t commute with D_s, X_s, E, and satisfy the commutation relations of algebra $\mathfrak{mp}(2n, \mathbb{R})$

$$
\begin{aligned}
[X_t, Y_t] &= H_t,\\
[H_t, X_t] &= 2X_t,\\
[H_t, Y_t] &= -2Y_t.
\end{aligned} \tag{8.35}
$$

The straightforward computation reveals

$$
\begin{array}{ll}
[Z_1, H_t] = -Z_1, & [Z_2, H_t] = Z_2,\\
[Z_1, X_t] = 0, & [Z_2, X_t] = iZ_1,\\
[Z_1, Y_t] = -iZ_2, & [Z_2, Y_t] = 0,\\
[Z_1, E] = -Z_1, & [Z_2, E] = -Z_2,
\end{array} \tag{8.36}
$$

$$
\begin{array}{ll}
[\partial_z, H_t] = -\partial_z, & [\partial_{\bar z}, H_t] = \partial_{\bar z},\\
[\partial_z, X_t] = 0, & [\partial_{\bar z}, X_t] = i\partial_z,\\
[\partial_z, Y_t] = -i\partial_{\bar z}, & [\partial_{\bar z}, Y_t] = 0,\\
[\partial_z, E] = \partial_z, & [\partial_{\bar z}, E] = \partial_{\bar z}.
\end{array} \tag{8.37}
$$

9. Rudiments of symplectic Clifford-Fourier transform

The central role in harmonic analysis on $\mathbb{R}^n$ is played by the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$, generated by the $\mathfrak{so}(n,\mathbb{R})$-invariant Laplace operator $\triangle$, the norm squared $|x|^2$ of the vector $x \in \mathbb{R}^n$ and the Euler operator. The classical integral Fourier transform,

$$F(f)(y) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x) \exp^{-\mathrm{i}\langle x,y\rangle} \, \mathrm{d}x, \quad \langle x, y\rangle = \sum_{i=1}^{n} x_i y_i, \tag{9.1}$$

can be equivalently represented by the operator exponential that contains the generators of $\mathfrak{sl}(2,\mathbb{C})$

$$\exp^{\frac{\mathrm{i}\pi n}{4}} \exp^{\frac{\mathrm{i}\pi}{4}(\triangle - |x|^2)}. \tag{9.2}$$

In particular the operators $\triangle$, $|x|^2$ have the same spectral properties. There are analogous results in the harmonic analysis for finite groups based on Dunkl operators, or Clifford analysis based on the Clifford algebra associated to a quadratic form and the Dirac operator $D = \sum_{j=1}^{n} e_j \partial_{x_j}$, written in a basis $e_1, \dots e_n$ of $\mathbb{R}^n$ with coordinates $x_1, \dots, x_n$. Cf., [7], [8] and [10].

In the present section, we discuss several basic questions in this direction, focusing on symplectic Clifford analysis and the associated symplectic Dirac operator in real dimension 2.

9.1 Eigenfunction decomposition for operator $D_s - cX_s$

The symplectic Fourier transform is based on the eigenvalue equation

$$(D_s - cX_s)f = \lambda f, \quad c \in \mathbb{R}, \lambda \in \mathbb{C}. \tag{9.3}$$

As already indicated, we shall stick to the real dimension 2 and look for solutions of this equation in terms of a linear combination of elements $g(X_s)m_k^s$, where $m_k^s \in M_k^s$ is a symplectic monogenic and g is a polynomial in the variable X_s. First we shall focus on the problem whether for a symplectic spinor φ valued in $\mathcal{S}(\mathbb{R})$ the relation $e^{\alpha X_s}\varphi \in \mathcal{S}(\mathbb{R})$ holds for $\alpha \in \mathbb{C}$.

Lemma 9.1.1. The following identity holds

$$e^{\alpha X_s} e^{-\frac{q^2}{2}} = e^{-\frac{q^2}{2}} e^{\frac{1}{2}\alpha(ix-y)(2q+\alpha y)}. \tag{9.4}$$

Proof. Writing the exponential as

$$e^{\alpha X_s} e^{-\frac{q^2}{2}} = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} X_s^k e^{-\frac{q^2}{2}},$$

we show by induction on $k \in \mathbb{N}_0$ that

$$X_s^k e^{-\frac{q^2}{2}} = e^{-\frac{q^2}{2}} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k!(ix-y)^{k-m} q^{k-2m} y^m}{m!(k-2m)!2^m}. \tag{9.5}$$

Recall, that $\lfloor \cdot \rfloor$ is the floor function. The equation is satisfied for $k = 0$ and for $k = 1$, $X_s e^{-\frac{q^2}{2}}$ is equal to $e^{-\frac{q^2}{2}} q(\mathrm{i}x - y)$. Assuming (9.5) holds for k, we aim to prove the identity for $k+1$. Let us start with odd k

$$(ixq + y\partial_q) e^{-\frac{q^2}{2}} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k!(ix-y)^{k-m} q^{k-2m} y^m}{m!(k-2m)!2^m} =$$
$$= e^{-\frac{q^2}{2}} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k!(ix-y)^{k+1-m} q^{k+1-2m} y^m}{m!(k-2m)!2^m} + e^{-\frac{q^2}{2}} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k!(ix-y)^{k-m} q^{k-1-2m} y^{m+1}}{m!(k-2m-1)!2^m},$$

and the shift $m \mapsto m-1$ in the second sum results into

$$e^{-\frac{q^2}{2}} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(k+1)!(ix-y)^{k+1-m} q^{k+1-2m} y^m}{m!(k+1-2m)!2^m} \left(\frac{k+1-2m}{k+1} + \frac{2m}{k+1} \right)$$
$$+ e^{-\frac{q^2}{2}} \frac{(k+1)!(ix-y)^{k+1-\frac{k+1}{2}} y^{\frac{k+1}{2}}}{\left(\frac{k+1}{2}\right)! 2^{\frac{k+1}{2}}} =$$
$$= e^{-\frac{q^2}{2}} \sum_{m=0}^{\lfloor \frac{k+1}{2} \rfloor} \frac{(k+1)!(ix-y)^{k+1-m} q^{k+1-2m} y^m}{m!(k+1-2m)!2^m}$$

which proves the induction step. For k even, $\lfloor \frac{k}{2} \rfloor = \lfloor \frac{k+1}{2} \rfloor$ and the second expression on the last display is zero. Then

$$e^{\alpha X_s} e^{-\frac{q^2}{2}} = e^{-\frac{q^2}{2}} \sum_{k=0}^{\infty} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{\alpha^k (ix-y)^{k-m} q^{k-2m} y^m}{m!(k-2m)!2^m}.$$

The change of the order in the last summation when keeping m fixed, gives

$$\sum_{k=2m}^{\infty} \frac{\alpha^k (ix-y)^{k-m} q^{k-2m} y^m}{m!(k-2m)!2^m} = \frac{\alpha^{2m} (ix-y)^m y^m}{m! 2^m} e^{\alpha q (ix-y)}, \tag{9.6}$$

and thus

$$e^{-\frac{q^2}{2}} \sum_{m=0}^{\infty} \frac{\alpha^{2m} (ix-y)^m y^m}{m! 2^m} e^{\alpha q (ix-y)} = e^{-\frac{q^2}{2}} e^{\frac{1}{2}\alpha(ix-y)(2q+\alpha y)}.$$

This completes the proof. □

By Lemma 9.1.1, we see that $e^{\alpha X_s}\varphi$, $\alpha \in \mathbb{C}$, is for $\varphi = e^{-\frac{q^2}{2}}$ a Schwartz function in the variable q and a non-polynomial function in the variables x, y. This property remains true for any $\varphi = p(x,y) e^{-\frac{q^2}{2}}$, where $p(x,y) \in \mathrm{Pol}(\mathbb{R}^2, \mathbb{C})$.

Taking as basis elements of the Schwartz space $q^j e^{-\frac{q^2}{2}} \in \mathcal{S}(\mathbb{R})$, $j \in \mathbb{N}_0$, we see that

$$e^{\alpha X_s} q^j e^{-\frac{q^2}{2}} = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} X_s^k q^j e^{-\frac{q^2}{2}} \tag{9.7}$$

is a Schwartz function in q, because in the expansion of $X_s^k q^j e^{-\frac{q^2}{2}}$ the maximal exponent of q is just $k+j$, cf., (9.5). Therefore, $e^{\alpha X_s} q^j e^{-\frac{q^2}{2}}$ grows as $q^j e^{-\frac{q^2}{2}} e^{\alpha q}$, $\alpha \in \mathbb{C}$, which is a characterizing property of Schwartz function class in the variable q.

It is easy to verify the following identities in the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$

$$\begin{aligned}
[E+n, X_s^k] &= kX_s^k, \\
[D_s, X_s^k] &= -i(E+n)X_s^{k-1} - iX_s(E+n)X_s^{k-2} - \ldots - iX_s^{k-1}(E+n) \\
&= -ik\frac{k-1}{2}X_s^{k-1} - ikX_s^{k-1}(E+n).
\end{aligned} \tag{9.8}$$

Then for $\alpha \in \mathbb{C}$, we get

$$\begin{aligned}
[D_s, e^{\alpha X_s}] &= \sum_{k=0}^{\infty} \frac{\alpha^k}{k!}[D_s, X_s^k] \\
&= -i\sum_{k=1}^{\infty} \frac{\alpha^k}{(k-1)!}X_s^{k-1}(E+n) - i\frac{\alpha^2}{2}X_s \sum_{k=2}^{\infty} \frac{\alpha^{k-2}}{(k-2)!}X_s^{k-2} \\
&= -i\alpha e^{\alpha X_s}(E+n) - i\frac{\alpha^2}{2}X_s e^{\alpha X_s}.
\end{aligned} \tag{9.9}$$

Let us mention the generalized Laguerre polynomials

$$L_j^{\beta}(x) = \sum_{l=0}^{j} (-1)^l \binom{j+\beta}{j-l} \frac{x^l}{l!}$$

defined by the formula

$$L_j^{\beta}(x) = \frac{x^{-\beta} e^x}{j!} \frac{d^j}{dx^j}(x^{x+\beta} e^{-x})$$

for $j \in \mathbb{N}_0$ and arbitrary real $\beta > -1$, see [33] for more details.

The spectral decomposition of our operator, which we call the symplectic spin harmonic oscillator, is summarized in the following Lemma.

Lemma 9.1.2. The operator $H = D_s - cX_s$, $c \in \mathbb{R}$, has a complete system of eigenfunctions (valued in the Segal-Shale-Weil representation) given by

$$f_k^j = e^{\sqrt{2ic}X_s} L_j^{2n+2k-1}(-2\sqrt{2ic}X_s) m_k^s, \tag{9.10}$$

where $m_k^s \in M_k^s$ is a symplectic monogenic, $L_j^{\alpha}(-2\sqrt{2ic}X_s)$ is the generalized Laguerre polynomial of operator $-2\sqrt{2ic}X_s$ and $j, k \in \mathbb{N}$. The corresponding eigenvalues are

$$\lambda_k^j = \sqrt{2ic}(n+j+k). \tag{9.11}$$

Proof. The substitution of

$$f = e^{\alpha X_s} g(X_s) m_k^s$$

into (9.3), where $m_k^s \in M_k^s$ is a symplectic monogenic and $g(X_s)$ is a polynomial in X_s, yields

$$D_s e^{\alpha X_s} g(X_s) m_k^s - c X_s e^{\alpha X_s} g(X_s) m_k^s = \lambda e^{\alpha X_s} g(X_s) m_k^s.$$

By (9.9) and because $e^{\alpha X_s}$ is an invertible operator, we get

$$D_s(g(X_s)m_k^s) - i\alpha(E+n)g(X_s)m_k^s - \big(c + i\frac{\alpha^2}{2}\big)X_s g(X_s)m_k^s = \lambda g(X_s)m_k^s.$$

Now we set $c = -i\frac{\alpha^2}{2}$, i.e., $\sqrt{2ic} = \alpha$ (we choose and fix one of the roots)

$$D_s\big(g(X_s)m_k^s\big) - i\alpha(E+n)g(X_s)m_k^s = \lambda g(X_s)m_k^s \tag{9.12}$$

and substitute

$$g(X_s) = g_k^j(X_s) = \sum_{l=0}^{j} \beta_l^{j,k} X_s^l. \tag{9.13}$$

Then (9.12) turns into recurrence relation

$$\lambda \sum_{l=0}^{j} \beta_l^{j,k} X_s^l m_k^s = -i\alpha \sum_{l=0}^{j} \beta_l^{j,k}(l+k+n) X_s^l m_k^s + \sum_{l=0}^{j} \beta_l^{j,k} D_s(X_s^l m_k^s).$$

Noting that $D_s(X_s^l m_k^s) = -i\frac{l}{2}(2k+2n+l-1)X_s^{l-1}m_k^s$, see (9.8), we have

$$\sum_{l=0}^{j}(\lambda + i\alpha(l+k+n))\beta_l^{j,k} X_s^l m_k^s = -i\sum_{l=0}^{j-1} \frac{l+1}{2}(2k+2n+l)\beta_{l+1}^{j,k} X_s^l m_k^s.$$

Finally, we obtain the following recurrence relations for $l = 0, 1, \ldots, j-1$

$$(\lambda + i\alpha(l+k+n))\beta_l^{j,k} = -i\frac{l+1}{2}(2k+2n+l)\beta_{l+1}^{j,k}, \tag{9.14}$$

$$(\lambda + i\alpha(l+k+n))\beta_j^{j,k} = 0. \tag{9.15}$$

In order $g(X_s)$ is a polynomial, we have $\lambda = -i\alpha(n+j+k)$, that is an eigenvalue. Hence, our recurrence relation becomes

$$\alpha(j-l)\beta_l^{j,k} = \frac{l+1}{2}(2k+2n+l)\beta_{l+1}^{j,k},$$

which results into

$$\beta_{l+1}^{j,k} = \frac{2\alpha(j-l)}{(l+1)(2k+2n+l)}\beta_l^{j,k} = \ldots = 2^{l+1}\alpha^{l+1}\binom{j}{l+1}\frac{(2k+2n-1)!}{(2k+2n+l)!}\beta_0^{j,k}.$$

Therefore, we conclude that

$$\beta_l^{j,k} = 2^l \alpha^l \binom{j}{l}\frac{(2k+2n-1)!}{(2k+2n-1+l)!}\beta_0^{j,k}. \tag{9.16}$$

We choose $\beta_0^{j,k} = 1$. Hence, we have

$$\begin{aligned} g_k^j(X_s) &= \sum_{l=0}^{l} 2^l \alpha^l \binom{j}{l} \frac{(2k+2n-1)!}{(2k+2n-1+l)!} X_s^l \\ &= j! \frac{(2k+2n-1)!}{(2k+2n-1+j)!} L_j^{2n+2k-1}(-2\alpha X_s), \end{aligned} \tag{9.17}$$

where L_j^{β} is the generalized Laguerre polynomial. □

Example 7. The simplest eigenfunction for $j = 0$ is $e^{\sqrt{2ic}X_s} e^{-\frac{q^2}{2}} \in M_0^s$, where $e^{-\frac{q^2}{2}}$ is a highest weight vector of the Segal-Shale-Weil representation.

10. Symplectic Fischer product and reproducing kernel on symplectic spinors

Let us briefly mention a motivation given by the classical orthogonal Fischer scalar product. For two complex polynomials valued in the Clifford algebra associated to a quadratic form, $f \otimes a,\ g \otimes b \in \mathrm{Pol}(\mathbb{R}^m, \mathbb{C}) \otimes Cl(\mathbb{R}^m)$, the Fischer scalar product is defined by

$$\langle f \otimes a, g \otimes b \rangle = [\overline{f(\partial_x)} g]_{x=0} [\overline{a} b]_0.$$

Here $\overline{f(\partial_x)}$ is a differential operator, where we substitute ∂_{x_j} for the variable x_j, $j = 1, \ldots, m$, and act by the resulting differential operator on a polynomial $g(x)$. As for the values, $[\ ]_0$ denotes the zero degree part of an element in $Cl(\mathbb{R}^m)$. The properties of scalar products are conveniently encoded in their reproducing kernels. For example, the space of homogeneous polynomials of homogeneity k satisfies

$$\left\langle \frac{\langle x, y \rangle^k}{k!}, g(x) \right\rangle = g(y)$$

for all $g \in \mathrm{Pol}_k(\mathbb{R}^m, \mathbb{C})$ and $\langle , \rangle$ the canonical scalar product on $\mathbb{R}^m$. Hence, the reproducing kernel for homogeneity k harmonic polynomials $\mathcal{H}_k$,

$$Z_k(x, y) = \mathrm{Proj}_{\mathcal{H}_k} \left(\frac{\langle x, y \rangle^k}{k!} \right), \tag{10.1}$$

can be expressed by the use of the so-called Gegenbauer polynomial. The interested reader can find more about this topic in, e.g., [6].

In what follows, we attempt to apply the concept of Fischer product and reproducing kernel to the space of symplectic spinors equipped with the action of the metaplectic Lie algebra. As in the previous section, after some general considerations we focus mostly on the real dimension 2.

10.1 Symplectic Fischer product and reproducing kernel for $n = 1$

Now we aim to define the Fischer product on the space of symplectic spinors. We construct the symplectic Fischer product on $\mathrm{Pol}(\mathbb{R}^{2n}, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R}^n)$ for $f \otimes \psi$, $g \otimes \phi$ with $f, g \in \mathrm{Pol}(\mathbb{R}^{2n}, \mathbb{C})$ and $\psi, \phi \in \mathcal{S}(\mathbb{R}^n)$, in the form

$$\langle f \otimes \psi, g \otimes \phi \rangle = \omega(f, g) \int_{\mathbb{R}^n} \overline{\psi(q)} \phi(q)\, \mathrm{d}q. \tag{10.2}$$

The integral is the inner product in the fiber variables $q_1, \ldots, q_n$ and $\omega(f, g)$ is the evaluation of a lift of the symplectic form to $Sym_k(\mathbb{R}^{2n})$, $k \in \mathbb{N}$. We put

$$\omega(v_1 \otimes \ldots \otimes v_n, w_1 \otimes \ldots \otimes w_n) = \sum_{(j_1, \ldots, j_n) \in S_n} \omega(v_1, w_{j_1}) \omega(v_2, w_{j_2}) \ldots \omega(v_n, w_{j_n}), \tag{10.3}$$

where $v_j, w_j \in \mathbb{R}^{2n}$ and we sum over all even permutations of the set $\{1, \dots, n\}$.

As already announced above, we focus on the real 2-dimensional case and for a moment, just on the part of the inner product on $\mathrm{Pol}(\mathbb{R}^2, \mathbb{C})$. We normalize the lift of the symplectic form to be $\omega(e_1, e_2) = 1$ for $v = xe_1 + ye_2 \in \mathbb{R}^2$, and define the Fourier symplectic transformation by

$$x \longleftrightarrow \partial_y, \quad y \longleftrightarrow -\partial_x.$$

Consequently, we get for $r, s, t, u \in \mathbb{N}_0$

$$\langle x^r y^s, x^t y^u \rangle = \omega(x^r y^s, x^t y^u) = (-1)^s \partial_y^r \partial_x^s x^t y^u = (-1)^s u! s! \delta_{r,u} \delta_{s,t}.$$

Thus, we have for $f = x^r y^s, g = x^t y^u$ and $r + s = t + u$

$$\begin{aligned} \omega(f, yg) &= \langle f, yg \rangle = (-1)^s \partial_y^r \partial_x^s x^t y^{u+1} = (-1)^s (u+1)! s! \delta_{r,u+1} \delta_{s,t}, \\ \omega(\partial_x f, g) &= \langle \partial_x f, g \rangle = r \langle x^{r-1} y^s, x^t y^u \rangle = (-1)^s r (r-1)! s! \delta_{r-1,u} \delta_{s,t}, \\ \omega(f, xg) &= \langle f, xg \rangle = (-1)^s (t+1)! s! \delta_{r,u} \delta_{s,t+1}, \\ \omega(\partial_y f, g) &= \langle \partial_y f, g \rangle = (-1)^{s-1} r! (s-1)! \delta_{r,u} \delta_{s-1,t}. \end{aligned}$$

Hence, there are relations

$$\langle \partial_x f, g \rangle = \langle f, yg \rangle, \quad -\langle \partial_y f, g \rangle = \langle f, xg \rangle. \tag{10.4}$$

Let us now summarize our definitions and basic properties in the 2-dimensional case.

Definition 10.1.1. The symplectic Fischer product for $f(x, y) \otimes \psi$, $g(x, y) \otimes \phi$, with $f, g \in \mathrm{Pol}(\mathbb{R}^2, \mathbb{C})$ and $\psi, \phi \in \mathcal{S}(\mathbb{R})$, is given by

$$\langle f \otimes \psi, g \otimes \phi \rangle = [f(\partial_y, -\partial_x) g(x, y)]_{x=y=0} \int_{-\infty}^{\infty} \overline{\psi(q)} \phi(q) \, \mathrm{d}q, \tag{10.5}$$

where the bar denotes the complex conjugation of a complex valued function.

Lemma 10.1.1. The bilinear form defined in (10.5) for all $a, b \in \mathrm{Pol}(\mathbb{R}^2, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R})$ satisfies

1. $\langle qa, b \rangle = \langle a, qb \rangle$, and $\langle iqa, b \rangle = \langle a, -iqb \rangle$,
2. $\langle \partial_q a, b \rangle = -\langle a, \partial_q b \rangle$ and $\langle i\partial_q a, b \rangle = \langle a, i\partial_q b \rangle$,
3. $\langle \partial_x a, b \rangle = \langle a, yb \rangle$,
4. $\langle \partial_y a, b \rangle = -\langle a, xb \rangle$,
5. $\langle xa, b \rangle = \langle a, \partial_y b \rangle$,
6. $\langle ya, b \rangle = -\langle a, \partial_x b \rangle$.

Now we compute the adjoints of operators D_s, X_s with respect to $\langle \, , \rangle$.

Lemma 10.1.2. The adjoint operator for the symplectic Dirac operator D_s with respect to the symplectic Fischer product is X_s, and vice versa. Thus, we have

$$\langle D_s a, b \rangle = \langle a, X_s b \rangle, \quad \langle X_s a, b \rangle = \langle a, D_s b \rangle,$$

for arbitrary $a, b \in \mathrm{Pol}(\mathbb{R}^2, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R})$.

Proof. A direct computation for $a, b \in \mathrm{Pol}(\mathbb{R}^2, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R})$ reveals

$$\langle D_s a, b\rangle = \langle (iq\partial_y - \partial_q\partial_x)a, b\rangle = \langle a, (iqx + \partial_q y)b\rangle,$$
$$\langle X_s a, b\rangle = \langle (iqx + \partial_q y)a, b\rangle = \langle a, (iq\partial_y - \partial_q\partial_x)b\rangle.$$

□

Consequently, we have the orthogonality relations for the symplectic Fischer decomposition,

$$\langle X_s^j m_k^s, X_s^l m_h^s\rangle \sim \delta_{j,l}\delta_{k,h}, \tag{10.6}$$

with symplectic monogenics $m_k^s \in M_k^s, m_h^s \in M_h^s$.

Lemma 10.1.3. The adjoint operators to the basis elements $\tilde{X}, \tilde{Y}$ and $\tilde{H}$ of $\mathfrak{mp}(2, \mathbb{R})$, cf., (8.15), with respect to the symplectic Fischer product are $-\tilde{X}, -\tilde{Y}$ and $-\tilde{H}$, respectively.

Now we pass to the construction of the reproducing kernel $K_k(\xi_1, \xi_2, x, y)$ for the bilinear form

$$(f(x, y), g(x, y)) = [f(\partial_y, -\partial_x)g(x, y)]_{x=y=0}$$

on the space of polynomials of homogeneity k. Inspired by the orthogonal case, we claim

$$K_k(\xi_1, \xi_2, x, y) = \frac{1}{k!}\big(-\xi_1 y + \xi_2 x\big)^k. \tag{10.7}$$

Indeed, we have

$$\big(K_k(\xi_1, \xi_2, x, y), p(x, y)\big) = \left(\frac{1}{k!}\big(-\xi_1 y + \xi_2 x\big)^k, p(x, y)\right)$$
$$= \frac{1}{k!}\big(\xi_1\partial_x + \xi_2\partial_y\big)^k p(x, y) = p(\xi_1, \xi_2) \tag{10.8}$$

for $p(x, y) \in \mathrm{Pol}_k(\mathbb{R}^2, \mathbb{C})$.

In order to adapt $K_k(\xi_1, \xi_2, x, y)$ to the reproducing kernel Z_k of the space of symplectic monogenics M_k^s, we shall regard $K_k(\xi_1, \xi_2, x, y)$ as an element in $\mathrm{Pol}_k(\mathbb{R}^2, \mathbb{C}) \otimes \mathrm{End}(\mathcal{S}(\mathbb{R}))$ with the value in $\mathrm{End}(\mathcal{S}(\mathbb{R}))$ given by the identity endomorphism on $\mathcal{S}(\mathbb{R})$. Moreover, we introduce the projector

$$\mathrm{Proj}_{sm}^k : \mathrm{Pol}_k(\mathbb{R}^2, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R}) \to M_k^s,$$
$$\mathrm{Proj}_{sm}^k = \sum_{j=1}^{k} a_j^k X_s^j D_s^j \;\in \mathrm{Pol}_k(\mathbb{R}^2, \mathbb{C}) \otimes \mathrm{End}(\mathcal{S}(\mathbb{R})), \tag{10.9}$$

to homogeneity k symplectic monogenics, see [9], and define the symplectic Fischer $\mathrm{End}(\mathcal{S}(\mathbb{R}))$-valued product for the elements in $\mathrm{Pol}_k(\mathbb{R}^2, \mathbb{C}) \otimes \mathrm{End}(\mathcal{S}(\mathbb{R}))$ by

$$\langle f(\xi_1, \xi_2, x, y, q, \partial_q), g(x, y, q)\rangle = [f(\xi_1, \xi_2, \partial_y, -\partial_x, q, \partial_q)g(x, y, q)]_{x=y=0}\,. \tag{10.10}$$

We remark that in (10.10) we use the same notation $\langle\,,\rangle$ for the symplectic Fischer $\mathrm{End}(\mathcal{S}(\mathbb{R}))$-valued product as for the $\mathbb{R}$-valued scalar product (10.2), and believe the attentive reader will not have a problem in distinguishing which of them is currently used. Another remark is that we exploit in (10.10) the well-known fact that any symplectic spinor $g(x, y, q) \in \mathrm{Pol}_k(\mathbb{R}^2, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R})$ can be regarded as an element in $\mathrm{Pol}_k(\mathbb{R}^2, \mathbb{C}) \otimes \mathrm{End}(\mathcal{S}(\mathbb{R}))$, because the space of Schwartz functions is a complex algebra.

Theorem 10.1.4. The projection operator Proj_{sm}^k and the reproducing kernel Z_k relate to the symplectic Fischer product as follows

1. Proj_{sm}^k is self-adjoint.
2. $Z_k(\xi_1, \xi_2, x, y, q, \partial_q) = \mathrm{Proj}_{sm}^k K_k(\xi_1, \xi_2, x, y)$ is the reproducing kernel for M_k^s.

Proof. Using this pairing, we first observe the self-adjointness property of Proj_{sm}^k, indeed,

$$\langle \mathrm{Proj}_{sm}^k f, g\rangle = \sum_{j=0}^k a_j^k \langle X_s^j D_s^j f, g\rangle = \sum_{j=0}^k a_j^k \langle f, X_s^j D_s^j g\rangle = \langle f, \mathrm{Proj}_{sm}^k g\rangle.$$

By (10.8), we have for $m_k^s \in M_k^s$

$$\langle Z_k(\xi_1, \xi_2, x, y, q, \partial_q), m_k^s(x, y, q)\rangle = \langle K_k(\xi_1, \xi_2, x, y), \mathrm{Proj}_{sm}^k m_k^s\rangle = m_k^s(\xi_1, \xi_2, q),$$

and for any $j \in \mathbb{N}$, the relation holds

$$\langle Z_k(\xi_1, \xi_2, x, y, q, \partial_q), X_s^j m_{k-j}^s(x, y, q)\rangle = \langle K_k(\xi_1, \xi_2, x, y), \mathrm{Proj}_{sm}^k X_s^j m_{k-j}^s\rangle = 0.$$

The proof is complete. □

Lemma 10.1.5. The reproducing kernel Z_k has the explicit form

$$Z_k(\xi_1, \xi_2, x, y, q, \partial_q) = \sum_{j=0}^k \mathrm{i}^j a_j^k \frac{1}{(k-j)!}\big(- \xi_1 y + \xi_2 x\big)^{k-j} X_s^j \xi_s^j, \tag{10.11}$$

where $\xi_s = -q\xi_1 + \mathrm{i}\partial_q \xi_2$.

Proof. First we need an explicit formula for $D_s^j K_k(\xi_1, \xi_2, x, y)$, $j = 1, \dots, k$. By the chain rule, we obtain

$$\begin{aligned} D_s K_k(\xi_1, \xi_2, x, y) &= (\mathrm{i}q\partial_y - \partial_q\partial_x)\frac{1}{k!}\big(- \xi_1 y + \xi_2 x\big)^k \\ &= K_{k-1}(\xi_1, \xi_2, x, y)(\mathrm{i}q\xi_1 - \partial_q\xi_2) = \mathrm{i}K_{k-1}(\xi_1, \xi_2, x, y)\xi_s. \end{aligned}$$

Therefore, $D_s^j K_k(\xi_1, \xi_2, x, y) = \mathrm{i}^j K_{k-j}(\xi_1, \xi_2, x, y)\xi_s^j$, and so

$$\begin{aligned} Z_k(\xi_1, \xi_2, x, y, q, \partial_q) &= \sum_{j=0}^k a_j^k X_s^j D_s^j K_k(\xi_1, \xi_2, x, y) \\ = \sum_{j=0}^k \mathrm{i}^j a_j^k X_s^j K_{k-j}(\xi_1, \xi_2, x, y)\xi_s^j &= \sum_{j=0}^k \mathrm{i}^j a_j^k X_s^j \frac{\big(- \xi_1 y + \xi_2 x\big)^{k-j}}{(k-j)!}\xi_s^j, \end{aligned}$$

which proves the assertion. □

11. Explicit bases of symplectic monogenics on $(\mathbb{R}^2, \omega)$

In the present section we construct some explicit bases for symplectic monogenics M^s_h in $\mathrm{Pol}(\mathbb{R}^2,\mathbb{C})\otimes\mathcal{S}(\mathbb{R})$ of homogeneity h, and prove several useful characterizing properties.

The first distinguished basis is written in the real coordinates x and y on $\mathbb{R}^2$ and the (topological) basis $q^j e^{-\frac{q^2}{2}}$, $j \in \mathbb{N}_0$, of the Schwartz space. The second distinguished basis for symplectic monogenics is written in the complex coordinates z and $\bar{z}$ on $\mathbb{R}^2 \simeq \mathbb{C}$ and the (topological) basis of Hermite functions $\psi_j(q)$, $j \in \mathbb{N}_0$, for $\mathcal{S}(\mathbb{R})$.

Lemma 11.0.6. The symplectic spinors of homogeneity h in the variables x, y and odd in the variable q for $k \in \mathbb{N}_0$, $k \geq h$,

$$\tilde{s}^h_{o,k} = e^{-\frac{q^2}{2}} \sum_{p=0}^{h} (-1)^p \frac{(2k+1)!!}{(2k-2p+1)!!} \binom{h}{p} q^{2k+1-2p} (x+iy)^{h-p} (iy)^p, \tag{11.1}$$

and even in variable q for $k \in \mathbb{N}_0$,

$$\tilde{s}^h_{e,k} = e^{-\frac{q^2}{2}} \sum_{p=0}^{h} (-1)^p \frac{(2k)!!}{(2k-2p)!!} \binom{h}{p} q^{2k-2p} (x+iy)^{h-p} (iy)^p, \tag{11.2}$$

form a topological basis of the odd and even part of the symplectic monogenics M^s_h, respectively.

Proof. Let us consider a polynomial symplectic spinor

$$f(x,y,q) = e^{-\frac{q^2}{2}} \sum_{j=0}^{\infty} q^j p_j(x,y),$$

where $p_j(x,y)$ are polynomials in the variables x, y. Solving the equation $0 = D_s f(x,y,q)$, we have

$$\begin{aligned} 0 &= (iq\partial_y - \partial_x\partial_q) f(x,y,q) \\ &= e^{-\frac{q^2}{2}} \sum_{j=0}^{\infty} \left(iq^{j+1}\partial_y p_j(x,y) + q^{j+1}\partial_x p_j(x,y) - jq^{j-1}\partial_x p_j(x,y) \right). \end{aligned}$$

Because the functions $e^{-\frac{q^2}{2}} q^j$, $j \in \mathbb{N}_0$ are linearly independent,

$$q^j \left((\partial_x + i\partial_y) p_{j-1}(x,y) - (j+1)\partial_x p_{j+1}(x,y) \right) = 0 \tag{11.3}$$

for each $j \in \mathbb{N}_0$. We get a system of recurrence equations, splitting into two subsystems of odd and even in the variable q and the solution follows. The functions $p_j(x,y)$ are polynomials, hence solutions.

For a fixed homogeneity h in the variables x and y, the systems $\tilde{s}^h_{o,k}$ and $\tilde{s}^h_{e,k}$ contain all powers q^j, $j \in \mathbb{N}_0$, for appropriate k and all possible combinations of x, y in $\mathrm{Pol}(\mathbb{R}^2,\mathbb{C})$ so that it is in $\mathrm{Ker}(D_s)$. Therefore, odd (11.1) and even (11.2) systems form a basis of M^s_h because $\{q^j e^{-\frac{q^2}{2}}\}_{j\in\mathbb{N}_0}$ is a topological basis of $\mathcal{S}(\mathbb{R})$. ∎

Retaining the notation of the previous Lemma, it is straightforward to prove.

Remark 10. The symplectic monogenics $\tilde{s}^h_{o,k}$ and $\tilde{s}^h_{e,k}$ are eigenfunctions for operators $\tilde{X}' = \tilde{X}$, $\tilde{H}' = \tilde{H} + 2i\tilde{X}$ and $\tilde{Y}' = \tilde{Y} - i\tilde{H} + \tilde{X}$, where $\tilde{X}$, $\tilde{Y}$ and $\tilde{H}$ are defined in (8.15).

In the complex coordinates $z = x + iy$, $\bar{z} = x - iy$, with $\partial_x = \partial_z + \partial_{\bar{z}}$ and $\partial_y = i(\partial_z - \partial_{\bar{z}})$, the symplectic Dirac operator in the complex coordinates $z, \bar{z}$ is given by

$$D_s = -(q + \partial_q)\partial_z + (q - \partial_q)\partial_{\bar{z}}. \tag{11.4}$$

See (6.32). Let us recall that the Hermite functions $\{\psi_k(q)\}_{k\in\mathbb{N}_0}$ are a topological basis of the Schwartz space $\mathcal{S}(\mathbb{R})$. The k-th Hermite function is

$$\psi_k(q) = \frac{1}{\sqrt{2^k k! \sqrt{\pi}}} e^{-\frac{q^2}{2}} H_k(q) = \frac{(-1)^k}{\sqrt{2^k k! \sqrt{\pi}}} \big(q - \partial_q\big)^k e^{-\frac{q^2}{2}},$$

where H_k is the k-th Hermite polynomial. The operators $(q + \partial_q)$ and $(q - \partial_q)$ act on the basis vectors by

$$\begin{aligned} (q + \partial_q)\psi_k &= \sqrt{2}\sqrt{k}\psi_{k-1}, \\ (q - \partial_q)\psi_k &= \sqrt{2}\sqrt{k+1}\psi_{k+1}. \end{aligned} \tag{11.5}$$

Together with the Euler operator acting by a multiple of identity on each ψ_k, they form a representation of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. We shall use the following easily verified formulas

$$\begin{aligned} (q^2 - \partial_q^2)\psi_k &= (2k + 1)\psi_k, \\ (q - \partial_q)^2\psi_k &= 2\sqrt{(k + 1)(k + 2)}\psi_{k+2}, \\ (q + \partial_q)^2\psi_k &= 2\sqrt{k(k - 1)}\psi_{k-2}. \end{aligned} \tag{11.6}$$

Lemma 11.0.7. The polynomial symplectic spinors of homogeneity h in the variables $z, \bar{z}$ and odd in the variable q for $k \in \mathbb{N}_0$,

$$s^h_{o,k} = \sum_{p=0}^{h} \sqrt{\frac{(2k + 2p)!!}{(2k + 2p + 1)!!}} \binom{h}{p} \psi_{2k+2p+1}(q)\bar{z}^{h-p} z^p, \tag{11.7}$$

form the basis of the odd part of the solution space of the symplectic Dirac operator D_s.

The polynomial symplectic spinors of homogeneity h in the variables $z, \bar{z}$ and even in the variable q for $k \in \mathbb{N}_0$,

$$s^h_{e,k} = \sum_{p=0}^{h} \sqrt{\frac{(2k + 2p - 1)!!}{(2k + 2p)!!}} \binom{h}{p} \psi_{2k+2p}(q)\bar{z}^{h-p} z^p, \tag{11.8}$$

and, for $k = -1, -2, \ldots, -h$,

$$s^h_{e,k} = \sum_{p=|k|}^{h} \sqrt{\frac{(2k + 2p - 1)!!}{(2k + 2p)!!}} \binom{h}{p} \psi_{2k+2p}(q)\bar{z}^{h-p} z^p, \tag{11.9}$$

form a basis of the even part of the solution space of the symplectic Dirac operator D_s.

Proof. Let us consider a symplectic spinor

$$f(z, \bar{z}, q) = \sum_{l=0}^{\infty} \psi_l(q) p_l(z, \bar{z}),$$

where $\psi_l(q)$ is the l-th Hermite function and $p_l(z, \bar{z})$ is a polynomial in the variables $z, \bar{z}$. The action of the symplectic Dirac operator is then

$$\begin{aligned} D_s f(z, \bar{z}, q) &= \big((q + \partial_q)\partial_z - (q - \partial_q)\partial_{\bar{z}}\big) f(z, \bar{z}, q) \\ &= \sqrt{2} \sum_{l=0}^{\infty} \sqrt{l}\psi_{l-1}(q)\partial_z p_l(z, \bar{z}) - \sqrt{l+1}\psi_{l+1}(q)\partial_{\bar{z}} p_l(z, \bar{z}). \end{aligned}$$

The Hermite functions are linearly independent, which implies

$$\psi_l(q)(\sqrt{l+1}\partial_z p_{l+1}(z, \bar{z}) - \sqrt{l}\partial_{\bar{z}} p_{l-1}(z, \bar{z})) = 0 \tag{11.10}$$

for each $l \in \mathbb{N}_0$. The system of recurrence equations is split into two subsystems with odd and even indexes in the variable q, each of which is easy so resolve.

For a fixed homogeneity h, the systems of symplectic polynomial spinors (11.7), (11.8) and (11.9) form a basis of symplectic monogenics M_k^s of homogeneity h. Because the Hermite functions form a topological basis of $\mathcal{S}(\mathbb{R})$ the above collection of symplectic monogenics is a topological basis of $\mathrm{Ker}(D_s)$. □

Now let us explore the properties of the symplectic Fischer product (10.5) applied to the basis elements discussed in the Proposition 11.0.6 and Proposition 11.0.7. The motivation for this question is the existence of a basis of symplectic monogenics, which is isotropic with respect to the product (10.5).

Lemma 11.0.8. The basis elements (11.1) and (11.2) of homogeneity 2 in the symplectic Fischer product (10.5) satisfy, for $k, l \in \mathbb{N}$, $k, l \geq 2$,

$$\begin{aligned} \langle \tilde{s}^2_{o,k}, \tilde{s}^2_{o,l} \rangle &= \frac{-3\sqrt{\pi}(2k + 2l - 5)}{2^{k+l-3}}, \\ \langle \tilde{s}^2_{e,k}, \tilde{s}^2_{e,l} \rangle &= \frac{-3\sqrt{\pi}(2k + 2l - 5)}{2^{k+l-2}}, \\ \langle \tilde{s}^2_{o,k}, \tilde{s}^2_{e,l} \rangle &= 0. \end{aligned}$$

Proof. Focusing just on the $\mathrm{Pol}(\mathbb{R}^2, \mathbb{C})$ part of the symplectic Fischer product (10.5), the only non zero combinations of x, y in homogeneity 2 are $\langle x^2, y^2 \rangle = 2$ and $\langle xy, xy \rangle = -1$. Then $\int_{-\infty}^{\infty} e^{-q^2} q^{2t}\, \mathrm{d}q = \frac{\sqrt{\pi}(2t+1)}{2^t}$ for $t \in \mathbb{N}_0$ and moreover, $\int_{-\infty}^{\infty} e^{-q^2} q^t\, \mathrm{d}q = 0$ for t odd. □

Therefore, we see that the symplectic Fischer product (10.5) of any two odd or even basis elements (11.1), (11.2) for $k, l \geq 2$ is non-zero (in fact, negative) for $k = l$. This implies that the symplectic Fischer product (10.5) does not seem to be a convenient candidate for the scalar product on $\mathrm{Pol}(\mathbb{R}^2, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R})$.

Let us rewrite the symplectic Fischer product (10.5) in the complex variables. In the variables $z, \bar{z}$, we have a non-trivial pairing for the pairs $z \longleftrightarrow -2\mathrm{i}\partial_{\bar{z}}$

and $\bar{z} \longleftrightarrow 2\mathrm{i}\partial_z$. Hence for $f(z,\bar{z}) \otimes \psi$, $g(z,\bar{z}) \otimes \phi$, with $f, g \in \mathrm{Pol}(\mathbb{R}^2, \mathbb{C})$ and $\psi, \phi \in \mathcal{S}(\mathbb{R})$,

$$\langle f \otimes \psi, g \otimes \phi \rangle = [f(-2\mathrm{i}\partial_{\bar{z}}, 2\mathrm{i}\partial_z) g(z,\bar{z})]_{z=\bar{z}=0} \int_{-\infty}^{\infty} \overline{\psi(q)} \phi(q) \, \mathrm{d}q. \tag{11.11}$$

Let us look at the symplectic Fischer product (11.11) for the low homogeneity basis elements $s^h_{o,k}$, (11.7), of odd part of the symplectic monogenics.

Example 8. In the homogeneity 2 and $k, l \in \mathbb{N}$, the relation holds

$$\langle s^2_{o,k}, s^2_{o,l} \rangle = -8\frac{(2k)!!}{(2k+1)!!}\delta_{2k+1,2l+5} + 16\frac{(2k+2)!!}{(2k+3)!!}\delta_{2k+2,2l+2} - 8\frac{(2k+4)!!}{(2k+5)!!}\delta_{2k+5,2l+1},$$

where just one of the (Kronecker) deltas on the previous display may be non-zero. We observe that for $k = l$ is $\langle s^2_{o,k}, s^2_{o,k} \rangle \neq 0$, because $\delta_{2k+2,2l+2} \neq 0$.

In the homogeneity 3 and $k, l \in \mathbb{N}$,

$$\begin{aligned} \langle s^3_{o,k}, s^3_{o,l} \rangle = & -48i\frac{(2k)!!}{(2k+1)!!}\delta_{2k+1,2l+7} - 16i\frac{(2k+2)!!}{(2k+3)!!}\delta_{2k+3,2l+5} \\ & + 16i\frac{(2k+4)!!}{(2k+5)!!}\delta_{2k+5,2l+3} + 48i\frac{(2k+6)!!}{(2k+7)!!}\delta_{2k+7,2l+1}, \end{aligned}$$

where again just one delta may be non-zero. For $k = l$ the symplectic Fischer product gives zero, $\langle s^3_{o,k}, s^3_{o,k} \rangle = 0$, and analogous conclusion $\langle s^h_{o,k}, s^h_{o,k} \rangle = 0$ can be made for all odd homogeneities h.

Let us now consider another skew-symmetric bilinear form on $\mathrm{Pol}(\mathbb{R}^2, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R})$, which is skew-symmetric on $\mathcal{S}(\mathbb{R})$ and possesses several remarkable properties. We use again the complex variables on $\mathbb{R}^2$.

Definition 11.0.2. Let us introduce a bilinear form $\langle \, , \rangle_1$ on symplectic spinors, defined on $f(z,\bar{z}) \otimes \psi$, $g(z,\bar{z}) \otimes \phi$ with $f, g \in \mathrm{Pol}(\mathbb{R}^2, \mathbb{C})$ and $\psi, \phi \in \mathcal{S}(\mathbb{R})$ by

$$\langle f \otimes \psi, g \otimes \phi \rangle_1 = \sqrt{2} \left[\frac{1}{h!} f(\partial_z, \partial_{\bar{z}}) g(z,\bar{z}) \right]_{z=\bar{z}=0} \int_{-\infty}^{\infty} \big(\partial_q \psi(q)\big) \phi(q) \, \mathrm{d}q, \tag{11.12}$$

where h denotes the homogeneity of the polynomial $f(z,\bar{z})$.

In the monomial basis, we have for $r, s, t, u \in \mathbb{N}_0$

$$\langle z^r \bar{z}^s \otimes \psi, z^t \bar{z}^u \otimes \phi \rangle_1 = \sqrt{2} \frac{r!s!}{(r+s)!} \delta_{r,t} \delta_{s,u} \int_{-\infty}^{\infty} \big(\partial_q \psi(q)\big) \phi(q) \, \mathrm{d}q, \tag{11.13}$$

where $\delta_{r,t}$ denotes the Kronecker delta. Moreover, for $a, b \in \mathrm{Pol}(\mathbb{R}^2, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R})$, the relations hold

$$\begin{aligned} \langle za, b \rangle_1 &= \langle a, \partial_z b \rangle_1, \qquad \langle \partial_z a, b \rangle_1 = \langle a, zb \rangle_1, \\ \langle \bar{z}a, b \rangle_1 &= \langle a, \partial_{\bar{z}} b \rangle_1, \qquad \langle \partial_{\bar{z}} a, b \rangle_1 = \langle a, \bar{z}b \rangle_1. \end{aligned} \tag{11.14}$$

Notice that the bilinear form $\langle\,,\rangle_1$ is not $\mathfrak{mp}(2,\mathbb{R})$-invariant on the whole space of symplectic spinors, because

$$\begin{aligned}
&\langle H_t(f\otimes\psi), g\otimes\phi\rangle_1 - \langle f\otimes\psi, H_t(g\otimes\phi)\rangle_1 \\
&\quad = [f,g]\int_{-\infty}^{\infty} q\psi(q)\phi(q)\,\mathrm{d}q, \\
&\langle X_t(f\otimes\psi), g\otimes\phi\rangle_1 - \langle f\otimes\psi, X_t(g\otimes\phi)\rangle_1 \\
&\quad = \frac{\mathrm{i}}{2}[f,g]\int_{-\infty}^{\infty}\Big(q\psi(q)\phi(q) + 2q\big(\partial_q\psi(q)\big)\big(\partial_q\phi(q)\big)\Big)\,\mathrm{d}q, \\
&\langle Y_t(f\otimes\psi), g\otimes\phi\rangle_1 - \langle f\otimes\psi, Y_t(g\otimes\phi)\rangle_1 \\
&\quad = \frac{\mathrm{i}}{2}[f,g]\int_{-\infty}^{\infty}\Big(q\psi(q)\phi(q) - 2q\big(\partial_q\psi(q)\big)\big(\partial_q\phi(q)\big)\Big)\,\mathrm{d}q,
\end{aligned}$$

with

$$[f,g] = \sqrt{2}\left[\frac{1}{h!}f(\partial_z,\partial_{\bar{z}})g(z,\bar{z})\right]_{z=\bar{z}=0}.$$

However, $\langle\,,\rangle_1$ is $\mathfrak{mp}(2,\mathbb{R})$-invariant when restricted to any of the two irreducible subspaces of symplectic spinors (given by the subspaces of even and odd Schwartz functions, respectively).

Now, let us define the elements

$$s^h_{E,l} = \frac{1}{2^h}\sum_{j=0}^{l} s^h_{e,j}, \tag{11.15}$$

which form a basis of even symplectic monogenics in the homogeneity h (as well as the set $s^h_{e,l}$, $l\in\mathbb{N}_0$, cf., (11.8).)

Lemma 11.0.9. The basis elements $s^h_{o,k}$, $s^h_{E,k}$, $k\in\mathbb{N}_0$, of polynomial symplectic monogenics of homogeneity h in the variables $z,\bar{z}$ form two isotropic subspaces of symplectic monogenics M^s_h with respect to the form defined in (11.12). Namely, the basis elements satisfy

$$\begin{aligned}
\langle s^h_{o,k}, s^h_{o,l}\rangle_1 &= 0, \qquad \langle s^h_{o,k}, s^h_{E,l}\rangle_1 = \delta_{k,l}, \\
\langle s^h_{E,k}, s^h_{E,l}\rangle_1 &= 0, \qquad \langle s^h_{E,l}, s^h_{o,k}\rangle_1 = -\delta_{k,l},
\end{aligned} \tag{11.16}$$

for $k,l\in\mathbb{N}_0$ and $h\in\mathbb{N}_0$. Moreover, the form is identically zero for symplectic monogenics of different homogeneities h,h'.

Proof. Let us remind the orthonormality relation $\int_{-\infty}^{\infty}\psi_k(q)\psi_l(q)\,\mathrm{d}q = \delta_{k,l}$ for Hermite functions. Then the relations in the first column (11.16) are obvious, because the derivative of a Hermite function $\psi_k(q)$ is

$$\partial_q\psi_k(q) = \sqrt{\frac{k}{2}}\psi_{k-1}(q) - \sqrt{\frac{k+1}{2}}\psi_{k+1}(q)$$

and consequently, the integral in the bilinear form is zero.

As for the proof of the relation $\langle s^h_{o,k}, s^h_{E,l}\rangle_1 = \delta_{k,l}$, we first prove

$$\langle s^h_{o,k}, s^h_{e,l}\rangle_1 = 2^h(\delta_{k,l} - \delta_{k+1,l})$$

for $k, l \in \mathbb{N}_0$. We use (11.13) to simplify the calculation, we get

$$\langle s^h_{o,k}, s^h_{e,l}\rangle_1 = \sum_{p=0}^{h} \sqrt{\frac{(2k+2p)!!(2l+2p-1)!!}{(2k+2p+1)!!(2l+2p)!!}} \binom{h}{p}^2 \frac{(h-p)!p!}{h!} \times$$
$$\times \left(\delta_{2k+2p,2l+2p}\sqrt{2k+2p+1} - \delta_{2k+2p+2,2l+2p}\sqrt{2k+2p+2}\right).$$

This is equal to $\sum_{p=0}^{h}\binom{h}{p} = 2^h$ for $k = l$, -2^h for $k+1 = l$ and zero otherwise. Then for the basis elements $s^h_{E,l}$ we have $\langle s^h_{o,k}, s^h_{E,l}\rangle_1 = \sum_{j=0}^{l}\delta_{k,j} - \sum_{j=0}^{l}\delta_{k+1,j}$, which is non-zero just for $k = l$. The last relation in (11.16) follows from the skew-symmetry of the product.

For different homogeneities, the statement easily follows from (11.13), and the proof is complete. □

We remark that for $k < 0$, the elements $s^h_{e,k}$ in (11.9) satisfy

$$\langle s^h_{e,l}, s^h_{e,k}\rangle_1 = 0, \quad \langle s^h_{o,j}, s^h_{e,k}\rangle_1 = 0, \quad \langle s^h_{E,j}, s^h_{e,k}\rangle_1 = 0,$$

for each $k, l \in \mathbb{Z}$, $-h \le k < 0$, $-h \le l$ and $j \in \mathbb{N}_0$.

11.1 Action of symmetries of D_s on basis of symplectic monogenics

In the present part we determine the action of the symmetry operators introduced in Section 8.2 on the basis of symplectic monogenics described in Lemma 11.0.7. We remark that the action of the symmetry operators on the basis of symplectic monogenics described in Lemma 11.0.6 is much more involved.

We shall start with the even component of the basis.

Lemma 11.1.1. The operators ∂_z and $\partial_{\bar z}$ decrease the homogeneity in $z, \bar z$ and preserve the elements of even basis (11.8), (11.9) of the kernel of the symplectic Dirac operator D_s. In particular, for $k \in \mathbb{Z}$, $k \le -h$,

$$\begin{aligned} \partial_z s^h_{e,k} &= h s^{h-1}_{e,k+1}, \\ \partial_{\bar z} s^h_{e,k} &= h s^{h-1}_{e,k}, \quad \text{for } k \neq -h, \quad \partial_{\bar z} s^h_{e,-h} = 0. \end{aligned} \tag{11.17}$$

Proof. We verify the relation $\partial_z s^h_{e,k} = h s^{h-1}_{e,k+1}$ for $k \in \mathbb{N}_0$, the others being analogous. We have

$$\partial_z \sum_{p=0}^{h} \sqrt{\frac{(2k+2p-1)!!}{(2k+2p)!!}} \binom{h}{p} \psi_{2k+2p}(q)\bar z^{h-p} z^p =$$
$$= \sum_{p=1}^{h} \sqrt{\frac{(2k+2p-1)!!}{(2k+2p)!!}} \frac{h!}{(h-p)!(p-1)!} \psi_{2k+2p}(q)\bar z^{h-p} z^{p-1}.$$

Shifting the summation index by one, we get

$$\sum_{p=0}^{h-1} \sqrt{\frac{(2k+2p+1)!!}{(2k+2p+2)!!}} \frac{h!}{(h-p-1)!p!} \psi_{2k+2p+2}(q)\bar z^{h-1-p} z^p = h s^{h-1}_{e,k+1}$$

as required. □

Lemma 11.1.2. The operators H_t, X_t and Y_t, see (8.34), preserve the span of even elements of the basis (11.8), (11.9) of the kernel of the symplectic Dirac operator, and the action on basis elements is, for $k \in \mathbb{Z}$, $k \leq -h$, given by

$$\begin{aligned} H_t s^h_{e,k} &= (h + 2k + \frac{1}{2}) s^h_{e,k}, \\ X_t s^h_{e,k} &= i(h + k + 1) s^h_{e,k+1}, \\ Y_t s^h_{e,k} &= i(k - \frac{1}{2}) s^h_{e,k-1}, \quad \text{for } k \neq -h, \quad Y_t s^h_{e,-h} = 0. \end{aligned} \tag{11.18}$$

Proof. This is again a straightforward computation. For example, let us prove the relation $X_t s^h_{e,k} = i(h + k + 1) s^h_{e,k+1}$, $k \in \mathbb{N}_0$. We have

$$\begin{aligned} &\Big(i\bar{z}\partial_z + \frac{i}{4}(q - \partial_q)^2 \Big) \sum_{p=0}^{h} \sqrt{\frac{(2k + 2p - 1)!!}{(2k + 2p)!!}} \binom{h}{p} \psi_{2k+2p}(q) \bar{z}^{h-p} z^p = \\ &= i \sum_{p=1}^{h} \sqrt{\frac{(2k + 2p - 1)!!}{(2k + 2p)!!}} \frac{h!}{(h - p - 1)!(p - 1)!} \psi_{2k+2p}(q) \bar{z}^{h-p+1} z^{p-1} \\ &+ \frac{i}{4} \sum_{p=0}^{h} \sqrt{\frac{(2k + 2p - 1)!!}{(2k + 2p)!!}} \binom{h}{p} 2\sqrt{(2k + 2p + 1)(2k + 2p + 2)} \psi_{2k+2p+2}(q) \bar{z}^{h-p} z^p. \end{aligned}$$

Shifting by one in the summation index in the first sum, we get

$$\begin{aligned} &= i \sum_{p=0}^{h} \sqrt{\frac{(2k + 2p + 1)!!}{(2k + 2p + 2)!!}} \binom{h}{p} \psi_{2k+2p+2}(q) \bar{z}^{h-p} z^p \big(h - p + \frac{1}{2}(2k + 2p + 2) \big) \\ &= i(h + k + 1) s^h_{e,k+1}. \end{aligned}$$

□

Lemma 11.1.3. The operators Z_1 and Z_2, see (8.32), increase the homogeneity by one in the variables $z, \bar{z}$ and preserve even elements of the basis (11.8), (11.9) of the kernel of the symplectic Dirac operator. The operators satisfy for $k \in \mathbb{Z}$, $k \leq -h$,

$$\begin{aligned} Z_1 s^h_{e,k} &= 2(h + 1)(h + k + 1) s^{h+1}_{e,k}, \\ Z_2 s^h_{e,k} &= (h + 1)(2k - 1) s^{h+1}_{e,k-1}. \end{aligned} \tag{11.19}$$

Proof. It follows from (8.32) that

$$\begin{aligned} Z_1 = &- \frac{1}{2} \Big(\big(q - \partial_q\big)^2 z^2 + 2\big(q^2 + \partial_q^2\big) z\bar{z} + \big(q + \partial_q\big)^2 \bar{z}^2 \Big) \partial_z \\ &+ \bar{z}(E + 1)(2E + 1) + \frac{1}{2} \Big(\big(q - \partial_q\big)^2 z + \big(q + \partial_q\big)\big(q - \partial_q\big) \bar{z} \Big)(2E + 1), \\ Z_2 = &- \frac{1}{2} \Big(\big(q - \partial_q\big)^2 z^2 + 2\big(q^2 + \partial_q^2\big) z\bar{z} + \big(q + \partial_q\big)^2 \bar{z}^2 \Big) \partial_{\bar{z}} \\ &- z(E + 1)(2E + 1) + \frac{1}{2} \Big(\big(q - \partial_q\big)\big(q + \partial_q\big) z + \big(q + \partial_q\big)^2 \bar{z} \Big)(2E + 1). \end{aligned}$$

Now using (11.5), (11.6) and $(2E+1)s^h_{e,k} = (2h+1)s^h_{e,k}$, $(E+1)s^h_{e,k} = (h+1)s^h_{e,k}$, we verify the relation $Z_2 s^h_{e,k} = (h+1)(2k-1)s^{h+1}_{e,k-1}$ for $k \in \mathbb{N}_0$. The expression

$$Z_2\left(\sum_{p=0}^{h}\sqrt{\frac{(2k+2p-1)!!}{(2k+2p)!!}}\binom{h}{p}\psi_{2k+2p}(q)\bar{z}^{h-p}z^p\right)$$

can be expanded in the following tree sums.

$$\begin{aligned}
=&-\frac{1}{2}\sum_{p=0}^{h-1}\sqrt{\frac{(2k+2p-1)!!}{(2k+2p)!!}}\binom{h}{p}(h-p)\times\\
&\qquad\times 2\Big(\sqrt{(2k+2p+1)(2k+2p+2)}\psi_{2k+2p+2}(q)\bar{z}^{h-p-1}z^{p+2}\\
&\qquad\quad+(4k+4p+1)\psi_{2k+2p}(q)\bar{z}^{h-p}z^{p+1}\\
&\qquad\quad+\sqrt{(2k+2p)(2k+2p-1)}\psi_{2k+2p-2}(q)\bar{z}^{h-p+1}z^p\Big)\\
&-\sum_{p=0}^{h}\sqrt{\frac{(2k+2p-1)!!}{(2k+2p)!!}}\binom{h}{p}(h+1)(2h+1)\psi_{2k+2p}(q)\bar{z}^{h-p}z^{p+1}\\
&+\frac{1}{2}\sum_{p=0}^{h}\sqrt{\frac{(2k+2p-1)!!}{(2k+2p)!!}}\binom{h}{p}(2h+1)2\Big((2k+2p)\psi_{2k+2p}(q)\bar{z}^{h-p}z^{p+1}\\
&\qquad\quad+\sqrt{(2k+2p)(2k+2p-1)}\psi_{2k+2p-2}(q)\bar{z}^{h-p+1}z^p\Big)
\end{aligned}$$

We reorganize the sum of summations to get the contributions to a given Hermite function as follows

$$\begin{aligned}
&\sum_{p=0}^{h-1}\sqrt{\frac{(2k+2p+1)!!}{(2k+2p+2)!!}}\psi_{2k+2p+2}(q)\bar{z}^{h-p-1}z^{p+2}\binom{h}{p}(p-h)(2k+2p+2)\\
&+\sum_{p=0}^{h}\sqrt{\frac{(2k+2p-1)!!}{(2k+2p)!!}}\psi_{2k+2p}(q)\bar{z}^{h-p}z^{p+1}\binom{h}{p}\times\\
&\qquad\times\big((p-h)(4k+4p+1)+(2h+1)(2k+2p-h-1)\big)\\
&+\sum_{p=0}^{h}\sqrt{\frac{(2k+2p-3)!!}{(2k+2p-2)!!}}\psi_{2k+2p-2}(q)\bar{z}^{h-p+1}z^p\binom{h}{p}(2k+2p-1)(h+p+1).
\end{aligned}$$

We do the appropriate shifts in summations and multiple expressions to produce the required combinatorial numbers

$$\begin{aligned}
&\sum_{p=0}^{h+1}\sqrt{\frac{(2k+2p-3)!!}{(2k+2p-2)!!}}\psi_{2k+2p-2}(q)\bar{z}^{h-p+1}z^p\frac{(h+1)!}{(h-p+1)!p!}\times\\
&\times\frac{1}{h-1}\big(-(p-1)p(2k+2p-2)+p(p-1-h)(4k+4p-3)\\
&\qquad+p(2h+1)(2k+2p-h-3)+(h-p+1)(2k+2p-1)(h+p+1)\big)\\
=&s^{h+1}_{e,k-1}(h+1)(2k-1).
\end{aligned}$$

The remaining equalities are analogous and the proof is complete. □

The proof of the analogous statement for the odd part of the basis of the kernel of the symplectic Dirac operator is analogous computation to the even part in the previous Lemma and so it is omitted. This result is summarized in the next Lemma.

Lemma 11.1.4.

1. The operators ∂_z and $\partial_{\bar{z}}$ decrease the homogeneity in the variables $z, \bar{z}$ by one and preserve odd elements of the basis (11.7), $k \in \mathbb{N}_0$, of the kernel of the symplectic Dirac operator D_s

$$\partial_z s^h_{o,k} = h s^{h-1}_{o,k+1},$$
$$\partial_{\bar{z}} s^h_{o,k} = h s^{h-1}_{o,k}. \tag{11.20}$$

2. The operators H_t, X_t and Y_t preserve odd elements of the basis (11.7), $k \in \mathbb{N}_0$, of the kernel of the symplectic Dirac operator

$$H_t s^h_{o,k} = (h + 2k + \frac{3}{2}) s^h_{o,k},$$
$$X_t s^h_{o,k} = i(h + k + \frac{3}{2}) s^h_{o,k+1},$$
$$Y_t s^h_{o,k} = ik s^h_{o,k-1}, \quad \text{for } k \neq 0, \quad Y_t s^h_{o,0} = 0. \tag{11.21}$$

3. The operators Z_1 and Z_2 increase the homogeneity by one in the variables $z, \bar{z}$, and map odd elements of the basis (11.7), $k \in \mathbb{N}_0$, to the elements of odd basis of the homogeneity plus one higher of the kernel of the symplectic Dirac operator,

$$Z_1 s^h_{o,k} = (1 + h)(2h + 2k + 3) s^{h+1}_{o,k},$$
$$Z_2 s^h_{o,k} = 2(h + 1) k s^{h+1}_{o,k-1}. \tag{11.22}$$

12. Extension map for symplectic spinors on $(\mathbb{R}^{2n}, \omega)$

Let us introduce a notation of this chapter for three symplecticaly invariant operators, endomorphisms of polynomial symplectic spinors, already defined in (7.6) and also in (8.4)

$$
\begin{aligned}
X_{s,n} &= \sum_{j=1}^{n} (y_j \partial_{q_j} + i x_j q_j), \\
D_{s,n} &= \sum_{j=1}^{n} (i q_j \partial_{y_j} - \partial_{x_j} \partial_{q_j}), \\
E_n &= \sum_{j=1}^{n} (x_j \partial_{x_j} + y_j \partial_{y_j}).
\end{aligned} \tag{12.1}
$$

They generate the representation of the $\mathfrak{sl}(2,\mathbb{C})$ on the space $\mathrm{Pol}(\mathbb{R}^{2n},\mathbb{C}) \otimes \mathcal{S}(\mathbb{R}^n)$. The non-canonical $\mathfrak{sl}(2,\mathbb{C})$ commutation relations of these operators are

$$
\begin{aligned}
[E_n + n, D_{s,n}] &= -D_{s,n}, \\
[E_n + n, X_{s,n}] &= X_{s,n}, \\
[X_{s,n}, D_{s,n}] &= i(E_n + n).
\end{aligned} \tag{12.2}
$$

A mathematical induction and previous commutators yield to following relations for the power of operators acting on $\varphi \in \mathrm{Pol}(\mathbb{R}^{2n},\mathbb{C}) \otimes \mathcal{S}(\mathbb{R}^n)$ and $\psi \in \mathrm{Ker}(D_{s,n}) \subset \mathrm{Pol}(\mathbb{R}^{2n},\mathbb{C}) \otimes \mathcal{S}(\mathbb{R}^n)$

$$
(E_n + n)(X_{s,n})^k \varphi = (X_{s,n})^k (E_n + n + k)\varphi, \tag{12.3}
$$

$$
D_{s,n}(X_{s,n})^k \varphi = (X_{s,n})^k D_{s,n}\phi - i(X_{s,n})^{k-1}\Big(k(E_n + n) + \frac{(k-1)k}{2}\Big)\varphi,
$$

$$
(D_{s,n})^l (X_{s,n})^k \psi = (-i)^l (X_{s,n})^{k-l} \prod_{j=0}^{l-1} \Big((k-j)(E_n + n) + \frac{(k-1-j)(k-j)}{2}\Big)\psi,
$$

where $k \in \mathbb{N}_0$.

12.1 Representation on tensor product

The Schwartz space of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^n)$ is a Fréchet space. Moreover, it is a nuclear space. Therefore there is a canonical isomorphism

$$
\mathcal{S}(\mathbb{R}^n) \hat{\otimes} \mathcal{S}(\mathbb{R}^m) \cong \mathcal{S}(\mathbb{R}^{n+m}) \tag{12.4}
$$

for $n, m \in \mathbb{N}$ as noticed in the Section 2.4.

Let ρ_n and ρ_m be two representations of Lie algebras $\mathfrak{sp}(2n,\mathbb{R})$ and $\mathfrak{sp}(2m,\mathbb{R})$ on $\mathrm{Pol}(\mathbb{R}^{2n},\mathbb{C}) \otimes \mathcal{S}(\mathbb{R}^n)$ and $\mathrm{Pol}(\mathbb{R}^{2m},\mathbb{C}) \otimes \mathcal{S}(\mathbb{R}^m)$, respectively derived from Segal-Shale-Weil representation of corresponding symplectic groups. We want to characterise a block diagonal injection of Lie algebras $\mathfrak{sp}(2n,\mathbb{R})\times\mathfrak{sp}(2m,\mathbb{R})\hookrightarrow\mathfrak{sp}(2(n+m),\mathbb{R})$ in the terms of representations.

For representation spaces of ρ_n and ρ_m we have

$$\big(\mathrm{Pol}(\mathbb{R}^{2n},\mathbb{C})\otimes\mathcal{S}(\mathbb{R}^n)\big)\hat{\otimes}\big(\mathrm{Pol}(\mathbb{R}^{2m},\mathbb{C})\otimes\mathcal{S}(\mathbb{R}^m)\big)\cong\mathrm{Pol}(\mathbb{R}^{2(n+m)},\mathbb{C})\otimes\mathcal{S}(\mathbb{R}^{n+m})$$

thanks to (12.4) and multiplicative structure on polynomials.

Let us consider tensor product of representations

$$\begin{aligned}\rho_n\boxtimes\rho_m\quad:\quad&\mathfrak{sp}(2n,\mathbb{R})\times\mathfrak{sp}(2m,\mathbb{R})\to\mathrm{End}\left(\mathrm{Pol}(\mathbb{R}^{2(n+m)},\mathbb{C})\right)\otimes\mathrm{End}\left(\mathcal{S}(\mathbb{R}^{n+m})\right)\\&\to\mathrm{End}\left(\mathrm{Pol}(\mathbb{R}^{2(n+m)},\mathbb{C})\otimes\mathcal{S}(\mathbb{R}^{n+m})\right)\end{aligned}$$

for $g_n\in\mathfrak{sp}(2n,\mathbb{R})$ and $g_m\in\mathfrak{sp}(2m,\mathbb{R})$ by

$$(\rho_n\boxtimes\rho_m)(g_n,g_m)=\rho_n(g_n)\otimes\mathrm{Id}_m+\mathrm{Id}_n\otimes\rho_m(g_m)$$

on the endomorphisms of the space $\big(\mathrm{Pol}(\mathbb{R}^{2n},\mathbb{C})\otimes\mathcal{S}(\mathbb{R}^n)\big)\hat{\otimes}\big(\mathrm{Pol}(\mathbb{R}^{2m},\mathbb{C})\otimes\mathcal{S}(\mathbb{R}^m)\big)$, where Id_n, Id_m denote appropriate identity operators.

12.2 Extension map

The construction of symplectic monogenics in $\mathrm{Pol}(\mathbb{R}^2,\mathbb{C})\otimes\mathcal{S}(\mathbb{R})$ is described in Chapter 6 and in Chapter 11 we have showed an explicit basis for symplectic monogenics. In this chapter we study the question whether there is a way to construct symplectic monogenics in $\mathrm{Ker}(D_{s,n})\subset\mathrm{Pol}(\mathbb{R}^{2n},\mathbb{C})\otimes\mathcal{S}(\mathbb{R}^n)$. We demonstrate a recursive construction from $\mathrm{Ker}(D_{s,n-1})$ and $\mathrm{Ker}(D_{s,1})$ to $\mathrm{Ker}(D_{s,n})$.

Moreover, from representations of algebras $\mathfrak{sp}(2(n-1),\mathbb{R})$ and $\mathfrak{sp}(2,\mathbb{R})$, we get the discrete part of a representation of algebra $\mathfrak{sp}(2n,\mathbb{R})$.

Let us denote

$$\begin{aligned}\tilde{X}_{s,n}&=y_n\partial_{q_n}+ix_nq_n,\\\tilde{D}_{s,n}&=iq_n\partial_{y_n}-\partial_{x_n}\partial_{q_n},\\\tilde{E}_n&=x_n\partial_{x_n}+y_n\partial_{y_n}\end{aligned}\tag{12.5}$$

operators generating $\mathfrak{sl}(2)$ algebra in variables x_n,y_n and q_n. The commutation relations are as in (12.2)

$$[\tilde{E}_n,\tilde{D}_{s,n}]=-\tilde{D}_{s,n},\quad[\tilde{E}_n,\tilde{X}_{s,n}]=\tilde{X}_{s,n},\quad[\tilde{X}_{s,n},\tilde{D}_{s,n}]=i(\tilde{E}_n+1).\tag{12.6}$$

Theorem 12.2.1. Let $m^s_h\in M^s_h(\mathbb{R}^{2n-2})$ be a symplectic monogenic and $\tilde{m}^s_{\tilde{h}}\in M^s_{\tilde{h}}(\mathbb{R}^2)$ be a symplectic monogenic in variables x_n,y_n and q_n. Then

$$\Psi_b:M^s_{\tilde{h}}(\mathbb{R}^2)\times M^s_h(\mathbb{R}^{2n-2})\to M^s_{\tilde{h}+h+b}(\mathbb{R}^{2n}),\tag{12.7}$$

$$\begin{aligned}&\Psi_b(\tilde{m}^s_{\tilde{h}},m^s_h)=\\&=\sum_{l=0}^{b}(-1)^l\frac{\prod_{j=0}^{l-1}\left((b-j)(h+n-1)+\frac{(b-j-1)(b-j)}{2}\right)}{\prod_{j=1}^{l}\left(j(\tilde{h}+1)+\frac{(j-1)j}{2}\right)}(\tilde{X}_{s,n})^l\tilde{m}^s_{\tilde{h}}(X_{s,n-1})^{b-l}m^s_h,\end{aligned}\tag{12.8}$$

where $b\in\mathbb{N}_0$.

Proof. Let us show that $D_{s,n}\Psi_b(\tilde{m}^s_{\tilde{h}}, m^s_h) = 0$, where $D_{s,n} = \tilde{D}_{s,n} + D_{s,n-1}$. By the relation (12.3) we have

$$\tilde{D}_{s,n}(\tilde{X}_{s,n})^l \tilde{m}^s_{\tilde{h}} = -\,i\left(l(\tilde{h}+1) + \frac{(l-1)l}{2}\right)(\tilde{X}_{s,n})^{l-1}\tilde{m}^s_{\tilde{h}},$$

$$D_{s,n-1}(X_{s,n-1})^{b-l} m^s_h = -\,i\left((b-l)(h+n-1) + \frac{(b-l-1)(b-l)}{2}\right)\times$$
$$\times\,(X_{s,n-1})^{b-l-1} m^s_h$$

whence $\tilde{D}_{s,n}\Psi_b(\tilde{m}^s_{\tilde{h}}, m^s_h) + D_{s,n-1}\Psi_b(\tilde{m}^s_{\tilde{h}}, m^s_h) = 0$ because

$$\sum_{l=1}^{b}(-1)^l \frac{\prod_{j=0}^{l-1}\left((b-j)(h+n-1) + \frac{(b-j-1)(b-j)}{2}\right)}{\prod_{j=1}^{l-1}\left(j(\tilde{h}+1) + \frac{(j-1)j}{2}\right)}(\tilde{X}_{s,n})^{l-1}\tilde{m}^s_{\tilde{h}}(X_{s,n-1})^{b-l} m^s_h =$$
$$= \sum_{l=0}^{b-1}(-1)^l \frac{\prod_{j=0}^{l}\left((b-j)(h+n-1) + \frac{(b-j-1)(b-j)}{2}\right)}{\prod_{j=1}^{l}\left(j(\tilde{h}+1) + \frac{(j-1)j}{2}\right)}(\tilde{X}_{s,n})^{l}\tilde{m}^s_{\tilde{h}}(X_{s,n-1})^{b-l-1} m^s_h.$$

□

It is possible to modify the "principle" of the map Ψ_b in order to obtain extension map $\Psi_{b,c}$ that increases the dimension by more than one.

Proposition 12.2.2. Let $c \in \mathbb{N}$, m^s_h is a symplectic monogenic of the homogeneity h and $\tilde{m}^s_{\tilde{h}}$ is a symplectic monogenic of the homogeneity $\tilde{h}$ in the variables $x_{n+1}, \ldots, x_{n+c}$, $y_{n+1}, \ldots, y_{n+c}$ and $q_{n+1}, \ldots, q_{n+c}$. Then

$$\Psi_{b,c}(\tilde{m}^s_{\tilde{h}}, m^s_h) : \mathrm{Ker}(D_{s,c}) \times \mathrm{Ker}(D_{s,n}) \to \mathrm{Ker}(D_{s,n+c}), \tag{12.9}$$

$$\Psi_{b,c}(\tilde{m}^s_{\tilde{h}}, m^s_h) = \tag{12.10}$$
$$= \sum_{l=0}^{b}(-1)^l \frac{\prod_{j=0}^{l-1}\left((b-j)(h+n) + \frac{(b-j-1)(b-j)}{2}\right)}{\prod_{j=1}^{l}\left(j(\tilde{h}+c) + \frac{(j-1)j}{2}\right)}(X_{s,n+c} - X_{s,n})^l \tilde{m}^s_{\tilde{h}}(X_{s,n})^{b-l} m^s_h,$$

where $b \in \mathbb{N}_0$.

Proof. The proof follows the idea of the proof of Theorem 12.2.1. □

Remark 11. By the decomposition (7.5) every polynomial symplectic spinor $p \in \mathrm{Pol}_h(\mathbb{R}^{2n-2}, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R}^{n-1})$ is possible to write as linear combination of $(X_{s,n-1})^j m^s_l$, where $j \in \mathbb{N}_0$, $j \le h$ and $m^s_l \in M^s_l \subset \mathrm{Ker}(D_{s,n-1})$ for $l \le h$. Therefore it is possible to write a variant of the mapping Ψ_b for which it is sufficient to know symplectic monogenics in $\mathrm{Pol}_h(\mathbb{R}^2, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R})$ only.

$$\Psi_a : M^s_{\tilde{h}}(\mathbb{R}^2) \times \mathrm{Pol}_h(\mathbb{R}^{2n-2}, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R}^{n-1}) \to M^s_{\tilde{h}+h-a}(\mathbb{R}^{2n}), \tag{12.11}$$

where $a = 0, 1, \ldots, h$. Mapping Ψ_a constructs a symplectic spinor in the kernel of the symplectic Dirac operator $D_{s,n}$ from the symplectic monogenic

$$\tilde{m}^s_{\tilde{h}} = \tilde{m}^s_{\tilde{h}}(x_n, y_n, q_n) \in M^s_{\tilde{h}}(\mathbb{R}^2)$$

and arbitrary symplectic spinor of homogeneity h

$$p = p(x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1}, q_1, \dots, q_{n-1}) \in \mathrm{Pol}_h(\mathbb{R}^{2n-2}, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R}^{n-1})$$

by the formula

$$\Psi_a(\tilde{m}^s_{\tilde{h}}, p) = \sum_{l=0}^{h} \frac{(-i)^l}{\prod_{j=1}^{l} \left(j(\tilde{h}+1) + \frac{(j-1)j}{2} \right)} (\tilde{X}_{s,n})^l \tilde{m}^s_{\tilde{h}} (D_{s,n-1})^{l+a} p. \qquad (12.12)$$

13. Symplectic Dirac and symplectic Twistor operators on complex tori

In the present chapter, we shall be interested in a class of examples given by even dimensional tori. So let Γ be a lattice in $\mathbb{C}^n$, i.e., a free $\mathbb{Z}$-module whose rank is the cardinality of a real basis of $\mathbb{C}^n$ over $\mathbb{R}$. The discrete group Γ acts on $\mathbb{C}^n$ by translations, and its action is proper and fixed point free. The quotient $\mathbb{C}^n/\Gamma$ is compact and called complex (or even dimensional) tori. There is a $\mathbb{R}$-linear automorphism of $\mathbb{C}^n$ sending Γ to $\mathbb{Z}^{2n}$, so that the complex tori is as a topological manifold isomorphic to $(\mathbb{R}/\mathbb{Z})^{2n} \simeq (S^1)^{2n}$. Since the standard Hermitian metric on $\mathbb{C}^n$ is preserved by Γ, it induces flat Kähler metric on $\mathbb{C}^n/\Gamma$. Generators of the lattice Γ will be denoted by $v_1, \dots, v_{2n}$.

We have the first homology class $H_1(\mathbb{C}^n/\Gamma, \mathbb{Z}_2) \simeq \mathbb{Z}_2^{2n}$ for n-dimensional complex tori. Its dual $H^1(\mathbb{C}^n/\Gamma, \mathbb{Z}_2) \simeq \mathbb{Z}_2^{2n}$ is a $\mathbb{Z}_2$-module generated by classes

$$\frac{1}{2}v_1^\star, \frac{1}{2}v_2^\star, \dots, \frac{1}{2}v_{2n}^\star \in (\frac{1}{2}\Gamma^\star)/\Gamma^\star. \tag{13.1}$$

Here we use the notation $v_1^\star, \dots, v_{2n}^\star$ for the generators of the dual lattice

$$\Gamma^\star = \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{Z}), \quad \Gamma^\star \simeq \mathbb{Z} < v_1^\star, \dots, v_{2n}^\star >, \tag{13.2}$$

fulfilling $v_i^\star(v_j) = \delta_{i,j}$, $i, j = 1, \dots, 2n$. Let $e_1, \dots, e_{2n}$ be a (real) basis of $\mathbb{C}^n$, $\Gamma \subset \mathbb{C}^n$ a lattice and B the matrix with column entries given by coefficients of the expansion of generators of Γ in the basis $e_1, \dots, e_{2n}$. Then the coefficients of analogous expansion for the dual lattice $\Gamma^\star \subseteq \mathbb{C}^{n\star}$ are given by matrix entries in $A = B(B^TB)^{-1} = (B^T)^{-1}$.

It remains to describe the function spaces for different choices of metaplectic structure. The smooth translation Γ-invariant functions on $\mathbb{C}^n$,

$$f_{v^\star}(x) = e^{2\pi i \langle v^\star, x\rangle}, \quad v^\star \in \Gamma^\star,\, x \in \mathbb{C}^n, \tag{13.3}$$

descend to $\mathbb{C}^n/\Gamma$ and form the topological basis of $\mathcal{C}^\infty(\mathbb{R}^{2n}/\Gamma, \mathbb{C})$. The tensor product with Segal-Shale-Weil representation gives the basis of the space of sections of symplectic spinor bundle for the trivial metaplectic structure.

The symplectic spinor bundle for a non-trivial metaplectic structure is realized by a twist of the trivial symplectic spinor bundle with a real line bundle L fulfilling $L \otimes L \simeq \mathcal{C}^\infty(M)$. The complexified line bundle $L^{\mathbb{C}} = L \otimes \mathbb{C}$ is topologically trivial because $H^2(\mathbb{C}^n/\Gamma, \mathbb{Z})$ has no $\mathbb{Z}_2$-torsion. The equivalence classes of L bijectively correspond to elements of $H^1(\mathbb{C}^{2n}/\Gamma, \mathbb{Z}_2) \simeq \mathbb{Z}_2^{2n}$, i.e., to the $2n$-tuples $\epsilon = (\epsilon_1, \dots, \epsilon_{2n}) \in \mathbb{Z}_2^{2n}$. Then the complete basis of the space of smooth sections of the vector bundle L on $\mathbb{R}^{2n}/\Gamma$ corresponding to metaplectic structure ϵ is parametrized by the lattice $\Gamma^\star + \sum_{j=1}^{2n} \frac{1}{2}\epsilon_j v_j^\star$, and elements

$$f_{v^\star+\epsilon}(x) = e^{2\pi i \langle v^\star + \sum_{j=1}^{2n} \frac{1}{2}\epsilon_j v_j^\star, x\rangle}, \quad v^\star \in \Gamma^\star \subset (\mathbb{R}^{2n})^\star,\, x \in \mathbb{R}^{2n}, \tag{13.4}$$

tensored with the Segal-Shale-Weil representation give a complete basis of the space of sections for the symplectic spinor bundle corresponding to the metaplectic structure ϵ. This is the result of the comparison isomorphism between two metaplectic bundles for two different metaplectic structures, cf. [17] for analogous considerations in the case of the classical (orthogonal) Dirac operator.

Notice that eigenfunctions and eigenvalues for the coordinate vector fields ∂_{x_j}, $j = 1, \ldots, n$, are

$$\partial_{x_j} e^{2\pi i \langle v^\star + \frac{1}{2}\sum_{j=1}^{2n} \epsilon_j v_j^\star, x\rangle} = 2\pi i (v^\star + \frac{1}{2}\epsilon)_j e^{2\pi i \langle v^\star + \frac{1}{2}\sum_{j=1}^{2n} \epsilon_j v_j^\star, x\rangle}, \tag{13.5}$$

where $(v^\star + \frac{1}{2}\epsilon)_j$ denotes j-th component of the vector $v^\star + \frac{1}{2}\epsilon$ in the dual basis $v_1^\star, \ldots, v_n^\star$.

13.1 Complex tori of dimension one: Elliptic curves

One dimensional complex tori correspond to elliptic curves. In its complex uniformization $\mathbb{C}/\Gamma_\tau$, the 2-dimensional torus $\mathbb{T}^2_\tau$ is a compact Riemann surface of genus one. Here the lattice $\Gamma_\tau \simeq \mathbb{Z} \oplus \tau\mathbb{Z}$, where τ is in the upper half plane $\mathcal{H} = \{\tau \in \mathbb{C} \,|\, \mathrm{Im}(\tau) > 0\}$, is isomorphic to $\pi_1(\mathbb{T}^2)$. Isomorphic choices for the generators of the lattice Γ_τ are related by elements in $\mathrm{SL}(2,\mathbb{Z})$, acting on $\mathcal{H}$ by the fractional linear transformations

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z}).$$

Another characterization of $\mathbb{T}^2_\tau$ up to an isomorphism is given by an equivalence class of a conformal structure of flat representative metric, (where x_1, y_1 are real coordinates on $\mathbb{C}$)

$$\mathrm{d}s^2 = e^f |\mathrm{d}x_1 + \tau \mathrm{d}y_1|^2_g, \quad g_{ab} = \rho \begin{pmatrix} 1 & \mathrm{Re}(\tau) \\ \mathrm{Re}(\tau) & \tau\bar{\tau} \end{pmatrix}, \quad f \in \mathcal{C}^\infty(\mathbb{T}, \mathbb{C}). \tag{13.6}$$

Being a Kähler manifold, the associated complex structure on $\mathbb{T}^2_\tau$ is given by

$$J_a{}^b = \sqrt{g}\, \varepsilon_{ac}\, g^{cb}, \tag{13.7}$$

for ε_{ab} the Levi-Civita symbol and $g = \det(g_{ab})$. The complex structure J is clearly invariant under the local Weyl transformations of the metric g_{ab} and so descends to its conformal class. The inverse of the metric g

$$g^{cb} = \frac{1}{\rho(\mathrm{Im}(\tau))^2} \begin{pmatrix} \tau\bar{\tau} & -\mathrm{Re}(\tau) \\ -\mathrm{Re}(\tau) & 1 \end{pmatrix}$$

implies

$$J_1{}^1 = \frac{-\mathrm{Re}(\tau)}{\mathrm{Im}(\tau)}, \quad J_1{}^2 = \frac{1}{\mathrm{Im}(\tau)}, \quad J_2{}^1 = \frac{-\tau\bar{\tau}}{\mathrm{Im}(\tau)}, \quad J_2{}^2 = \frac{\mathrm{Re}(\tau)}{\mathrm{Im}(\tau)}$$

with $J_a{}^b J_b{}^c = -\mathrm{Id}_a{}^c$. In the real basis $\binom{1}{0}, \binom{0}{i}$ of $\mathbb{C}/\mathbb{R}$, the lattice $\Gamma_\tau \simeq \mathbb{Z}\langle 1, \tau\rangle$ has the matrix form

$$B = \begin{pmatrix} 1 & \mathrm{Re}(\tau) \\ 0 & \mathrm{Im}(\tau) \end{pmatrix}, \quad \mathrm{Im}(\tau) > 0, \quad \tau = \mathrm{Re}(\tau) + i\mathrm{Im}(\tau). \tag{13.8}$$

Its inverse

$$B^{-1} = \begin{pmatrix} 1 & -\frac{\mathrm{Re}(\tau)}{\mathrm{Im}(\tau)} \\ 0 & \frac{1}{\mathrm{Im}(\tau)} \end{pmatrix} \tag{13.9}$$

is the matrix of the dual lattice $\Gamma^\star_\tau \subset \mathbb{C}$ and due to $1 - i\frac{\mathrm{Re}(\tau)}{\mathrm{Im}(\tau)} = \frac{-i\tau}{\mathrm{Im}(\tau)}$,

$$\Gamma^\star_\tau \simeq \mathbb{Z}\left\langle \frac{i}{\mathrm{Im}(\tau)}, \frac{-i\tau}{\mathrm{Im}(\tau)} \right\rangle \simeq \mathbb{Z}\left\langle \frac{i}{\mathrm{Im}(\tau)}, \frac{i\tau}{\mathrm{Im}(\tau)} \right\rangle. \tag{13.10}$$

Then the columns of the matrix

$$B(B^T B)^{-1} = (B^T)^{-1} = \begin{pmatrix} 1 & 0 \\ -\frac{\mathrm{Re}(\tau)}{\mathrm{Im}(\tau)} & \frac{1}{\mathrm{Im}(\tau)} \end{pmatrix}$$

give the coordinates of generators of $\Gamma^\star_\tau$. An element $v^\star \in \Gamma^\star_\tau$ is then of the form

$$v^\star = m\epsilon^1 - m\frac{\mathrm{Re}(\tau)}{\mathrm{Im}(\tau)}\epsilon^2 + n\frac{1}{\mathrm{Im}(\tau)}\epsilon^2 \quad m, n \in \mathbb{Z}, \tag{13.11}$$

where $\{\epsilon^1\}^2_{j=1}$ is the symplectic coframe dual to the symplectic frame $\{e_j\}^2_{j=1}$, $\epsilon^j(e_k) = \delta_{j,k}$.

13.2 Symplectic Dirac operator on elliptic curves

We use the notation x_1, y_1 for the real coordinates on $\mathbb{C}$, the universal covering space of an elliptic curve. The coordinate on the 1-dimensional real vector space underlying the Schwartz space $\mathcal{S}(\mathbb{R})$ is denoted by q.

A smooth symplectic spinor $f \in \mathcal{C}^\infty(\mathbb{T}^2_\tau, \mathcal{S}(\mathbb{R}))$ can be written as a convergent series

$$\begin{aligned} f(x_1, y_1, q) &\equiv \sum_{v^\star \in \Gamma^\star} B_{v^\star} f_{v^\star}(x_1, y_1, q) \\ f_{v^\star}(x_1, y_1, q) &\equiv \sum_{j=0}^{\infty} A_{j,v^\star} e^{2\pi i\langle v^\star, \boldsymbol{x}\rangle} q^j e^{-\frac{q^2}{2}} \end{aligned} \tag{13.12}$$

for $A_{j,v^\star}, B_{v^\star} \in \mathbb{C}$ and $\boldsymbol{x} \equiv (x_1, y_1)$. The symplectic Dirac operator D_s, written on $\mathbb{C}$ in the real coordinates x_1, y_1, is

$$D_s = iq\partial_{y_1} - \partial_q \partial_{x_1}. \tag{13.13}$$

It descends to $\mathbb{T}^2_\tau$, and we use the same notation for the corresponding operator on $\mathbb{T}^2_\tau$.

Lemma 13.2.1. A smooth symplectic spinor $f \in \mathcal{C}^\infty(\mathbb{T}^2_\tau, \mathcal{S}(\mathbb{R}))$ for the trivial metaplectic structure on $\mathbb{T}^2$ is in the kernel of the symplectic Dirac operator D_s if and only if $f(x_1, y_1, q)$ is independent of x_1, y_1.

Proof. We have

$$D_s f(x_1, y_1, q) = (2\pi i) \sum_{v^\star \in \Gamma^\star} B_{v^\star} \sum_{j=0}^{\infty} A_{j,v^\star} v_2^\star i q^{j+1} e^{2\pi i \langle v^\star, \boldsymbol{x} \rangle} e^{-\frac{q^2}{2}} \\ - (2\pi i) \sum_{v^\star \in \Gamma^\star} B_{v^\star} \sum_{j=0}^{\infty} A_{j,v^\star} v_1^\star (j q^{j-1} - q^{j+1}) e^{2\pi i \langle v^\star, \boldsymbol{x} \rangle} e^{-\frac{q^2}{2}}, \tag{13.14}$$

where $v_1^\star, v_2^\star$ are components of $v^\star$ and we used

$$\partial_{x_1} e^{2\pi i \langle v^\star, \boldsymbol{x} \rangle} = 2\pi i v_1^\star e^{2\pi i \langle v^\star, \boldsymbol{x} \rangle}, \quad \partial_{y_1} e^{2\pi i \langle v^\star, z \rangle} = 2\pi i v_2^\star e^{2\pi i \langle v^\star, \boldsymbol{x} \rangle}.$$

First of all, for $v^\star = 0$ we get the constant function in x_1, y_1, so it is in the solution space of D_s. The elements in the kernel of the symplectic Dirac operator are characterized by

$$-A_{1,v^\star} v_1^\star = 0, \\ A_{0,v^\star}(v_1^\star + i v_2^\star) - 2A_{2,v^\star} v_1^\star = 0, \\ \dots \\ A_{j,v^\star}(v_1^\star + i v_2^\star) - (j+2) A_{j+2,v^\star} v_1^\star = 0, \qquad j \in \mathbb{N}_0. \tag{13.15}$$

For $v_1^\star = 0$, we get $A_{j,v^\star} = 0$ for all $j \in \mathbb{N}_0$ and this yields the trivial solution. For $v_1^\star \neq 0$, we get $A_{1,v^\star} = 0$ and therefore all odd coefficients $A_{2k+1,v^\star} = 0$, $k \in \mathbb{N}_0$. So there is no non-trivial solution supported by odd coefficients.

For even $j = 2k$, $k \in \mathbb{N}_0$, the coefficients fulfil the recursion relation

$$\frac{v_1^\star + i v_2^\star}{(j+2) v_1^\star} A_{j,v^\star} = \frac{w}{(j+2)} A_{j,v^\star} = A_{j+2,v^\star},$$

where we introduced the shorthand notation $w = \frac{v_1^\star + i v_2^\star}{v_1^\star}$ ($v_1^\star, v_2^\star$ are components of $v^\star$) with the real part of w equal to 1. The recursion relates even coefficients in the expansion of $f_{v^\star}$, i.e., $A_{2k,v^\star}$ is determined by $A_{0,v^\star}$ for all $k \in \mathbb{N}_0$ and $v^\star \in \Gamma^\star$

$$A_{2k,v^\star} = \frac{1}{k!} \left(\frac{w}{2}\right)^k A_{0,v^\star}. \tag{13.16}$$

Its unique solution is the even function in the variable q,

$$f_{v^\star}(x_1, y_1, q) = e^{q^2 \left(\frac{w}{2} - \frac{1}{2}\right)} e^{2\pi i \langle v^\star, \boldsymbol{x} \rangle}. \tag{13.17}$$

However, $f_{v^\star}$ is not in $\mathcal{C}^\infty(\mathbb{T}^2_\tau, \mathcal{S}(\mathbb{R}))$ because $\mathrm{Re}\left(\frac{w}{2} - \frac{1}{2}\right) = 0$, and so there is no even solution in the required analytical class. □

Let us consider a general metaplectic structure $\epsilon = (\epsilon_1, \epsilon_2)$. Denoting by $e_1^\star, e_2^\star$ the generators of $\Gamma_\tau^\star$, we have

$$\left((v_1^\star + \frac{1}{2}\epsilon_1 e_1^\star) + (v_2^\star + \frac{1}{2}\epsilon_2 e_2^\star)\right)\left(x_1 e_1 + y_1 e_2\right) = (v_1^\star + \frac{1}{2}\epsilon_1) x_1 + (v_2^\star + \frac{1}{2}\epsilon_2) y_1,$$

and so

$$\partial_{x_1} e^{2\pi i\langle v^\star+\frac{1}{2}(\epsilon_1 e_1^\star+\epsilon_2 e_2^\star),\boldsymbol{x}\rangle} = 2\pi i\left(v_1^\star+\frac{1}{2}\epsilon_1\right) e^{2\pi i\langle v^\star+\frac{1}{2}(\epsilon_1 e_1^\star+\epsilon_2 e_2^\star),\boldsymbol{x}\rangle},$$
$$\partial_{y_1} e^{2\pi i\langle v^\star+\frac{1}{2}(\epsilon_1 e_1^\star+\epsilon_2 e_2^\star),\boldsymbol{x}\rangle} = 2\pi i\left(v_2^\star+\frac{1}{2}\epsilon_2\right) e^{2\pi i\langle v^\star+\frac{1}{2}(\epsilon_1 e_1^\star+\epsilon_2 e_2^\star),\boldsymbol{x}\rangle}.$$

Lemma 13.2.2. The solution space of the symplectic Dirac operator D_s acting on smooth sections of symplectic spinor bundle on $\mathbb{T}^2_\tau$ is trivial for each non-trivial metaplectic structure $\epsilon = (\epsilon_1, \epsilon_2) \neq 0 \in \mathbb{Z}_2^2$.

Proof. For each non-trivial metaplectic structure ϵ on $\mathbb{T}^2$, we get from an equation $D_s f(x_1, y_1, q) = 0$ the condition analogous to (13.15)

$$\begin{aligned} A_{1,v^\star}\left(v_1^\star+\frac{1}{2}\epsilon_2\right) &= 0,\\ &\dots,\\ A_{j,v^\star}\left((v_1^\star+\frac{1}{2}\epsilon_1)+i(v_2^\star+\frac{1}{2}\epsilon_2)\right)-(j+2)A_{j+2,v^\star}\left(v_1^\star+\frac{1}{2}\epsilon_2\right) &= 0, \end{aligned} \tag{13.18}$$

$j \in \mathbb{N}_0$. In the case $v_1^\star + \frac{1}{2}\epsilon_2 = 0$ it follows $A_{j,v^\star} = 0$ for all $j \in \mathbb{N}_0$, hence the solution is trivial. Also the odd part in the case $v_1^\star+\frac{1}{2}\epsilon_2 \neq 0$ is trivial. Otherwise, we define

$$w' = \frac{v_1^\star + i v_2^\star + \frac{1}{2}(\epsilon_1 + i\epsilon_2)}{v_1^\star + \frac{1}{2}\epsilon_1}.$$

Then in the exponent of the unique solution of our recursion relation, we get $\frac{w'-1}{2}$. Because $\mathrm{Re}(w') = 1$, the solution is not in $\mathcal{C}^\infty(\mathbb{T}^2_\tau, \mathcal{S}(\mathbb{R}))$ again.

For each non-trivial metaplectic structures $\epsilon = (\epsilon_1, \epsilon_2) \neq (0, 0)$, there is no $v^\star \in \Gamma^\star$ such that $v^\star + \frac{1}{2}(\epsilon_1 e_1^\star + \epsilon_2 e_2^\star) = 0$. Applying the same line of reasoning as in the case of the trivial metaplectic structure, we see that the only solution is the trivial one. □

Lemma 13.2.3. A smooth symplectic spinor satisfying the eigen-equation

$$D_s g = \alpha\, g$$

for $\alpha \in \mathbb{C}$, $\alpha \neq 0$, does not belong to the function space $\mathcal{C}^\infty(\mathbb{T}^2_\tau, \mathcal{S}(\mathbb{R}))$.

Proof. Any smooth symplectic spinor $g \in \mathcal{C}^\infty(\mathbb{T}^2_\tau, \mathcal{S}(\mathbb{R}))$ can be written as

$$g(x_1, y_1, q) \equiv \sum_{m,n\in\mathbb{Z}} a_{m,n}(q,\tau) e^{2\pi i\left(m x_1 - m\frac{\mathrm{Re}(\tau)}{\mathrm{Im}(\tau)}y_1 + n\frac{1}{\mathrm{Im}(\tau)}y_1\right)},$$

where $a_{m,n}(q,\tau) \in \mathcal{S}(\mathbb{R})$ and $\langle v^\star, \boldsymbol{x}\rangle$ was expanded by (13.11) as $mx_1 - m\frac{\mathrm{Re}(\tau)}{\mathrm{Im}(\tau)}y_1 + n\frac{1}{\mathrm{Im}(\tau)}y_1$, $v^\star \in \Gamma^\star_\tau$ and $m, n \in \mathbb{Z}$.

Solving the equation $D_s g(x_1, y_1, q) = \alpha\, g(x_1, y_1, q)$, we recall $D_s = iq\partial_{y_1} - \partial_q\partial_{x_1}$ and so

$$\sum_{m,n\in\mathbb{Z}}\left(iq a_{m,n}(q,\tau)2\pi i\left(-m\frac{\mathrm{Re}(\tau)}{\mathrm{Im}(\tau)} + \frac{n}{\mathrm{Im}(\tau)}\right) - \partial_q a_{m,n}(q,\tau)2\pi i m\right)\times$$
$$\times\, e^{2\pi i\left(m x_1 - m\frac{\mathrm{Re}(\tau)}{\mathrm{Im}(\tau)}y_1 + n\frac{1}{\mathrm{Im}(\tau)}y_1\right)} = \alpha\sum_{m,n\in\mathbb{Z}} a_{m,n}(q,\tau) e^{2\pi i\left(m x_1 - m\frac{\mathrm{Re}(\tau)}{\mathrm{Im}(\tau)}y_1 + n\frac{1}{\mathrm{Im}(\tau)}y_1\right)}.$$

Because the vectors $e^{2\pi i\left(mx_1 - m\frac{\mathrm{Re}(\tau)}{\mathrm{Im}(\tau)}y_1 + n\frac{1}{\mathrm{Im}(\tau)}y_1\right)}$ are linearly independent for different pairs $m, n \in \mathbb{Z}$, this reduces to

$$a_{m,n}(q,\tau)\left(-2\pi q\left(\frac{n}{\mathrm{Im}(\tau)} - \frac{m\mathrm{Re}(\tau)}{\mathrm{Im}(\tau)}\right) - \alpha\right) = 2\pi i m\big(\partial_q a_{m,n}(q,\tau)\big). \quad (13.19)$$

For $m = 0$, we have $a_{1,n}(q,\tau) = 0$, while for $m \neq 0$,

$$a_{m,n}(q,\tau) = e^{\frac{i\alpha}{2\pi m}q + i\frac{n - m\mathrm{Re}(\tau)}{m\mathrm{Im}(\tau)}\frac{q^2}{2}}c_{m,n}(\tau), \quad (13.20)$$

where $c_{m,n}(\tau)$ is a smooth function of τ. Therefore, we get

$$g(x_1,y_1,q) = \sum_{m,n\in\mathbb{Z}} c_{m,n}(\tau)e^{\frac{i\alpha}{2\pi m}q + i\frac{n-m\mathrm{Re}(\tau)}{m\mathrm{Im}(\tau)}\frac{q^2}{2}}e^{2\pi i\left(mx_1 - m\frac{\mathrm{Re}(\tau)}{\mathrm{Im}(\tau)}y_1 + n\frac{1}{\mathrm{Im}(\tau)}y_1\right)}. \quad (13.21)$$

However, this function is not in $\mathcal{C}^\infty(\mathbb{T}^2_\tau, \mathcal{S}(\mathbb{R}))$, because

$$\mathrm{Re}\Big(\frac{i\alpha}{2\pi m}q + i\frac{n - m\mathrm{Re}(\tau)}{m\mathrm{Im}(\tau)}\frac{q^2}{2}\Big) = \frac{-\mathrm{Im}(\alpha)}{2\pi m}q.$$

and for $m, n \in \mathbb{Z}$, it holds

$$\int_{-\infty}^{\infty}\left|c_{m,n}(\tau)e^{\frac{i\alpha}{2\pi m}q + i\frac{n-m\mathrm{Re}(\tau)}{m\mathrm{Im}(\tau)}\frac{q^2}{2}}e^{2\pi i\left(mx_1 - m\frac{\mathrm{Re}(\tau)}{\mathrm{Im}(\tau)}y_1 + n\frac{1}{\mathrm{Im}(\tau)}y_1\right)}\right|^2 \mathrm{d}q =$$

$$= \int_{-\infty}^{\infty}|c_{m,n}(\tau)|^2 e^{2\mathrm{Re}\left(\frac{i\alpha}{2\pi m}q + i\frac{n-m\mathrm{Re}(\tau)}{m\mathrm{Im}(\tau)}\frac{q^2}{2}\right)}\,\mathrm{d}q = \int_{-\infty}^{\infty}|c_{m,n}(\tau)|^2 e^{\frac{-\mathrm{Im}(\alpha)}{\pi m}q}\,\mathrm{d}q.$$

However, this does not converge for any choice of $\alpha \in \mathbb{C}$. In conclusion, there is no solution in the required analytical class $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R})$. □

13.3 Symplectic Twistor operator on elliptic curves

We now pass to analogous questions for the symplectic twistor operator T_s. The symplectic twistor operator has two components at covectors ϵ^1 and ϵ^2, see (5.16) and (5.17). By abuse of notation we denote T_s a component by covector ϵ^1,

$$T_s = \partial_{x_1} - q\partial_q\partial_{x_1} + iq^2\partial_{y_1}, \quad (13.22)$$

and call it the symplectic twistor operator. This terminology is justified by the following property. A smooth symplectic spinor $f \in \mathcal{C}^\infty(\mathbb{T}^2_\tau, \mathcal{S}(\mathbb{R}))$ is in the kernel of the symplectic twistor operator if and only if it fulfils the equation $(\partial_{x_1} - q\partial_q\partial_{x_1} + iq^2\partial_{y_1})f = 0$.

The symplectic twistor operator descends as in the case of the symplectic Dirac operator to $\mathbb{T}^2_\tau$. Acting on a smooth symplectic spinor f, cf., (13.12), we get

$$T_s f(x_1,y_1,q) = 2\pi i\Big(\sum_{v^\star\in\Gamma^\star} B_{v^\star}\sum_{j=0}^{\infty}\Big(A_{j,v^\star}v_1^\star(1-j)q^j + A_{j,v^\star}(v_1^\star + iv_2^\star)q^{j+2}\Big)e^{2\pi i\langle v^\star, x\rangle}e^{-\frac{q^2}{2}}\Big). \quad (13.23)$$

Lemma 13.3.1. A smooth symplectic spinor $f \in C^{\infty}(\mathbb{T}^2_{\tau}, \mathcal{S}(\mathbb{R}))$,

$$f(x_1, y_1, q) = \sum_{v^\star \in \Gamma^\star} B_{v^\star} f_{v^\star}(x_1, y_1, q)$$

for the trivial metaplectic structure on $\mathbb{T}^2$, $B_{v^\star} \in \mathbb{C}$, $v^\star \in \Gamma^\star$, is in the kernel of the symplectic twistor operator T_s, acting on smooth sections of symplectic spinor bundle on $\mathbb{T}^2_{\tau}$, if and only if $f(x_1, y_1, q) = B_0 f_0(q)$, $f_0(q)$ being independent on x_1, y_1.

Proof. The symplectic spinors $f(x_1, y_1, q)$ independent on x_1 and y_1 are obviously in the solution space of the symplectic twistor operator.

The symplectic spinor (13.12) is a solution of the symplectic twistor operator provided the following recursion relation

$$A_{0,v^\star} v_1^\star = 0, \quad A_{1,v^\star} v_1^\star (1-1) = 0, \quad \ldots, \quad A_{j+2,v^\star} v_1^\star (1-j) + A_{j,v^\star}(v_1^\star + i v_2^\star) = 0,$$

$$A_{j+2,v^\star} = \frac{w}{j-1} A_{j,v^\star} \tag{13.24}$$

is satisfied for each $j \in \mathbb{N}_0$, where $A_{0,v^\star} = 0$ and $A_{1,v^\star} \in \mathbb{C}$. Here we introduced the shorthand notation $w = \frac{v_1^\star + i v_2^\star}{v_1^\star}$. In the case $v_1^\star = 0$ we get $f_{v^\star}(x_1, y_1, q) = 0$.

For even $j = 2k$, we get the trivial solution $A_{2k,v^\star} = 0$ for all $k \in \mathbb{N}_0$.

For odd $j = 2k+1$, we choose the value of the coefficient $A_{1,v^\star}$ and the recursion yields

$$A_{2k+1,v^\star} = \frac{1}{k!} \left(\frac{w}{2}\right)^k A_{1,v^\star}, \quad k \in \mathbb{N}_0. \tag{13.25}$$

The resulting odd solution in the variable q is

$$f_{v^\star}(x_1, y_1, q) = q e^{q^2 \left(\frac{w-1}{2}\right)} e^{2\pi i \langle v^\star, x \rangle}, \tag{13.26}$$

and because $\mathrm{Re}(w) = 1$, $f_{v^\star}(x_1, y_1, q)$ is moreover not square integrable in the fibre variable and so there is no solution from $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R})$. This completes the proof. ◻

Lemma 13.3.2. For each non-trivial metaplectic structure $\epsilon = (\epsilon_1, \epsilon_2) \neq (0,0) \in \mathbb{Z}_2^2$ on $\mathbb{T}^2$, there is no non-trivial smooth solution of the symplectic twistor operator T_s.

Proof. For each non-trivial metaplectic structure, $T_s f(x_1, y_1, q) = 0$ implies the conditions analogous to (13.23) for the trivial metaplectic structure

$$A_{j,v^\star} \left((v_1^\star + \frac{1}{2}\epsilon_1) + i(v_2^\star + \frac{1}{2}\epsilon_2) \right) - (j-1) A_{j+2,v^\star} \left(v_1^\star + \frac{1}{2}\epsilon_2 \right) = 0, \quad j \in \mathbb{N}_0. \tag{13.27}$$

The metaplectic structure $\epsilon = (\epsilon_1, \epsilon_2) \in \mathbb{Z}_2^2$ and the variable

$$w' = \frac{v_1^\star + i v_2^\star + \frac{1}{2}(\epsilon_1 + i\epsilon_2)}{v_1^\star + \frac{1}{2}\epsilon_1},$$

allow to write the exponent of the solution as $\frac{w'-1}{2}$. Thus, $\mathrm{Re}(w') = 1$ implies that the solution is not in $\mathcal{S}(\mathbb{R})$. ◻

14. Symplectic analogues of classical theta functions

The content of the present chapter is motivated by definition of classical theta functions, there we construct a generalization of classical theta functions for the symplectic Dirac operator. As it is based on the symplectic Dirac operator, which is less restrictive than the Cauchy-Riemann operator, it allows to construct non-holomorphic examples in the hierarchy of symplectic theta functions. The questions on divisors, zeroes and the modular behaviour of symplectic theta functions are postponed to further research. We shall call these objects symplectic theta functions.

Let $\Gamma \subset \mathbb{C}^n$ be a lattice and D a pull-back of a positive divisor on $\mathbb{C}^n/\Gamma$ to $\mathbb{C}^n$. A theta function of type (T, J) on the complex torus $\mathbb{C}^n/\Gamma$ is a non-zero entire function $F : \mathbb{C}^n \to \mathbb{C}$ whose zeroes are in D, and such that for all $z \in \mathbb{C}^n$, $\gamma \in \Gamma$,

$$F(z + \gamma) = e^{2i\pi(T(z,\gamma)+J(\gamma))} F(z), \tag{14.1}$$

where $T : \mathbb{C}^n \times \Gamma \to \mathbb{C}$ is $\mathbb{C}$-linear in the first variable and $J : \Gamma \to \mathbb{C}$ is a function. The element $e(\gamma, z) = e^{2i\pi(T(z,\gamma)+J(\gamma))}$ is called the automorphy factor for the theta function.

In the case $n = 1$, the classical theta function $\theta(z, \tau)$ is a complex valued holomorphic (or, analytic) function, depending on $z \in \mathbb{C}$ and a lattice $\Gamma = \Gamma_\tau \subset \mathbb{C}$ determined by a point in the upper half space $\mathcal{H} = \{\tau \in \mathbb{C} | \operatorname{Im}(\tau) > 0\}$

$$\theta(z, \tau) = \sum_{n \in \mathbb{Z}} e^{i\pi n^2 \tau + 2\pi i n z}, \tag{14.2}$$

satisfying

$$\theta(z + 1, \tau) = \theta(z, \tau), \quad \theta(z + \tau, \tau) = e^{-i\pi\tau - 2i\pi z} \theta(z, \tau). \tag{14.3}$$

For introduction and motivation of the classical theta function we refer to [34].

We shall start with a prospective definition in the case $n = 1$.

Definition 14.0.1. The *symplectic theta function* is a smooth function

$$\theta_s \in \mathcal{C}^\infty(\mathbb{R}^2 \times \mathcal{H}, \mathcal{S}(\mathbb{R})),$$

satisfying

1. $D_s \theta_s(x_1, x_1, \operatorname{Re}(\tau), \operatorname{Im}(\tau), q) = 0$,
2. $\theta_s(x_1 + 1, y_1, \operatorname{Re}(\tau), \operatorname{Im}(\tau), q) = \theta_s(x_1, y_1, \operatorname{Re}(\tau), \operatorname{Im}(\tau), q)$,
3. $\theta_s(x_1 + \operatorname{Re}(\tau), y_1 + \operatorname{Im}(\tau), \operatorname{Re}(\tau), \operatorname{Im}(\tau), q) =$
   $= C(x_1, y_1, \operatorname{Re}(\tau), \operatorname{Im}(\tau), q)\theta_s(x_1, y_1, \operatorname{Re}(\tau), \operatorname{Im}(\tau), q)$,
   where $C(x_1, y_1, \operatorname{Re}(\tau), \operatorname{Im}(\tau), -) \in \mathcal{C}^\infty(\mathbb{R}^2 \times \mathcal{H} \times \mathbb{R}, \operatorname{Aut}(\mathcal{S}(\mathbb{R})))$, i.e., $Cl_s(\mathbb{R}^2, \omega)$-valued invertible smooth function on $\mathbb{R}^2$.

Throughout the chapter we shall assume

$$C(x_1, y_1, \mathrm{Re}(\tau), \mathrm{Im}(\tau), q) = e^{A(\mathrm{Re}(\tau),\mathrm{Im}(\tau),q)+B(\mathrm{Re}(\tau),\mathrm{Im}(\tau),q)x_1+\tilde{B}(\mathrm{Re}(\tau),\mathrm{Im}(\tau),q)y_1}, \tag{14.4}$$

where e denotes the operator exponential and operators $A(\mathrm{Re}(\tau), \mathrm{Im}(\tau), q)$, $B(\mathrm{Re}(\tau), \mathrm{Im}(\tau), q)$ and $\tilde{B}(\mathrm{Re}(\tau), \mathrm{Im}(\tau), q) \in \mathcal{C}^\infty(\mathcal{H}, \mathrm{End}(\mathcal{S}(\mathbb{R})))$. In particular, we do not assume θ_s to be a holomorphic function of both the base variables x_1, y_1 and the moduli of lattices variables $\mathrm{Re}(\tau), \mathrm{Im}(\tau)$.

14.1 Holomorphic symplectic theta functions

Now let us assume that a symplectic theta function in Definition 14.0.1 is in addition holomorphic in variables τ and $z = x_1 + iy_1$, namely $\theta_s = \theta_s(z, \tau, q)$ and $\partial_{\bar{z}}\theta_s(z, \tau, q) = 0$. The first and the second conditions in Definition 14.0.1 together with the holomorphy assumption imply

$$\theta_s(z, \tau, q) = \sum_{n\in\mathbb{Z}} a_n(\tau, q)e^{2\pi inz}, \tag{14.5}$$

and

$$D_s\theta_s(z, \tau, q) = 0 \quad \Longleftrightarrow \quad (q + \partial_q)a_n(\tau, q) = 0 \quad \text{for all } n \in \mathbb{Z}. \tag{14.6}$$

Finally, the last condition in Definition 14.0.1 describes the quasi-periodicity for $z \mapsto z + \tau$. Thus we are looking for differential operators $A(\tau, q), B(\tau, q) \in \mathrm{End}(\mathcal{S}(\mathbb{R}))$ such that

$$\theta_s(z + \tau, \tau, q) = e^{A(\tau,q)+B(\tau,q)z}\theta_s(z, \tau, q). \tag{14.7}$$

For $B(\tau, q) = -2\pi i$, we get

$$\begin{aligned}\theta_s(z + \tau, \tau, q) &= e^{A(\tau,q)-2\pi iz}\theta_s(z, \tau, q) = \\ &= e^{A(\tau,q)-2\pi iz}\sum_{n\in\mathbb{Z}} a_n(\tau, q)e^{2\pi inz} = \sum_{n\in\mathbb{Z}} e^{A(\tau,q)}a_n(\tau, q)e^{2\pi i(n-1)z},\end{aligned}$$

and the comparison with

$$\theta_s(z + \tau, \tau, q) = \sum_{n\in\mathbb{Z}} a_{n-1}(\tau, q)e^{2\pi i(n-1)z}e^{2\pi i(n-1)\tau}$$

reveals the recursion relation

$$e^{A(\tau,q)}a_n(\tau, q) = e^{2\pi i(n-1)\tau}a_{n-1}(\tau, q), \quad n \in \mathbb{Z},$$

which is by invertibility of $e^{A(\tau,q)}$ equivalent to

$$a_n(\tau, q) = e^{-A(\tau,q)}e^{2\pi i(n-1)\tau}a_{n-1}(\tau, q), \quad n \in \mathbb{Z}. \tag{14.8}$$

It remains to find a collection of $\mathcal{S}(\mathbb{R})$-valued functions $a_n(\tau, q), n \in \mathbb{Z}$, subordinate to (14.6) and (14.8). The unique solution of (14.6) is

$$a_n(q, \tau) = b_n(\tau)e^{-\frac{q^2}{2}}, \quad n \in \mathbb{Z}, \tag{14.9}$$

$b_n(\tau)$ depending only on τ for all $n \in \mathbb{Z}$, and the equation (14.8) turns into

$$e^{-\frac{q^2}{2}} b_n(\tau) = e^{-A(\tau,q)} e^{2\pi i(n-1)\tau} e^{-\frac{q^2}{2}} b_{n-1}(\tau). \tag{14.10}$$

Let us now assume that $A(\tau, q)$ does not depend on the variable q, i.e., $A(\tau, q) = A(\tau)$ is a multiple of the identity endomorphism of $\mathcal{S}(\mathbb{R})$. Then (14.10) reduces to

$$b_n(\tau) = e^{-A(\tau)} e^{2\pi i(n-1)\tau} b_{n-1}(\tau). \tag{14.11}$$

Introducing the notation $b_n(\tau) = e^{2\pi i d_n(\tau)}$, $2\pi i a(\tau) = -A(\tau)$, the previous recursion relation reduces to

$$d_n(\tau) = a(\tau) + (n-1)\tau + d_{n-1}(\tau), \tag{14.12}$$

whose unique solution is

$$\begin{aligned} d_{m+1}(\tau) &= a(\tau) + m\tau + d_m(\tau) = (m+1)a(\tau) + \frac{(m+1)m}{2}\tau + d_0(\tau), \\ d_{-(m+1)}(\tau) &= -a(\tau) + (m+1)\tau + d_{-m}(\tau) = \\ &- (m+1)a(\tau) + \frac{(-m-1)(-m-2)}{2}\tau + d_0(\tau) \end{aligned} \tag{14.13}$$

for all $m \in \mathbb{N}$. Let us summarize the previous considerations into

Proposition 14.1.1. Let $a(\tau)$, $d_0(\tau)$ be smooth functions such that the sum

$$\theta_s(z, \tau, q) = \sum_{n \in \mathbb{Z}} e^{2\pi i\left(na(\tau) + \frac{n(n-1)}{2}\tau + d_0(\tau)\right)} e^{2\pi i n z} e^{-\frac{q^2}{2}} \tag{14.14}$$

is uniformly convergent on compact subsets of $\mathbb{C}$. Then $\theta_s \in \mathcal{C}^\infty(\mathbb{C} \times \mathcal{H}, \mathcal{S}(\mathbb{R}))$ and satisfies

1. $D_s \theta_s(z, \tau, q) = 0$,
2. $\theta_s(z+1, \tau, q) = \theta_s(z, \tau, q)$,
3. $\theta_s(z+\tau, \tau, q) = e^{-2\pi i a(\tau) - 2\pi i z} \theta_s(z, \tau, q)$.

For example, the choice $a(\tau) = \frac{\tau}{2}$ and $d_0(\tau) = 0$ leads to the classical theta function $\theta(z, \tau)$, the function $\theta_s(z, \tau, q) = \theta(z, \tau) e^{-\frac{q^2}{2}}$ is in the kernel of the symplectic Dirac operator D_s.

Moreover, for a general element $(r\tau + s) \in \Gamma_\tau$, $r, s \in \mathbb{Z}$, we have

$$\begin{aligned} \theta_s(z + r\tau + s, \tau, q) &= \sum_{n \in \mathbb{Z}} e^{2\pi i\left(na(\tau) + \frac{n(n-1)}{2}\tau + d_0(\tau)\right)} e^{2\pi i n z + 2\pi i n r\tau + 2\pi i n s} e^{-\frac{q^2}{2}} = \\ &= \sum_{n \in \mathbb{Z}} e^{2\pi i\left((n+r)a(\tau) - ra(\tau) + \frac{(n+r)(n+r-1)}{2}\tau - \frac{r^2 - r}{2}\tau + d_0(\tau) + (n+r)z - rz\right)} e^{2\pi i n s} e^{-\frac{q^2}{2}} = \\ &= e^{-\pi i(r^2 - r)\tau - 2\pi i r a(\tau) - 2\pi i r z} \theta_s(z, \tau, q), \end{aligned} \tag{14.15}$$

and so the automorphy factor of this symplectic theta function for $\kappa = r\tau + s \in \Gamma_\tau$ is given by

$$e_{a(\tau)}(\kappa, z) = e^{-\pi i\left((r^2 - r)\tau + 2ra(\tau) + 2rz\right)}. \tag{14.16}$$

Lemma 14.1.2. Let $\kappa_1, \kappa_2 \in \Gamma_\tau$. Then the automorphy factor of the symplectic theta function (14.14) fulfils

$$e_{a(\tau)}(\kappa_1 + \kappa_2, z) = e_{a(\tau)}(\kappa_2, z + \kappa_2) e_{a(\tau)}(\kappa_1, z). \tag{14.17}$$

Proof. Let $\kappa_1 = r_1\tau + s_1, \kappa_2 = r_2\tau + s_2$ for some $r_1, r_2, s_1, s_1 \in \mathbb{Z}$. Then

$$\begin{aligned} e_{a(\tau)}(\kappa_1 + \kappa_2, z) &= e_{a(\tau)}\big((r_1 + r_2)\tau + (s_1 + s_2), z\big) \\ &= e^{-\pi i\big((r_1+r_2)^2\tau - (r_1+r_2)\tau + 2(r_1+r_2)a(\tau) + 2(r_1+r_2)z\big)} \\ &= e^{-\pi i\big((r_2^2-r_2)\tau + 2r_2a(\tau) + r_2(z+r_1\tau)\big)} e^{-\pi i\big((r_1^2-r_1)\tau + 2r_1a(\tau) + 2r_1z\big)} \end{aligned}$$

and the claim follows. □

Lemma 14.1.3. Let $a(\tau) \in \{\frac{2n+1}{2}\tau \,|\, n \in \mathbb{Z}\}$ and $d_0(\tau) \in \mathbb{C}$ a constant. Then

$$\theta_s(z + \tfrac{1}{2}, \tau + 1, q) = \theta_s(z, \tau, q). \tag{14.18}$$

Proof. A direct computation, for example for $a(\tau) = \frac{3\tau}{2}$, reveals

$$\begin{aligned} \theta_s(z + \tfrac{1}{2}, \tau + 1, q) &= \textstyle\sum_{n\in\mathbb{Z}} e^{2\pi i\left(n\frac{3\tau}{2} + n\frac{3}{2} + \frac{n(n-1)}{2}\tau + \frac{n(n-1)}{2} + d_0(\tau)\right)} e^{2\pi inz + \pi in} e^{-\frac{q^2}{2}} = \\ &= \sum_{n\in\mathbb{Z}} e^{2\pi i\left(n\frac{3\tau}{2} + \frac{n(n-1)}{2}\tau + d_0(\tau)\right)} e^{2\pi inz} e^{-\frac{q^2}{2}} e^{\pi in(n+3)} = \theta_s(z, \tau, q), \end{aligned}$$

because $n(n + 3)$ is even for all $n \in \mathbb{Z}$. The proof for other values of $a(\tau)$ is analogous. □

There is no symplectic theta function depending on the anti-holomorphic variable $\bar{z}$ only, given by the anti-holomorphic counterpart of (14.6)

$$(-q + \partial_q)a_n(\tau, q) = 0, \quad n \in \mathbb{Z}, \tag{14.19}$$

whose unique solution $a_n(\tau, q) = b_n(\tau)e^{\frac{q^2}{2}}$ is not in $\mathcal{S}(\mathbb{R})$ for every $n \in \mathbb{Z}$.

It is also straightforward to generalize all preceding considerations to symplectic theta functions for generalized theta characteristics $\xi = (c, b)$, $b, c \in \mathbb{R}$. We shall state here just the definition, which can be completed to the results parallel to the trivial theta characteristic discussed in the previous exposition.

Definition 14.1.1. Let $\xi = b + c\tau$ for $b, c \in \mathbb{R}$. A symplectic theta function with theta characteristic ξ is defined by

$$\theta_{s\xi}(z, \tau, q) = \sum_{n\in\mathbb{Z}} e^{2\pi i\left((n+c)a(\tau) + \frac{(n+c)(n+c-1)}{2}\tau + d_0(\tau)\right)} e^{2\pi i(n+c)(z+b)} e^{-\frac{q^2}{2}}, \tag{14.20}$$

where $a(\tau)$ and $d_0(\tau)$ are smooth functions such that the sum is uniformly convergent on compact subsets of $\mathbb{C}$.

14.2 Holomorphic symplectic theta functions and non-trivial endomorphisms of $\mathcal{S}(\mathbb{R})$

In the previous part, we considered symplectic theta functions, characterized by periodicity which is realized by the scalar valued transitions on their values in $\mathcal{S}(\mathbb{R})$, see Proposition 14.1.1. However, the algebra $\mathrm{End}(\mathcal{S}(\mathbb{R}))$ is infinite-dimensional and contains the symplectic Clifford (or, Weyl) algebra $Cl_s(\mathbb{R}^2, \omega)$. Consequently, it offers variety of potential possibilities to be considered. Despite the fact that we do not have a clear evidence for a class of endomorphisms together with a reasonable characterization useful in the construction of symplectic theta functions, we present a non-trivial example.

Let us now assume $A(\tau, q) = c_1 q + c_2\partial_q$ for $c_1 = c_1(\tau), c_2 = c_2(\tau) \in \mathcal{C}^\infty(\mathcal{H})$, so we emphasize that $A(\tau, q) \in \mathrm{End}(\mathcal{S}(\mathbb{R}))$ is not a multiple of identity map. We first recall the Baker-Hausdorff-Campbell formula for the composition of exponentials of two operators X, Y

$$e^X e^Y = e^{Y+[X,Y]+\frac{1}{2!}[X,[X,Y]]+\frac{1}{3!}[X,[X,[X,Y]]]+\cdots} e^X, \tag{14.21}$$

and apply it to $X = c_1 q + c_2\partial_q, Y = -\frac{A}{2}q^2$ for some $A \in \mathbb{C}$. Because the only non-trivial iterated commutators are

$$[c_1 q + c_2\partial_q, -\frac{A}{2}q^2] = -Ac_2 q, \quad [c_1 q + c_2\partial_q, [c_q q + c_2\partial_q, -\frac{A}{2}q^2]] = -Ac_2^2,$$

we get

$$e^{c_1 q+c_2\partial_q} e^{-\frac{A}{2}q^2} = e^{-\frac{A}{2}q^2 - Ac_2 q - \frac{A}{2!}c_2^2} e^{c_1 q+c_2\partial_q} \tag{14.22}$$

and so it remains to evaluate

$$e^{c_1 q+c_2\partial_q} 1 = \sum_{k=0}^{\infty} \frac{(c_1 q + c_2\partial_q)^k}{k!} 1$$

with 1 the q-constant function.

Lemma 14.2.1. For all $k \in \mathbb{N}_0$, we have

$$(c_1 q + c_2\partial_q)^k 1 = \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k!}{m!(k-2m)!2^m} c_1^{k-m} c_2^m q^{k-2m}. \tag{14.23}$$

Proof. The proof is by induction on $k \in \mathbb{N}_0$. For $k = 0, 1$, the claim is obvious. Assuming the formula holds for a given k, we apply to it the operator $c_1 q + c_2\partial_q$

$$(c_1 q + c_2\partial_q) \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k!}{m!(k-2m)!2^m} c_1^{k-m} c_2^m q^{k-2m} =$$

$$= \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k!}{m!(k-2m)!2^m} c_1^{k+1-m} c_2^m q^{k+1-2m}$$

$$+ \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k!}{m!(k-2m)!2^m} c_1^{k-m} c_2^{m+1} (k-2m) q^{k-1-2m},$$

and sum up the coefficients with the same power of q by shifting the summation index in the second sum by 1. It is easy to check that the resulting formula corresponds to (14.23) for k replaced by $k + 1$. This completes the proof. □

The last result claims

$$e^{c_1 q + c_2 \partial_q} 1 = \sum_{k=0}^{\infty} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{1}{m!(k-2m)!2^m} c_1^{k-m} c_2^m q^{k-2m}. \tag{14.24}$$

Now we are allowed to change the order of summations by the Fubini theorem, because there exists an absolute convergent majorization

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \left| \frac{c_1^{k-m} c_2^m q^{k-2m}}{m!(k-2m)!2^m} \right| \leq \sum_{m}^{\infty} \sum_{k=2m}^{\infty} \frac{|c_1|^{k-m} |c_2|^m |q|^{k-2m}}{m!(k-2m)!2^m} \leq$$
$$\leq \sum_{m}^{\infty} \frac{|c_1|^m |c_2|^m}{m! 2^m} \sum_{k=2m}^{\infty} \frac{|c_1|^{k-2m} |q|^{k-2m}}{(k-2m)!} = \sum_{m}^{\infty} \frac{|c_1|^m |c_2|^m}{m! 2^m} e^{|c_1||q|} = e^{|c_1||q| + \frac{|c_1||c_2|}{2}} < \infty$$

By changing the order of summation, we get

$$\sum_{m}^{\infty} \sum_{k=2m}^{\infty} \frac{c_1^{k-m} c_2^m q^{k-2m}}{m!(k-2m)!2^m} = \sum_{m}^{\infty} \frac{c_1^m c_2^m}{m! 2^m} \sum_{k=2m}^{\infty} \frac{c_1^{k-2m} q^{k-2m}}{(k-2m)!} =$$
$$= \sum_{m}^{\infty} \frac{c_1^m c_2^m}{m! 2^m} e^{c_1 q} = e^{c_1 q} e^{\frac{c_1 c_2}{2}}.$$

The substitution of the last result into (14.22) yields

$$e^{c_1 q + c_2 \partial_q} e^{-\frac{A}{2} q^2} = e^{-\frac{A}{2} q^2 + (c_1 - A c_2) q + \frac{c_1 - A c_2}{2} c_2}. \tag{14.25}$$

In particular, for $A = 1$ we get

$$e^{c_1 q + c_2 \partial_q} e^{-\frac{q^2}{2}} b_n(\tau) = e^{2\pi i (n-1)\tau} a_{n-1}(\tau, q) \quad \Longleftrightarrow$$
$$a_{n-1}(\tau, q) = e^{-\frac{q^2}{2} + (c_1 - c_2) q + \frac{c_1 - c_2}{2} - 2\pi i (n-1)\tau} b_n(\tau),$$

and $a_{n-1}(\tau, q)$ satisfies $(q - \partial_q) a_{n-1}(\tau, q) = 0$ if and only if $c_1 = c_2$. Let us summarize our previous considerations.

Proposition 14.2.2. Let $a(\tau)$, $d_0(\tau)$ and $c_1(\tau)$ be smooth functions such that the sum

$$\theta_s(z, \tau, q) = \sum_{n \in \mathbb{Z}} e^{2\pi i \left(n a(\tau) + \frac{n(n-1)}{2} \tau + d_0(\tau) \right)} e^{2\pi i n z} e^{-\frac{q^2}{2}} \tag{14.26}$$

is uniformly convergent on compact subsets of $\mathbb{C}$. Then $\theta_s \in \mathcal{C}^\infty(\mathbb{C} \times \mathcal{H}, \mathcal{S}(\mathbb{R}))$ satisfies

1. $D_s \theta_s(z, \tau, q) = 0$,
2. $\theta_s(z + 1, \tau, q) = \theta_s(z, \tau, q)$,
3. $\theta_s(z + \tau, \tau, q) = e^{-2\pi i a(\tau) - 2\pi i z + c_1(\tau)(q + \partial_q)} \theta_s(z, \tau, q)$.

14.3 Vector valued non-holomorphic symplectic theta functions

Recall that in section 14.1 we constructed holomorphic symplectic theta functions. The aim of the present subsection is the construction of a series of vector-valued non-holomorphic symplectic theta functions as certain extensions of holomorphic symplectic theta function. More precisely, a class of symplectic theta functions in question is realized inside finite tensor powers of the Segal-Shale-Weil representation equipped with the diagonal action of the symplectic Dirac operator. Up to a multiple by invertible functions, the automorphy factors are in the basis of non-holomorphic descendants characterized by unipotent matrices of the rank equal to the power of the Segal-Shale-Weil representation.

Proposition 14.3.1. Let $k \in \mathbb{N}_0$. Then the 1-parameter family (given by $\beta \in \mathbb{C}$ such that $\mathrm{Re}(\beta) > 0$) of functions on $\mathbb{R}^2$, $(x_1, y_1) \in \mathbb{R}^2$,

$$\begin{aligned} &\theta_{k,\beta}(x_1, y_1, \tau, q) = \\ &= \sum_{m\in\mathbb{Z}} \Big(\frac{q^2}{2} + 2\pi m y_1 + \pi m^2 \mathrm{Im}(\tau)\Big)^k e^{-\frac{\beta q^2}{2}} e^{2\pi i m(x_1 + i\beta y_1)} e^{\pi i m^2(\mathrm{Re}(\tau) + i\beta \mathrm{Im}(\tau))}, \end{aligned} \tag{14.27}$$

is in the kernel of the symplectic Dirac operator D_s and belongs to $\mathcal{S}(\mathbb{R})$ with respect to the variable q. These functions are invariant with respect to translation $x_1 \mapsto x_1 + 1$ and transform as

$$\begin{aligned} &\theta_{k,\beta}(x_1 + \mathrm{Re}(\tau), y_1 + \mathrm{Im}(\tau), \tau, q) = \\ &= e^{-2\pi i(x_1 + i\beta y_1) - \pi i(\mathrm{Re}(\tau) + i\beta \mathrm{Im}(\tau))} \sum_{j=0}^{k} \binom{k}{j} \big(- 2\pi y_1 - \pi \mathrm{Im}(\tau)\big)^j \theta_{(k-j),\beta}(x_1, y_1, \tau, q) \end{aligned} \tag{14.28}$$

for $(x_1, y_1) \mapsto (x_1, y_1) + (\mathrm{Re}(\tau), \mathrm{Im}(\tau))$.

Proof. The periodicity of $\theta_{k,\beta}(x_1, y_1, \tau, q)$ for $(x_1, y_1) \mapsto (x_1, y_1) + (1, 0)$ is obvious. Let us introduce the notation $Z = \big(\frac{q^2}{2} + 2\pi m y_1 + \pi m^2 \mathrm{Im}(\tau)\big)$ and show the property

$$\begin{aligned} &(iq\partial_{y_1} - \partial_q \partial_{x_1})\theta_{k,\beta}(x_1, y_1, \tau, q) = \\ &\quad = \sum_{m\in\mathbb{Z}} \Big(iq\big(2\pi k Z^{k-1} - 2\pi m\beta Z^k\big) e^{-\frac{\beta q^2}{2}} e^{2\pi i m(x_1 + i\beta y_1)} e^{\pi i m^2(\mathrm{Re}(\tau) + i\beta \mathrm{Im}(\tau))} \\ &\qquad - 2\pi i m\big(qkZ^{k-1} + Z^k(-\beta q)\big) e^{-\frac{\beta q^2}{2}} e^{2\pi i m(x_1 + i\beta y_1)} e^{\pi i m^2(\mathrm{Re}(\tau) + i\beta \mathrm{Im}(\tau))} \Big) = 0. \end{aligned}$$

Then the translation $(x_1, y_1) \mapsto (x_1, y_1) + (\mathrm{Re}(\tau), \mathrm{Im}(\tau))$ results into

$$\begin{aligned}
&\theta_{k,\beta}(x_1 + \mathrm{Re}(\tau), y_1 + \mathrm{Im}(\tau), \tau, q) = \\
&= \sum_{m\in\mathbb{Z}} \Big(\frac{q^2}{2} + 2\pi m y_1 + 2\pi m \mathrm{Im}(\tau) + \pi m^2 \mathrm{Im}(\tau)\Big)^k \times \\
&\quad \times e^{-\frac{\beta q^2}{2}} e^{2\pi i m(x_1 + \mathrm{Re}(\tau) + i\beta(y_1 + \mathrm{Im}(\tau))) + \pi i m^2(\mathrm{Re}(\tau) + i\beta \mathrm{Im}(\tau))} \\
&= \sum_{m\in\mathbb{Z}} \Big(\frac{q^2}{2} + 2\pi(m+1)y_1 + \pi(m+1)^2\mathrm{Im}(\tau) - 2\pi y_1 - \pi \mathrm{Im}(\tau)\Big)^k \times \\
&\quad \times e^{-\frac{\beta q^2}{2}} e^{2\pi i(m+1)(x_1 + i\beta y_1) + \pi i (m+1)^2(\mathrm{Re}(\tau) + i\beta\mathrm{Im}(\tau))} e^{-2\pi i(x_1 + i\beta y_1) - \pi i(\mathrm{Re}(\tau) + i\beta \mathrm{Im}(\tau))} \\
&= \sum_{m\in\mathbb{Z}} \sum_{j=0}^{k} \binom{k}{j} \Big(\frac{q^2}{2} + 2\pi(m+1)y_1 + \pi(m+1)^2\mathrm{Im}(\tau)\Big)^{k-j} (-2\pi y_1 - \pi\mathrm{Im}(\tau))^k \times \\
&\quad \times e^{-\frac{\beta q^2}{2}} e^{2\pi i(m+1)(x_1 + i\beta y_1) + \pi i (m+1)^2(\mathrm{Re}(\tau) + i\beta\mathrm{Im}(\tau))} e^{-2\pi i(x_1 + i\beta y_1) - \pi i(\mathrm{Re}(\tau) + i\beta \mathrm{Im}(\tau))} \\
&= e^{-2\pi i(x_1 + i\beta y_1) - \pi i(\mathrm{Re}(\tau) + i\beta\mathrm{Im}(\tau))} \sum_{j=0}^{k} \binom{k}{j} \big(- 2\pi y_1 - \pi \mathrm{Im}(\tau)\big)^j \theta_{(k-j),\beta}(x_1, y_1, \tau, q)
\end{aligned}$$

and the proof is complete. □

In fact, the function

$$\theta_{0,1}(x_1, y_1, \tau, q) = \sum_{m\in\mathbb{Z}} e^{-\frac{q^2}{2}} e^{2\pi i m(x_1 + i y_1)} e^{\pi i m^2(\mathrm{Re}(\tau) + i\mathrm{Im}(\tau))}$$

is precisely the one introduced in Proposition 14.1.1 for the choice $a(\tau) = 0$ and $d_0(\tau) = 0$.

For further simplification we associate to $\kappa \in \Gamma_\tau$, $\kappa = s(1,0) + r(\mathrm{Re}(\tau), \mathrm{Im}(\tau))$, $s, r \in \mathbb{Z}$,

$$\begin{aligned}
\eta_\beta(\kappa, \boldsymbol{x}) &= e^{-2\pi i r(x_1 + i\beta y_1) - \pi i r^2(\mathrm{Re}(\tau) + i\beta\mathrm{Im}(\tau))}, \\
\varpi(\kappa, \boldsymbol{x}) &= -2\pi r y_1 - \pi r^2 \mathrm{Im}(\tau).
\end{aligned} \tag{14.29}$$

Proposition 14.3.2. Let $\kappa \in \Gamma_\tau$, $\kappa = s(1,0) + r(\mathrm{Re}(\tau), \mathrm{Im}(\tau))$, $s, r \in \mathbb{Z}$, $\boldsymbol{x} \equiv (x_1, y_1) \in \mathbb{R}^2$ and $k \in \mathbb{N}_0$. Then $\{\theta_{l,\beta}(\boldsymbol{x}, \tau, q)\}_{l\le k}$ for given $\beta \in \mathbb{C}$, $\mathrm{Re}(\beta) > 0$, defines a vector valued function satisfying

$$\begin{pmatrix} \theta_{0,\beta} \\ \theta_{1,\beta} \\ \theta_{2,\beta} \\ \vdots \\ \theta_{k,\beta} \end{pmatrix} (\boldsymbol{x} + \kappa, \tau, q) = e_\beta(\kappa, \boldsymbol{x}) \cdot \begin{pmatrix} \theta_{0,\beta} \\ \theta_{1,\beta} \\ \theta_{2,\beta} \\ \vdots \\ \theta_{k,\beta} \end{pmatrix} (\boldsymbol{x}, \tau, q), \tag{14.30}$$

where

$$e_\beta(\kappa, \boldsymbol{x}) = \eta_\beta \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ \varpi & 1 & 0 & \dots & 0 \\ \varpi^2 & 2\varpi & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ \varpi^k & \binom{k}{1}\varpi^{k-1} & \dots & \binom{k}{k-1}\varpi & 1 \end{pmatrix} \tag{14.31}$$

for $\eta_\beta \equiv \eta_\beta(\kappa, \boldsymbol{x}), \varpi \equiv \varpi(\kappa, \boldsymbol{x})$ associated to $\kappa \in \Gamma_\tau$ via (14.29). The set of matrix valued functions $e_\beta(\kappa, \boldsymbol{x})$ satisfies

$$\begin{pmatrix} \theta_{0,\beta} \\ \theta_{1,\beta} \\ \theta_{2,\beta} \\ \vdots \\ \theta_{k,\beta} \end{pmatrix} (\boldsymbol{x} + \kappa_1 + \kappa_2, \tau, q) = e_\beta(\kappa_2, \boldsymbol{x} + \kappa_1) \cdot e_\beta(\kappa_1, \boldsymbol{x}) \cdot \begin{pmatrix} \theta_{0,\beta} \\ \theta_{1,\beta} \\ \theta_{2,\beta} \\ \vdots \\ \theta_{k,\beta} \end{pmatrix} (\boldsymbol{x}, \tau, q), \quad (14.32)$$

hence they are the automorphy factors for these vector-valued non-holomorphic symplectic theta functions with the matrix multiplication rule

$$e_\beta(\kappa_1 + \kappa_2, \boldsymbol{x}) = e_\beta(\kappa_2, \boldsymbol{x} + \kappa_1) \cdot e_\beta(\kappa_1, \boldsymbol{x}). \quad (14.33)$$

Proof. We first prove the equation (14.30) for fixed k and a translation given by the lattice element $\kappa \in \Gamma_\tau$, $\kappa = s(1,0) + r(\mathrm{Re}(\tau), \mathrm{Im}(\tau))$ for $s, r \in \mathbb{Z}$

$$\begin{aligned}
&\theta_{k,\beta}(x_1 + s + r\mathrm{Re}(\tau), y_1 + r\mathrm{Im}(\tau), \tau, q) = \\
&= \sum_{m\in\mathbb{Z}} \Big(\frac{q^2}{2} + 2\pi m y_1 + 2\pi m r \mathrm{Im}(\tau) + \pi m^2 \mathrm{Im}(\tau)\Big)^k \times \\
&\quad \times e^{-\frac{\beta q^2}{2}} e^{2\pi i m(x_1 + i\beta y_1) + 2\pi i m(s + r\mathrm{Re}(\tau) + i\beta r \mathrm{Im}(\tau)) + \pi i m^2(\mathrm{Re}(\tau) + i\beta\mathrm{Im}(\tau))} \\
&= \sum_{m\in\mathbb{Z}} \Big(\frac{q^2}{2} + 2\pi(m+r)y_1 + \pi(m+r)^2\mathrm{Im}(\tau) - 2\pi r y_1 - \pi r^2 \mathrm{Im}(\tau)\Big)^k \times \\
&\quad \times e^{-\frac{\beta q^2}{2}} e^{2\pi i(m+r)(x_1+i\beta y_1) - 2\pi i r(x_1 + i\beta y_1)} e^{\pi i(m+r)^2(\mathrm{Re}(\tau)+i\beta\mathrm{Im}(\tau)) - \pi i r^2(\mathrm{Re}(\tau)+i\beta\mathrm{Im}(\tau))} \\
&= e^{-2\pi i r(x_1+i\beta y_1) - \pi i r^2(\mathrm{Re}(\tau)+i\beta\mathrm{Im}(\tau))} \times \\
&\quad \times \sum_{j=0}^{k} \binom{k}{j} \big(-2\pi r y_1 - \pi r^2 \mathrm{Im}(\tau)\big)^j \theta_{(k-j),\beta}(x_1, y_1, \tau, q). \quad (14.34)
\end{aligned}$$

Now we prove equation (14.32). It holds for $\theta_{0,\beta}$, because

$$\begin{aligned}
&\eta_\beta(\kappa_1 + \kappa_2, \boldsymbol{x}) = \\
&\quad = e^{-2\pi i(r_1+r_2)(x_1+i\beta y_1) - \pi i(r_1+r_2)^2(\mathrm{Re}(\tau)+i\beta\mathrm{Im}(\tau))} \\
&\quad = e^{-2\pi i r_2(x_1+i\beta y_1) - \pi i(2r_1r_2 + r_2^2)(\mathrm{Re}(\tau)+i\beta\mathrm{Im}(\tau))} e^{-2\pi i r_1(x_1+i\beta y_1) - \pi i r_1^2(\mathrm{Re}(\tau)+i\beta\mathrm{Im}(\tau))} \\
&\quad = e^{-2\pi i r_2(x_1+s_1+r_1\mathrm{Re}(\tau)+i\beta(y_1+r_1\mathrm{Im}(\tau))) - \pi i r_2^2(\mathrm{Re}(\tau)+i\beta\mathrm{Im}(\tau))} \eta_\beta(\kappa_1, \boldsymbol{x}) \\
&\quad = \eta_\beta(\kappa_2, \boldsymbol{x} + \kappa_1)\eta_\beta(\kappa_1, \boldsymbol{x}).
\end{aligned}$$

For $\varpi(\kappa_j, \boldsymbol{x})$ defined by (14.29), we get

$$\begin{aligned}
\varpi(\kappa_1 + \kappa_2, \boldsymbol{x}) &= -2\pi(r_1+r_2)y_1 - \pi(r_1+r_2)^2\mathrm{Im}(\tau) \\
&= -2\pi r_2(y_1 + r_1\mathrm{Im}(\tau)) - \pi r_2^2 \mathrm{Im}(\tau) - 2\pi r_1 y_1 - \pi r_1^2 \mathrm{Im}(\tau) \\
&= \varpi(\kappa_2, \boldsymbol{x} + \kappa_1) + \varpi(\kappa_1, \boldsymbol{x}).
\end{aligned}$$

Therefore, we have for any $l \in \mathbb{Z}$

$$\varpi(\kappa_1 + \kappa_2, \boldsymbol{x})^l = \sum_{j=0}^{l} \binom{l}{j} \varpi(\kappa_2, \boldsymbol{x} + \kappa_1)^j \varpi(\kappa_1, \boldsymbol{x})^{l-j},$$

hence the matrix

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ \varpi(\kappa_1+\kappa_2,\boldsymbol{x}) & 1 & 0 & \dots & 0 \\ \varpi(\kappa_1+\kappa_2,\boldsymbol{x})^2 & 2\varpi(\kappa_1+\kappa_2,\boldsymbol{x}) & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ \varpi(\kappa_1+\kappa_2,\boldsymbol{x})^k & \binom{k}{1}\varpi(\kappa_1+\kappa_2,\boldsymbol{x})^{k-1} & \dots & \binom{k}{k-1}\varpi(\kappa_1+\kappa_2,\boldsymbol{x}) & 1 \end{pmatrix}$$

equals to the matrix multiplication product (denoted $\circ$)

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ \varpi(\kappa_2,\boldsymbol{x}+\kappa_1) & 1 & 0 & \dots & 0 \\ \varpi(\kappa_2,\boldsymbol{x}+\kappa_1)^2 & 2\varpi(\kappa_2,\boldsymbol{x}+\kappa_1) & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ \varpi(\kappa_2,\boldsymbol{x}+\kappa_1)^k & \binom{k}{1}\varpi(\kappa_2,\boldsymbol{x}+\kappa_1)^{k-1} & \dots & \binom{k}{k-1}\varpi(\kappa_2,\boldsymbol{x}+\kappa_1) & 1 \end{pmatrix} \circ$$

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ \varpi(\kappa_1,\boldsymbol{x}) & 1 & 0 & \dots & 0 \\ \varpi(\kappa_1,\boldsymbol{x})^2 & 2\varpi(\kappa_1,\boldsymbol{x}) & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ \varpi(\kappa_1,\boldsymbol{x})^k & \binom{k}{1}\varpi(\kappa_1,\boldsymbol{x})^{k-1} & \dots & \binom{k}{k-1}\varpi(\kappa_1,\boldsymbol{x}) & 1 \end{pmatrix}. \tag{14.35}$$

This, together with the transformation property for η_β, gives the assertion for the automorphy factor $e_\beta(\kappa_1+\kappa_2,\boldsymbol{x})$. □

In fact, the vector valued symplectic theta functions in Proposition 14.3.1 are not unique. There are more general possibilities for the coefficient at $\mathrm{Re}(\tau)+i\beta\mathrm{Im}(\tau)$ in the exponent and at $\mathrm{Im}(\tau)$ in the bracket, cf., Proposition 14.1.1.

Proposition 14.3.3. Let $k\in\mathbb{N}_0$ and $\beta,a_1,a_2\in\mathbb{C}$ such that $\mathrm{Re}(\beta)>0$ and let $d_1(\tau)$ and $d_2(\tau)$ be smooth functions such that

$$\theta_{k,\beta,a}(x_1,y_1,\tau,q)=\sum_{m\in\mathbb{Z}}\Big(\frac{q^2}{2}+2\pi m y_1+2\pi\Big(ma_2+\frac{m(m-1)}{2}\Big)\mathrm{Im}(\tau)+d_2(\tau)\Big)^k \times e^{-\frac{\beta q^2}{2}}e^{2\pi i m(x_1+i\beta y_1)}e^{2\pi i\left(ma_1+\frac{m(m-1)}{2}\right)(\mathrm{Re}(\tau)+i\beta\mathrm{Im}(\tau))+d_1(\tau)} \tag{14.36}$$

is uniformly convergent on compact subsets of $\mathbb{C}$. Then $\theta_{k,\beta,a}\in\mathcal{C}^\infty(\mathbb{R}^2\times\mathcal{H},\mathcal{S}(\mathbb{R}))$, and satisfies

1. $D_s\theta_{k,\beta,a}(x_1,y_1,\tau,q)=0$,
2. for $s,r\in\mathbb{Z}$

$$\begin{aligned}\theta_{k,\beta,a}(x_1+r\mathrm{Re}(\tau)+s,y_1+r\mathrm{Im}(\tau),\tau,q)=\\ =e^{-2\pi i r(x_1+i\beta y_1)-\pi i(r^2-r+2ra_1)(\mathrm{Re}(\tau)+i\beta\mathrm{Im}(\tau))}\times\\ \times\sum_{j=0}^{k}\binom{k}{j}\big(-2\pi r y_1-(r^2-r+2ra_2)\pi\mathrm{Im}(\tau)\big)^j\theta_{(k-j),\beta,a}(x_1,y_1,\tau,q).\end{aligned} \tag{14.37}$$

Moreover, a_1 and a_2 can be suitable smooth function of $(\mathrm{Re}(\tau) + i\beta\mathrm{Im}(\tau))$ and $\mathrm{Im}(\tau)$.

The following Lemma is similar to Lemma 14.1.3.

Lemma 14.3.4. Take $a_1, d_1(\tau)$ from (14.36) so that $a_1 = \frac{1}{2}m$, for same $m \in \mathbb{Z}$, and $d_1(\tau) \in \mathbb{C}$, (or smooth periodic function of τ with period 1). Then

$$\theta_{k,\beta,a}(x_1 + \tfrac{1}{2}, y_1, \tau + 1, q) = \theta_{k,\beta,a}(x_1, y_1, \tau, q). \tag{14.38}$$

14.4 Symplectic vector valued non-holomorphic theta functions on higher even dimensional tori

In the present section we find a modification of the vector valued theta function introduced in the previous section for higher dimensional even dimensional symplectic tori. This means that instead of pair of variables x_1, y_1, we shall work with $2n$-tuple of symplectic variables $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}^{2n}$ with symplectic pairing between x_j, y_j, $j = 1, \ldots, n$. The symplectic spinor variable q is replaced by n-tuple $q_1, \ldots, q_n$. The higher dimensional analogue of τ from the upper half space $\mathcal{H}$ is complex valued $n \times n$ symmetric matrix Υ whose imaginary part is positive definite. The space of all such Υ is called the Siegel upper half space $\mathbb{H}^n$. Notice that, we consider flat tori.

Let us review the symplectic Dirac operator in dimension $2n$,

$$D_s = \sum_{j=1}^{n} \big(iq_j\partial_{y_j} - \partial_{q_j}\partial_{x_j}\big).$$

Let us also introduce a shorthand notation,

$$\begin{aligned} \mathbf{x} &= (x_1, \ldots, x_n)^T \in \mathbb{R}^n, \qquad & \mathbf{y} &= (y_1, \ldots, y_n)^T \in \mathbb{R}^n, \\ \mathbf{q} &= (q_1, \ldots, q_n)^T \in \mathbb{R}^n, \qquad & \mathbf{m} &= (m_1, \ldots, m_n)^T \in \mathbb{Z}^n, \end{aligned} \tag{14.39}$$

and finally β for a diagonal matrix with $\beta_1, \ldots, \beta_n \in \mathbb{C}$ fulfilling $\mathrm{Re}(\beta_j) > 0$, $j = 1 \ldots, n$, on the diagonal. The canonical basis of $\mathbb{R}^n$ is denoted e_j, $j = 1, \ldots, n$.

Proposition 14.4.1. Let $\beta_1, \ldots, \beta_n \in \mathbb{C}$, $\mathrm{Re}(\beta_j) > 0$ for $j = 1 \ldots, n$, are such that the sum

$$\theta_{k,\beta}(\mathbf{x}, \mathbf{y}, \Upsilon, \mathbf{q}) = \sum_{\mathbf{m}\in\mathbb{Z}^n} \Big(\sum_{l=1}^{n} \frac{q_l^2}{2} + 2\pi\mathbf{m}^T\mathbf{y} + \pi\mathbf{m}^T\mathrm{Im}(\Upsilon)\mathbf{m}\Big)^k \\ e^{-\sum_{l=1}^{n}\frac{\beta_l q_l^2}{2}} e^{2\pi i(\mathbf{m}^T\mathbf{x} + i\mathbf{m}^T\beta\mathbf{y}) + \pi i\mathbf{m}^T(\mathrm{Re}(\Upsilon) + i\beta\mathrm{Im}(\Upsilon))\mathbf{m}} \tag{14.40}$$

is uniformly convergent on compact subset of $\mathbb{R}^{2n}$. Then $\theta_{k,\beta}$ is a smooth function in $\mathcal{C}^\infty(\mathbb{R}^{2n} \times \mathbb{H}^n, \mathcal{S}(\mathbb{R}^n))$, with $\mathbb{H}^n$ the Siegel upper half space, satisfying

1. $D_s\theta_{k,\beta}(\mathbf{x}, \mathbf{y}, \Upsilon, \mathbf{q}) = 0$,
2. $\theta_{k,\beta}(\mathbf{x} + e_j, \mathbf{y}, \Upsilon, \mathbf{q}) = \theta_{k,\beta}(\mathbf{x}, \mathbf{y}, \Upsilon, \mathbf{q})$ for all $j = 1, \ldots, n$,

3. For $\mathbf{r} = (r_1, \ldots, r_n)^T \in \mathbb{Z}^n$,

$$\theta_{k,\beta}(\mathbf{x} + \mathrm{Re}(\Upsilon)\mathbf{r}, \mathbf{y} + \mathrm{Im}(\Upsilon)\mathbf{r}, \Upsilon, \mathbf{q}) = e^{-2\pi i \mathbf{r}^T(\mathbf{x}+i\beta\mathbf{y}) - \pi i \mathbf{r}^T(\mathrm{Re}(\Upsilon)+i\beta\mathrm{Im}(\Upsilon))\mathbf{r}} \times$$

$$\times \sum_{j=0}^{k} \binom{k}{j} \Big(-2\pi\mathbf{r}^T\mathbf{y} - \pi\mathbf{r}^T\mathrm{Im}(\Upsilon)\mathbf{r}\Big)^j \theta_{k-j,\beta}(\mathbf{x}, \mathbf{y}, \Upsilon, \mathbf{q}).$$

Proof. Let us introduce the notation

$$Z_{\mathbf{m}} = \Big(\sum_{l=1}^{n} \frac{q_l^2}{2} + 2\pi\mathbf{m}^T\mathbf{y} + \pi\mathbf{m}^T\mathrm{Im}(\Upsilon)\mathbf{m}\Big),$$

$$E_{\mathbf{m}} = e^{-\sum_{l=1}^{n} \frac{\beta_l q_l^2}{2} + 2\pi i(\mathbf{m}^T\mathbf{x} + i\mathbf{m}^T\beta\mathbf{y}) + \pi i(\mathbf{m}^T\mathrm{Re}(\Upsilon)\mathbf{m} + i\mathbf{m}^T\beta\mathrm{Im}(\Upsilon)\mathbf{m})}.$$

Then

$$\begin{aligned} D_s\theta_{k,\beta}(\mathbf{x}, \mathbf{y}, \Upsilon, \mathbf{q}) &= \sum_{j=1}^{n} \big(iq_j\partial_{y_j} - \partial_{q_j}\partial_{x_j}\big) \sum_{\mathbf{m}\in\mathbb{Z}^n} Z_{\mathbf{m}} E_{\mathbf{m}} \\ &= \sum_{\mathbf{m}\in\mathbb{Z}^n} \sum_{j=1}^{n} \Big(iq_j\big(Z_{\mathbf{m}}^{k-1} 2\pi m_j + 2\pi i i m_j\beta_j\big)E_{\mathbf{m}} \\ &\qquad - 2\pi i m_j\big(q_j Z_{\mathbf{m}}^{k-1} - q_j\beta_j\big)E_{\mathbf{m}}\Big) = 0. \end{aligned}$$

The second property is obvious, because $\theta_{k,\beta}(\mathbf{x} + e_j\mathbf{y}, \Upsilon, \mathbf{q}) = e^{2\pi i}\theta_{k,\beta}(\mathbf{x}, \mathbf{y}, \Upsilon, \mathbf{q})$. In the proof of the third property, we first consider the exponent

$$2\pi i(\mathbf{m}^T\mathbf{x} + i\mathbf{m}^T\beta\mathbf{y}) + \pi i(\mathbf{m}^T\mathrm{Re}(\Upsilon)\mathbf{m} + i\mathbf{m}^T\beta\mathrm{Im}(\Upsilon)\mathbf{m}),$$

and substitute $\mathbf{x} \mapsto \mathbf{x} + \mathrm{Re}(\Upsilon)\mathbf{r}$ and $\mathbf{y} \mapsto \mathbf{y} + \mathrm{Im}(\Upsilon)\mathbf{r}$ into it. We get

$$\begin{aligned} &2\pi i\mathbf{m}^T(\mathbf{x} + i\beta\mathbf{y}) + 2\pi i\mathbf{m}^T(\mathrm{Re}(\Upsilon) + i\beta\mathrm{Im}(\Upsilon))\mathbf{r} + \pi i\mathbf{m}^T(\mathrm{Re}(\Upsilon) + i\beta\mathrm{Im}(\Upsilon))\mathbf{m} = \\ &= 2\pi i(\mathbf{m} + \mathbf{r})^T(\mathbf{x} + i\beta\mathbf{y}) - 2\pi i\mathbf{r}^T(\mathbf{x} + i\beta\mathbf{y}) \\ &\quad + \pi i(\mathbf{m} + \mathbf{r})^T(\mathrm{Re}(\Upsilon) + i\beta\mathrm{Im}(\Upsilon))(\mathbf{m} + \mathbf{r}) - \pi i\mathbf{r}^T(\mathrm{Re}(\Upsilon) + i\beta\mathrm{Im}(\Upsilon))\mathbf{r} \end{aligned}$$

because the matrices $\mathrm{Re}(\Upsilon)$, $\mathrm{Im}(\Upsilon)$ and β are symmetric and so is $(\mathrm{Re}(\Upsilon) + i\beta\mathrm{Im}(\Upsilon))$. Finally, we substitute $\mathbf{y} \mapsto \mathbf{y} + \mathrm{Im}(\Upsilon)\mathbf{r}$ into

$$\Big(\sum_{l=1}^{n} \frac{q_l^2}{2} + 2\pi\mathbf{m}^T\mathbf{y} + \pi\mathbf{m}^T\mathrm{Im}(\Upsilon)\mathbf{m}\Big)^k,$$

with the result

$$\begin{aligned} &\Big(\sum_{l=1}^{n} \frac{q_l^2}{2} + 2\pi\mathbf{m}^T\mathbf{y} + 2\pi\mathbf{m}^T\mathrm{Im}(\Upsilon)\mathbf{r} + \pi\mathbf{m}^T\mathrm{Im}(\Upsilon)\mathbf{m}\Big)^k = \\ &= \Big(\sum_{l=1}^{n} \frac{q_l^2}{2} + 2\pi(\mathbf{m} + \mathbf{r})^T\mathbf{y} + \pi(\mathbf{m} + \mathbf{r})^T\mathrm{Im}(\Upsilon)(\mathbf{m} + \mathbf{r}) - 2\pi\mathbf{r}^T\mathbf{y} - \pi\mathbf{r}^T\mathrm{Im}(\Upsilon)\mathbf{r}\Big)^k. \end{aligned}$$

The rest of proof is based on the binomial expansion and the shift $\mathbf{m} \mapsto \mathbf{m} + \mathbf{r}$ in the sum over $m \in \mathbb{Z}^n$. $\square$

References

[1] Baum, H., Friedrich, T., Kath, I., Gruenewald, F. *Twistors and killing spinors on Riemannian manifolds*. B.G. Teubner, 1991. ISBN 3-8154-2014-8.

[2] Britten, D. J., Lemire,F. W. *On modules of bounded multiplicities for the symplectic algebras*. Trans. Amer. Math. Soc. 351, 1999, 3413–3431.

[3] Cannas da Silva, A. *Lectures on Symplectic Geometry*. Springer-Verlag, 2006.

[4] Choquet-Bruhat, Y., DeWitt-Morette, C. *Analysis, Manifolds and Physics*. Revised Editional. Amsterodam: North-Holland, 1982. ISBN 0-444-86017-7.

[5] Crumeyrolle, A. *Orthogonal and Symplectic Clifford Algebras: Spinor Structures*. Springer Netherlands, 2009. ISBN-13 978-9048140596.

[6] Dai, F., Xu, Y. *Approximation Theory and Harmonic Analysis on Spheres and Balls (Springer Monographs in Mathematics)*. Springer, 2013. ISBN 978-1-4614-6659-8.

[7] De Bie, H., Ørsted, B., Somberg, P., Souček, V. *Dunkl operators and a family of realizations of* $\mathfrak{osp}(1|2)$. Trans. Amer. Math. Soc. 364, 2012, 3875–3902.

[8] De Bie, H., Ørsted, B., Somberg, P., Souček, V. *The Clifford deformation of the Hermite semigroup*. SIGMA 9, 2013, 010.

[9] De Bie, H., Somberg, P., Souček, V. *The Howe Duality and Polynomial Solutions for the Symplectic Dirac Operator*. Journal of Geometry and Physics 75, 2014, 120–128.

[10] De Bie, H., Xu, Y. *On the Clifford-Fourier transform*. Int. Math. Res., no. 22, 2011, 5123–5163.

[11] Delanghe, R., Sommen, F., Souček, V. *Clifford Algebra and Spinor-valued Functions*. Mathematics and Its Applications 53, Kluwer Academic Publishers, 1992. ISBN 0-7923-0229-X.

[12] Dostálová, M., Somberg, P. *Symplectic twistor operator and its solution space on* $\mathbb{R}^2$. Archivum Math. 49, No. 3, 2013, 161-185.

[13] Dostálová, M., Somberg, P. *Symplectic Twistor Operator on* R^{2n} *and the Segal–Shale–Weil Representation*. Complex Anal. Oper. Theory 8, no. 2, 2014, 513–528.

[14] Dostálová, M. *Twistorový operátor v symplektické spinorové geometrii*. Master's thesis, Charles University Prague, Prague, 2011.

[15] Eastwood, M. *Higher symmetries of the Laplacian*. Annals of Math., 161, 2005, 1645–1665.

[16] FOLLAND, G. *Harmonic Analysis in Phase Space.* Princeton, Princeton University Press, 1989. ISBN 0-691-08527-7.

[17] FRIEDRICH, T. *Zur Abhängigkeit des Dirac-Operators von der Spin-Struktur.* Colloq. Math. 48, 1984, 57–62.

[18] FRIEDRICH, T. *Dirac Operators in Riemannian Geometry.* AMS, 2000. ISBN-13 978-0821820551.

[19] FULTON, W., HARRIS, J. *Representation Theory: A First Course (Graduate Texts in Mathematics / Readings in Mathematics).* Springer, 1991. ISBN-13 978-0387974958.

[20] GELFAND, I., RETAKH, V., SHUBIN, M. *Fedosov manifolds.* Adv. Math. 136, No.1., 1998, 104–140.

[21] HABERMANN, K. *The Dirac operator on symplectic spinors.* Ann. Global Anal. 13, no. 2, 1995, 155 - 168.

[22] HABERMANN, K., HABERMANN, L. *Introduction to Symplectic Dirac Operators, Lecture Notes in Math.* Berlin-Heidelberg: Springer-Verlag, 2006. ISBN 978-3-540-33421-7.

[23] HATCHER, A. *Algebraic Topology.* Cambridge: Cambridge University Press, 2002. ISBN 0-521-79540-0.

[24] KADLČÁKOVÁ, L. *Dirac operator in parabolic contact symplectic geometry.* Ph.D. thesis, Charles University Prague, Prague, 2001.

[25] KNAPP, A. W. *Representation Theory of Semisimple Groups.* Princeton: Princeton University Press, 1986. ISBN 0-691-09089-0.

[26] KOBAYASHI, T., ØRSTED, B., SOMBERG, P., SOUČEK, V. *Branching laws for Verma modules and applications in parabolic geometry. I.* Advances in Mathematics 285 (2015) 1-57, Archive http://xxx.lanl.gov/abs/1305.6040.

[27] KOLÁŘ, I., MICHOR, P. W., SLOVÁK, J. *Natural Operations in Differential Geometry.* Electronic edition. Originally published by Springer-Verlag, Berlin Heidelberg, 1993. ISBN 3-540-56235-4, ISBN 0-387-56235-4.

[28] KOSTANT, B. *Symplectic Spinors.* Rome Symposia, XIV, 1974, 139–152.

[29] KRÝSL, S. *Complex of twistor operators in spin symplectic geometry.* Monatsh. Math. 161, no. 4, 2010, 381—398.

[30] KRÝSL, S. *Howe duality for metaplectic group acting on symplectic spinor valued forms.* Journal of Lie theory, 22(4), 2012, 1049–1063.

[31] KRÝSL, S. *Symplectic spinor valued forms and operators acting between them.* Archivum Math. Brno, 42, 2006, 279–290.

[32] KRÝSL, S. *Classification of* 1*st order symplectic spinor operators in contact projective geometries.* Diff. Geom. Appl. 26, no. 5, 2008, 553—565.

[33] LEBEDEV, N. N. *Special Functions and their Application.* New York: Dover, 1989. ISBN 0-486-60624-4.

[34] MUMFORD, D. *Tata Lectures on Theta I.* Birkhauser, Boston, 1983. ISBN 3-7643-3109-7.

[35] MUMFORD, D., NORI, M., NORMAN P. *Tata Lectures on Theta III.* Birkhauser, Boston, 1991. ISBN 3-7643-3440-1.

[36] ROBINSON, P. L., RAWNSLEY, J. H. *The metaplectic representation, Mp^c structures and geometric quantization.* Mem. Amer. Math. Soc. 81, no. 410, 1989.

[37] SHALE, D. *Linear symmetries of free boson fields.* Trans. Amer. Math. Soc. 103, 1962, 149—167.

[38] TONDEUR, P. *Affine Zusammenhange auf Mannigfaltigkeiten mit fast-symplektischer Struktur.* Comment. Math. Helv. 36 1961 234—244.

[39] TREVES, F. *Topological Vector Spaces, Distributions and Kernels.* Academic Press inc., New York, 1967. ISBN 0-12-699450-1.

[40] WEIL, A. *Sur certains groupes d'opérateurs unitaires.* Acta Math, 111, 1964, 143–211.

[41] WELLS, R. O. *Differential Analysis on Complex Manifolds.* Springer Science + Business Media, 2007. ISBN 978-0-387-90419-0.