Symplectic Killing spinors

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Segal-Shale-Weil representation

$(\mathbb{V}, \omega)$ real symplectic vector space of dimension $2l$

$\mathbb{L}, \mathbb{L}'$ Lagrangian subspaces of $(\mathbb{V}, \omega)$ such that

$\mathbb{V} = \mathbb{L} \oplus \mathbb{L}'$

$G := Sp(\mathbb{V}, \omega) \simeq Sp(2l, \mathbb{R})$ symplectic group

$K :=$ maximal compact subgroup of G, $K \simeq U(l)$

$\pi_1(G) \simeq \pi_1(K) \simeq \mathbb{Z} \Longrightarrow \exists\ 2:1$ covering of G

$\lambda \ : \ \tilde{G} \ \stackrel{2:1}{\rightarrow} \ G \quad \tilde{G} \ =: \ Mp(\mathbb{V}, \omega) \ \simeq \ Mp(2l, \mathbb{R})$
metaplectic group

($\tilde{G}$ is not simply connected)

Real Heisenberg group

$H_l := (\mathbb{L} \oplus \mathbb{L}') \oplus \mathbb{R}$

$$(v,t).(v',t') := (v+v', t+t'+\frac{1}{2}\omega(v,v')),$$

$(v,t),(v',t') \in H_l$

$(v,t)^{-1} = (-v,-t)$, $e = (0,0)$

Schödinger representation

$$\pi : H_l \to \mathcal{U}(L^2(\mathbb{L})),$$

$\mathcal{U}(\mathbf{W})$ unitary operators on a Hilbert space $\mathbf{W}$

$$(\pi(((p,q),t))f)(p') := e^{-\imath(t+\omega(q,p'-\frac{1}{2}))} f(p-p'),$$

$((p,q),t) \in H_l$, $p' \in \mathbb{L}$, $f \in L^2(\mathbb{L})$

Stone-von Neumann Theorem: Up to a unitary equivalence, there is exactly one irreducible unitary representation of H_l on $L^2(\mathbb{L})$

$$\pi : H_l \rightarrow \mathcal{U}(L^2(\mathbb{L}))$$

satisfying $\pi(0,t) = e^{-\imath t} id_{L^2(\mathbb{L})}$, $t \in \mathbb{R}$.

From the Schrödinger representation of the Heisenberg group H_l, we would like to build a representation of the metaplectic group $Mp(\mathbb{V}, \omega)$.

$Sp(V, \omega) \times H_l \rightarrow H_l$

$(g, (v,t)) \mapsto (gv, t)$, $g \in Sp(\mathbb{V}, \omega)$, $(v,t) \in H_l$

Twisting of the Schrödinger representation π by the previous action, we get $\pi^g(v,t) := \pi(gv,t)$,

$$\pi^g : H_l \rightarrow \mathcal{U}(L^2(\mathbb{L}))$$

$\pi^g(0,t) = e^{-\imath t} id_{L^2(\mathbb{L})}$.

Use the Stone-von Neumann theorem $\Longrightarrow$

$\pi^g(v,t) = U(g)\pi(v,t)U(g)$ for some unitary $U(g)$.

The prescription $g \mapsto U(g)$ gives

$$U : Sp(\mathbb{V}, \omega) \to \mathcal{U}(L^2(\mathbb{L}))$$

(unitary) Schur lemma $\Longrightarrow$

$$U(gh) = c(g,h)U(g)U(h)$$

for some $c(g,h) \in S^1$.

Thus U is a projective unitary representation of the symplectic group $Sp(\mathbb{V}, \omega)$ on the Hilbert space $L^2(\mathbb{L})$ of the complex valued square Lebesgue integrable functions on the Lagrangian subspace $\mathbb{L}$.

André Weil / Berezin: U lifts to $\tilde{G} = Mp(\mathbb{V}, \omega)$, i.e.,

$$\begin{array}{ccc} Mp(\mathbb{V},\omega) = \tilde{G} & & \\ \lambda \downarrow & \overset{\text{SSW}}{\searrow} & \\ Sp(\mathbb{V},\omega) = G & \xrightarrow{U} & \mathcal{U}(L^2(\mathbb{L})) \end{array},$$

where SSW : $\tilde{G} \to \mathcal{U}(L^2(\mathbb{L}))$ is a "true" representation of $\tilde{G} = Mp(\mathbb{V}, \omega)$.

Call $L^2(\mathbb{L})$ the space of L^2-symplectic spinors.

SSW - Segal-Shale-Weil representation of $\tilde{G}$.

$L^2(\mathbb{L}) = L^2(\mathbb{L})_+ \oplus L^2(\mathbb{L})_-$ decomposition into $\tilde{G}$-invariant irreducible subspaces (even and odd L^2-functions).

Analytical aspect

Schmid: Existence of an adjoint functor mg (so called minimal globalization) to the forgetful Harish-Chandra functor HC.

$$L^2(\mathbb{L}) \stackrel{HC}{\to} \odot^\bullet \mathbb{L} \stackrel{mg}{\to} \mathbf{S}$$

Elements of $\mathbf{S}$ - symplectic spinors. Denote this representation of $\tilde{G}$ by

$$\text{meta} : \tilde{G} \longrightarrow \text{Aut}(\mathbf{S}).$$

(Only an analytical derivate of the SSW representation.)

$$\odot^\bullet \mathbb{L} \stackrel{\mathcal{C}^\infty-\text{globalization}}{\longrightarrow} \mathcal{S}(\mathbb{L})$$

$$\odot^\bullet \mathbb{L} \stackrel{L^2-\text{globalization}}{\longrightarrow} L^2(\mathbb{L})$$

$\mathbf{S} = \mathbf{S}_+ \oplus \mathbf{S}_-$

Why should we call the symplectic spinors spinors?

1. orthogonal spinors:

   1.1. $(\mathbb{W}, B)$ even dimensional real Euclidean vector space, $(\mathbb{W}^{\mathbb{C}}, B^{\mathbb{C}})$ complexification, $\dim_{\mathbb{C}} \mathbb{W}^{\mathbb{C}} = 2l$.

   1.2. $G' = SO(\mathbb{W}, B)$, $\mathfrak{g}'^{\mathbb{C}} = \mathfrak{so}(\mathbb{W}^{\mathbb{C}}, B^{\mathbb{C}})$.

   1.3. Take an isotropic subspace $\mathbb{M}$ of dimension l.

   1.4. $\mathbb{S} = \bigwedge^{\bullet} \mathbb{M}$ is the space of spinors ... exterior power

2. symplectic spinors:

   2.1. $L^2(\mathbb{L})$, $\mathfrak{g} = \mathfrak{sp}(\mathbb{V}, \omega)$, $K \simeq \tilde{U(l)}$

   2.2. Harish-Chandra $(\mathfrak{g}, K)$-module of $L^2(\mathbb{L})$ is $\mathbb{C}[x^1, \ldots, x^l] \simeq \odot^{\bullet} \mathbb{L}$ ... symmetric power

   2.3. Highest weights of
   $\mathbf{S}_+$, $\lambda_+ = (-\frac{1}{2}, \ldots, -\frac{1}{2})$
   $\mathbf{S}_-$, $\lambda_- = (-\frac{1}{2}, \ldots, -\frac{1}{2}, -\frac{3}{2})$ wr. to the standard $\{\epsilon^i\}_{i=1}^{l}$-basis.

Thus, the notions are parallel; (super)symmetric wr. to the simultaneous change of symplectic - orthogonal and symmetric - exterior.

Symplectic Clifford multiplication

In Physics: Schrödinger quantization prescription.

Aim: We would like to multiply symplectic spinors from $\mathbf{S}$ by vectors from $\mathbb{V}$. For our purpose, $\hbar = 1.$

$. : \mathbb{V} \times \mathbf{S} \to \mathbf{S}$. For $f \in \mathbf{S} \subseteq \mathcal{S}(\mathbb{L})$

$$(e_i.f)(x) := x^i f(x),$$
$$(e_{i+l}.f)(x) := \imath \frac{\partial f}{\partial x^i}(x), i = 1, \ldots, l,$$

$x \in \mathbb{L}$.

Extend linearly to $\mathbb{V}$.

Howe-type duality

1. Schur duality $G := GL(\mathbb{V})$

$$\rho_k : G \to \mathsf{Aut}(\mathbb{V}^{\otimes k})$$

$$\rho_k(g)(v_1 \otimes \ldots \otimes v_k) := gv_1 \otimes \ldots \otimes gv_k,$$

$g \in G, v_i \in \mathbb{V}, i = 1, \ldots, k.$

$$\sigma_k : \mathfrak{S}_k \to \mathsf{Aut}(\mathbb{V}^{\otimes k})$$

$$\sigma_k(\tau)(v_1 \otimes \ldots \otimes v_k) := v_{\tau(1)} \otimes \ldots \otimes v_{\tau(k)},$$

$\tau \in \mathfrak{S}_k$, $v_i \in \mathbb{V}$, $i = 1, \ldots, k$.

Easy:

$$\sigma_k(\tau)\rho_k(g) = \rho_k(g)\sigma_k(\tau)$$

$g \in G, \tau \in \mathfrak{S}_k$.

Not so easy = Schur duality: $T\rho_k(g) = \rho_k(g)T \Rightarrow T \in \mathbb{C}[\sigma_k(\mathfrak{S}_k)]$ (the group algebra of $\sigma_k(\mathfrak{S}_k)$.) $\mathfrak{S}_k$ is called the Schur dual of $GL(\mathbb{V})$ for $\mathbb{V}^{\otimes k}$.

Leads to Young diagrams.

2.) Another type of duality: spinor valued forms, $\tilde{G} = Spin(\mathbb{V}, B)$

Space: $\bigwedge^\bullet \mathbb{V} \otimes \mathbb{S}$, where $\mathbb{S}$ is the space of (orthogonal) spinors

$\mathrm{End}_{\tilde{G}}(\bigwedge^\bullet \mathbb{V} \otimes \mathbb{S}) := \{T : \bigwedge^\bullet \mathbb{V} \otimes \mathbb{S} \to \bigwedge^\bullet \mathbb{V} \otimes \mathbb{S} | \text{for all} g \in G T\rho(g) = \rho(g)T\}$.

Result:

$\mathrm{End}_{\tilde{G}}(\bigwedge^\bullet \mathbb{V} \otimes \mathbb{S}) = \langle \sigma(\mathfrak{sl}(2, \mathbb{C})) \rangle$ for certain representation σ of $\mathfrak{sl}(2, \mathbb{C})$. Thus, $\mathfrak{sl}(2, \mathbb{C})$ is a Howe type dual of $Spin(\mathbb{V}, B)$ on $\bigwedge^\bullet \mathbb{V} \otimes \mathbb{S}$.

Leads to a systematic treatment of some questions on Dirac operators and their higher spin analogues.

Lefschetz decomposition on Kähler manifolds and $\mathfrak{sl}(2, \mathbb{C})$.

3. Symplectic spinor valued forms, i.e., $\tilde{G} = Mp(\mathbb{V}, \omega)$ on $\bigwedge^{\bullet} \mathbb{V} \otimes \mathbf{S}$.

Consider the representation ρ of $Mp(\mathbb{V}, \omega)$

$$\rho : \tilde{G} \to \mathrm{Aut}(\bigwedge^{\bullet} \mathbb{V} \otimes \mathbb{S})$$

$$\rho(g)(\alpha \otimes s) := \lambda(g)^{*\wedge r} \alpha \otimes \mathrm{meta}(g) s,$$

where $g \in \tilde{G}$, $\alpha \in \bigwedge^r \mathbb{V}^*$ and $s \in \mathbb{S}$.

Decomposition of symplectic spinor valued forms

Using results of Britten, Hooper, Lemire [1], one can prove

Theorem:

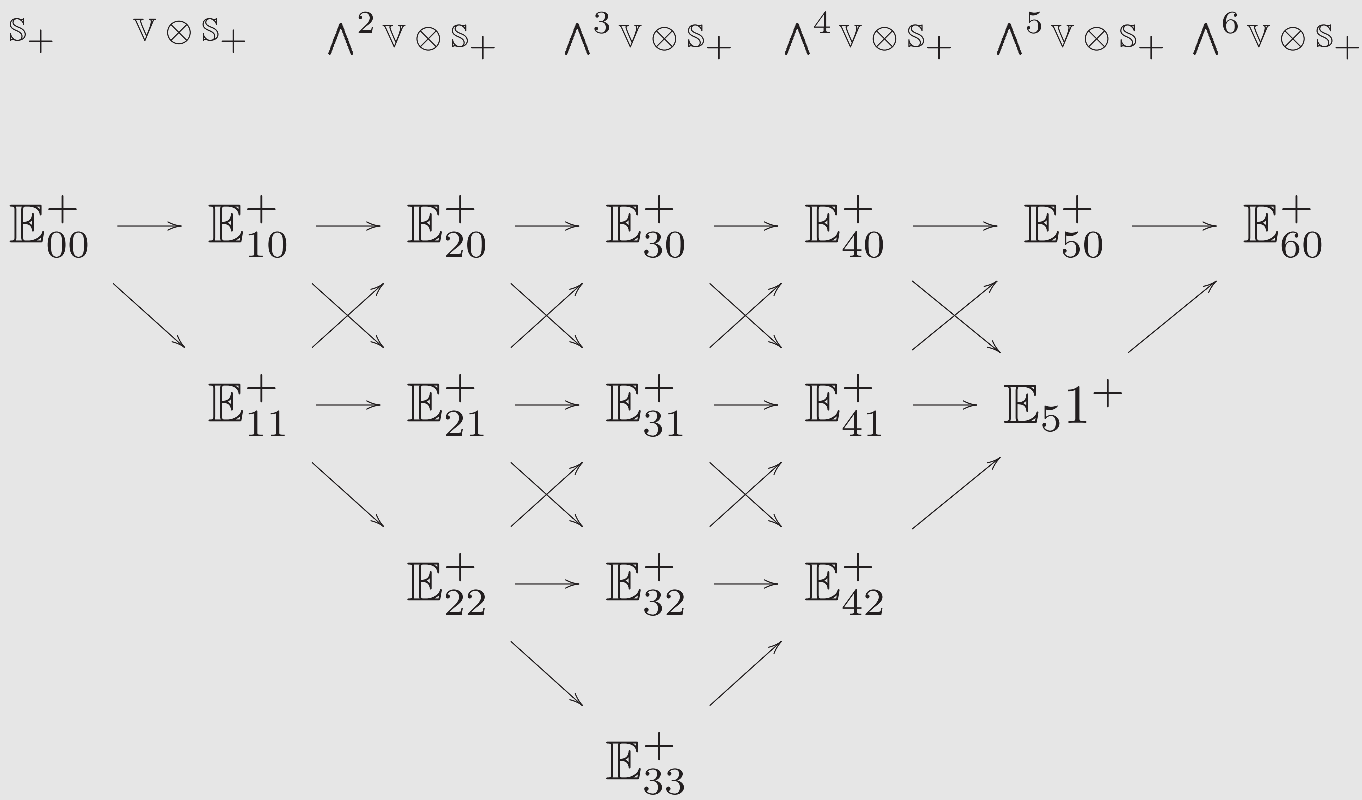
$$\bigwedge^{i} \mathbb{V} \otimes \mathbf{S}_{\pm} \simeq \bigoplus_{(i,j)\in\Xi} \mathbf{E}^{\pm}_{ij},$$

where $i = 0, \ldots, 2l$, $\Xi := \{(i,j)|i = 0, \ldots, l; j = 0, \ldots, i\} \cup \{(i,j)|i = l+1, \ldots, 2l; j = 0, \ldots, 2l-i\}$ and the infinitesimal $(\mathfrak{g}, \tilde{K})$-structure $\mathbb{E}^{\pm}_{ij}$ of $\mathbf{E}^{\pm}_{ij}$ satisfies

$$\mathbb{E}^{\pm}_{ij} \simeq L(\underbrace{\frac{1}{2}, \ldots, \frac{1}{2}}_{j}, \underbrace{-\frac{1}{2}, \ldots, -\frac{1}{2}}_{l-j-1}, -1+\frac{1}{2}(-1)^{i+j+sgn(\pm)}),$$

$sgn(\pm) := \pm 1.$

Example: $\dim \mathbb{V} = 2l = 6$, i.e., $l = 3$.



The orthosymplectic Lie super algebra $\mathfrak{osp}(1|2)$

Ortho-symplectic super Lie algebra $\mathfrak{osp}(1|2) = \langle f^+, f^-, h, e^+, e^- \rangle$.

Relations

$$[h, e^\pm] = \pm e^\pm \qquad [e^+, e^-] = 2h,$$

$$[h, f^\pm] = \pm \frac{1}{2} f^\pm \qquad \{f^+, f^-\} = \frac{1}{2} h,$$

$$[e^\pm, f^\mp] = -f^\pm \qquad \{f^\pm, f^\pm\} = \pm \frac{1}{2} e^\pm,$$

Consider the following mapping.

$$\sigma : \mathfrak{osp}(1|2) \to \mathrm{End}(\bigwedge^{\bullet} \mathbb{V} \otimes \mathbf{S})$$
$$\sigma(f^{\pm}) := F^{\pm},$$
$$\sigma(h) := 2\{F^{+}, F^{-}\},$$
$$\sigma(e^{\pm}) := \pm 2\{F^{\pm}, F^{\pm}\},$$

where the lowering and rising operators $F^{\pm}$ are defined as follows:

$$F^{\pm} : \bigwedge^{r} \mathbb{V}^{*} \otimes \mathbb{S} \to \bigwedge^{r\pm 1} \mathbb{V}^{*} \otimes \mathbb{S},$$

$r = 0, \ldots, 2l.$

$$F^{+}(\alpha \otimes s) := \sum_{i=1}^{l} \epsilon^{i} \wedge \alpha \otimes e_i.s$$

$$F^{-}(\alpha \otimes s) := \sum_{i=1}^{l} \iota_{\check{e}_i} \alpha \otimes e_i.s,$$

where $\alpha \otimes s \in \bigwedge^{\bullet} \mathbb{V}^* \otimes \mathbb{S}$ and $\{\check{e}_i\}_{i=1}^{2l}$ is the ω-dual basis to the symplectic basis $\{e_i\}_{i=1}^{2l}$.

Theorem: The mapping $\sigma : \mathfrak{osp}(1|2) \rightarrow \mathrm{End}(\bigwedge^{\bullet} \mathbb{V} \otimes \mathbf{S})$ is a super Lie algebra representation.

Theorem: The image $\mathrm{Im}(\sigma)$ of the representation σ satisfies $\mathrm{Im}(\sigma) \subseteq \mathrm{End}_{\tilde{G}}(\bigwedge^{\bullet} \mathbb{V}^* \otimes \mathbf{S})$.

Moreover, the space $\mathrm{End}_{\tilde{G}}(\bigwedge^{\bullet} \mathbb{V}^* \otimes \mathbf{S})$ of $\tilde{G}$-invariants is generated as an associative algebra by $\sigma(\mathfrak{osp}(1|2))$. Thus $\mathfrak{osp}(1|2)$ is the Howe dual of the metaplectic group $\tilde{G}$ acting on $\bigwedge^{\bullet} \mathbb{V}^* \otimes \mathbf{S}$ by the representation ρ introduced above.

Moreover, we have the following 2-folded Howe type decomposition:

Theorem :

$$\dot{\bigwedge} \mathbb{V}^* \otimes \mathbf{S} \simeq \bigoplus_{i=0}^{l} [(\mathbf{E}_+^{ii} \otimes G_i) \oplus (\mathbf{E}_-^{ii} \otimes G_i)]$$

as an $(Mp(\mathbb{V}, \omega) \times \mathfrak{osp}(1|2))$-module.

The spaces G_i are certain irreducible finite dimensional super Lie algebra representations of the super Lie algebra $\mathfrak{osp}(1|2)$.

Geometric part

(M, ω) symplectic manifold of dimension $2l$. $\mathcal{R}$ bundle of symplectic bases in TM, i.e.,

$\mathcal{R} := \{(e_1, \ldots, e_{2l})$ is a symplectic basis of$(T_m, \omega_m)|m \in M\}$.

$p_1 : \mathcal{R} \to M$, the foot-point projection, is a principal $Sp(2l, \mathbb{R})$-bundle.
$p_2 : \mathcal{P} \to M$ be a principal $Mp(2l, \mathbb{R})$-bundle.
$\Lambda : \mathcal{P} \to \mathcal{R}$ be a surjective bundle morphism over the identity on M.

Definition: We say that $(\mathcal{P}, \Lambda)$ is a metaplectic structure if

$$
\begin{array}{ccccc}
Mp(2l,\mathbb{R}) \times \mathcal{Q} & \longrightarrow & \mathcal{Q} & & \\
 & & & \searrow^{p_2} & \\
\Big\downarrow \lambda\times\Lambda & & \Big\downarrow \Lambda & & M \\
 & & & \nearrow_{p_1} & \\
Sp(2l,\mathbb{R}) \times \mathcal{P} & \longrightarrow & \mathcal{P} & &
\end{array}
$$

commutes. The horizontal arrows are the actions of the respective groups.

Symplectic spinors

$$\mathcal{S} := \mathcal{P} \times_{\mathsf{meta}} \mathbf{S}.$$

Elements of $\Gamma(M, \mathcal{S})$ symplectic spinors (Kostant)

Symplectic connection = torsion-free affine connection ∇ satisfying $\nabla\omega = 0$. It gives rise to a principal bundle connection Z on $p_1 : \mathcal{R} \to M$. Take a lift $\hat{Z}$ of Z to the metaplectic structure $p_2 : \mathcal{P} \to M$. Consider the associated covariant derivative on $\mathcal{S} \Longrightarrow$ symplectic spinor derivative $\nabla^{\mathcal{S}}$.

Remark. With help of $\nabla^{\mathcal{S}}$, one can define the symplectic Dirac operator and do, e.g., harmonic analysis for symplectic spinors (Habermann).

Manifolds admitting a metaplectic structure:

1.) phase spaces $(T^*N, d\theta)$, N orientable,

2.) complex projective spaces $\mathbb{P}^{2k+1}\mathbb{C}$, $k \in \mathbb{N}_0$,

3.) Grassmannian $Gr(2,4)$ e.t.c.

Symplectic curvature tensor

(M, ω) symplectic manifold

∇ symplectic connection (no uniqueness)

$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$

$$S(X,Y,Z,U) := \omega(R(X,Y)Z, U)$$

for $X, Y, Z, U \in \Gamma(M, TM)$. (different from Vaisman)

Symmetries of the symplectic curvature tensor S

1.) $S(X,Y,Z,U) = -S(Y,X,Z,U)$

2.) $S(X,Y,Z,U) = S(X,Y,U,Z)$

3.) $S(X,Y,Z,U) + S(Y,Z,X,U) + S(Z,X,Y,U) = 0$
(Bianchi)

Symplectic Ricci tensor

$$sRic(X,Y) := S(e_j, X, Y, e_i)\omega^{ij}$$

(Einstein summation convention)

$$\begin{aligned}\widetilde{sRic}(X,Y,Z,U) &:= \frac{1}{2l+2}(\omega(X,Z)sRic(Y,Z) - \\ -\omega(X,U)sRic(Y,Z) &- \omega(Y,Z)sRic(X,U) - \\ -\omega(Y,U)sRic(X,Z) &+ 2\omega(X,Y)sRic(Z,U))\end{aligned}$$

In general, we define $\tilde{T}$ for each $(2,0)$-covariant tensor (field) T.

Symplectic Weyl tensor

$$sW(X,Y,Z,U) := S(X,Y,Z,U) - \widetilde{sRic}(X,Y,Z,U)$$

Theorem (Vaisman): Let $\mathcal{C} \subseteq \bigotimes^4 \mathbb{V}$ be a subspace satisfying $(1), (2)$ and (3). Then $\mathcal{C} = \mathcal{C}^0 \oplus \mathcal{C}^r$ is an $Sp(\mathbb{V}, \omega)$-irreducible decomposition, where

$$\mathcal{C}^0 := \{T \in \mathcal{C} | \widetilde{T} = 0\}$$
$$\mathcal{C}^r := \{T \in \mathcal{C} | \exists K \in \bigotimes^2 \mathbb{V}, T = \widetilde{K}\}.$$

Remark: No nontrivial inner (=symplectic) traces.

Theorem: (M, ω) symplectic manifolds admitting a metaplectic structure Λ and ∇ a symplectic connection. $\mathcal{S}$ symplectic spinor bundle d^{∇^S} symplectic spinor exteriror derivative associated to ∇. Then for each $(i, j) \in \Xi$ we have

$$d^{\nabla^S} : \Gamma(M, \mathcal{E}^{ij}_{\pm}) \to \Gamma(M, \mathcal{E}^{i+1,j-1}_{\pm} \oplus \mathcal{E}^{i+1,j}_{\pm} \oplus \mathcal{E}^{i+1,j+1}_{\pm}),$$

where $\mathcal{E}^{ij}_{\pm}$ is the associated bundle to the principal $Mp(\mathbb{R}, 2l)$-bundle via the representation $\mathbf{E}^{ij}_{\pm}$ of $\tilde{G}$.

Back to the picture.

Symplectic Killing spinors

(M, ω) symplectic manifold admitting a metaplectic structure

$$\nabla^S \phi = \lambda F^+ \phi,$$

$\phi \in \Gamma(\mathcal{S}, M) \Longrightarrow$ call ϕ symplectic Killing spinor. λ is called symplectic Killing number. Equivalently,

$$\nabla^S_X \phi = \lambda X.\phi$$

for each $X \in \Gamma(TM, M)$.

Example: $(\mathbb{R}^2, \omega_0)$. Symplectic Killing spinor equation equivalent to

$$\frac{\partial \psi}{\partial t} = \lambda \frac{\partial \psi}{\partial x}$$

$$\frac{\partial \psi}{\partial s} = \lambda \imath x \psi,$$

where

$\psi : \mathbb{R}^3 \to \mathbb{C}$ such that $(\mathbb{R} \ni x \mapsto \psi(s,t,x)) \in \mathcal{S}(\mathbb{R})$.

Then the symplectic Killing spinor is a constant, i.e., there exists $f \in \mathcal{S}(\mathbb{R})$ such that for each $(s,t) \in \mathbb{R}^2$ we have $\phi(s,t) := f$.

Remark: The same is true for $\mathbb{R}^2$ and the standard Euclidean structure in the Riemannian spin-geometry.

Use of symplectic Killing spinor

- Sepectra embedding (Obstruction to a linear embedding of the spectrum of the symplectic Dirac into the spectrum of the symplectic Rarita-Schwinger operator.)

- Existence of a (nontrivial) symplectic Killing spinor $\Rightarrow$ symmetry.

Symplectic Dirac operator

$$\mathfrak{D}_1 := -F^- \circ D_1$$

Symplectic Rarita-Schwinger operator

$$\mathfrak{R}_1 := -F^- \circ R_1$$

$$
\begin{array}{ccccc}
\mathbb{E}_{00}^{+} & \xrightarrow{D_1} & \mathbb{E}_{10}^{+} & & \\
& \searrow^{T_1} & & \searrow^{T_2} & \\
& & \mathbb{E}_{11}^{+} & \xrightarrow{R_1} & \mathbb{E}_{21}^{+}
\end{array}
$$

Theorem: (M, ω) symplectic manifold of dimension $2l$ admitting a metaplectic structure Λ. Let ∇ be a Weyl flat symplectic connection.

1.) If $\lambda \in \mathrm{Spec}(\mathfrak{D})$ and $-\imath l\lambda$ not a symplectic Killing number. Then $\frac{l-1}{l}\lambda \in \mathrm{Spec}(\mathfrak{R})$.

2.) If ϕ is an eigenvector of $\mathfrak{D}$ and not a symplectic Killing spinor. Then ϕ is an eigenvector of $\mathfrak{R}$.

Existence of symplectic Killing spinor $\Rightarrow$ rigidity (symmetry) of (M, ω, ∇).

Lemma: If ψ is a symplectic Killing spinor, which is not identically zero. Then ψ is nowhere zero.

Proof. Method of characteristics.

Theorem: (M, ω) Weyl-flat symplectic manifold, Λ metaplectic structure, ∇ symplectic connecton, ψ nonzero symplectic Killing spinor with constant energy, i.e., $H^{sRic}\psi = \tilde{\lambda}\psi$ for a $\tilde{\lambda} \in \mathbb{C}$. Then (M, ω, ∇) is flat, i.e., $R^{\nabla} = 0$.

$H^{sRic}(\psi) := \frac{1}{2} sRic_{ij} e_i . e_j . \psi$ (energy),