Highlights

Families of elliptic complexes in symplectic spinor geometry

- Elliptic complexes has usually convenient analytic features as is the existence of a parametrix
- We gain elliptic complexes on infinite rank bundles induced by symplectic connections that need not be flat
- Obtained complexes may serve as objects for an analytically or homologically oriented research in the field of complexes on infinite rank bundles

Families of elliptic complexes in symplectic spinor geometry

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Abstract

For a symplectic manifold admitting a metaplectic structure and equipped with a symplectic Weyl-flat connection, we introduce complexes of symplectic twistor operators and prove that they are elliptic.

Keywords: Elliptic complex, Segal–Shale–Weil representation, symplectic spinor representation, Fedosov manifold, symplectic Weyl-flat connection, symplectic Ricci-type connection, metaplectic structure
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1. Introduction

Twistor operators, known from Riemannian and pseudo-Riemannian geometry, are first order differential operators having origin in the twistor theory established by R. Penrose. See Penrose [30] and Penrose, Rindler [31]. In the article, we study their symplectic counterparts, namely the *symplectic twistor operators*, defined for those symplectic manifolds (M, ω) that admit a metaplectic structure, a symplectic analogue of the spin structure, introduced by Kostant in [21]. In contrary to the Riemannian and pseudo-Riemannian twistor operators, the symplectic ones are defined on sections of vector bundles of infinite rank. The vector bundles are associated to the total space of the metaplectic structure using the *symplectic spinor representation* ρ of the connected double cover $\widetilde{G}$ of the symplectic group G on a separable Hilbert space, that is denoted by E in our paper. The symplectic spinor representation (ρ, E) is also called the metaplectic, Kostant's, Segal–Shale–Weil or the oscillator representation. (See Borel, Wallach [3] and Habermann, Habermann [17].) In particular, $\rho : \widetilde{G} \to U(E)$ is a homomorphism of $\widetilde{G}$, called the metaplectic group, into the group of unitary maps on E. It is known that E

decomposes as a representation of $\widetilde{G}$ into the orthogonal sum $E_+ \oplus E_-$ of two inequivalent irreducible subrepresentations.

Let (M, ω) be a $2n$-dimensionional symplectic manifold, (V, ω_0) be a real symplectic vector space of the same dimension and $G \subseteq GL(V)$ be the symplectic group of (V, ω_0). In our article the symplectic twistor operators are defined with the help of tensor products of the symplectic spinor representation with natural representations of the metaplectic group $\widetilde{G}$ on the wedge products $\bigwedge^i V^*$, that are induced by the covering homomorphism $\lambda : \widetilde{G} \to G$. In this way we get a representation on E-valued antisymmetric i-forms, i.e., on $E_i = \bigwedge^i V^* \otimes E$. As a representation of $\widetilde{G}$, each E_i decomposes into irreducible $\widetilde{G}$-submodules, denoted by $E_{ij,\pm}$. The representations of $\widetilde{G}$ on E_i and on $E_{ij} = E_{ij,+} \oplus E_{ij,-}$ are associated to the total space of the metaplectic structure and the resulting vector bundles are denoted by $\mathcal{E}_i$ and $\mathcal{E}_{ij}$, respectively.

If ∇ is a symplectic connection on a symplectic manifold (M, ω), it induces the exterior covariant derivatives $\nabla_i : \Gamma(\mathcal{E}_i) \to \Gamma(\mathcal{E}_{i+1})$ on M. Let us denote the restrictions to $\Gamma(\mathcal{E}_{ij})$ of ∇_i by ∇_{ij}. In particular, ∇_{ij} maps $\Gamma(\mathcal{E}_{ij})$ into $\Gamma(\mathcal{E}_{i+1})$. The symplectic twistor operators $T_{ij}^{\pm} : \Gamma(\mathcal{E}_{ij}) \to \Gamma(\mathcal{E}_{i+1,j\pm 1})$ are defined as compositions of ∇_{ij} with the projections of $\Gamma(\mathcal{E}_{i+1})$ onto $\Gamma(\mathcal{E}_{i+1,j\pm 1})$. Let us recall that if ∇ is torsion-free, the triple (M, ω, ∇) is called a Fedosov manifold. It is known that in this case, the symplectic twistor operators form complexes if, in addition, ∇ is symplectic Weyl-flat (also called symplectic Ricci-type). See Krýsl [22, 27]. We remark that this type of connections is investigated by Vaisman [39], Cahen, Gutt, Schwachhöfer [7] and Bieliavsky et al. [2], among others.

In the article, we consider two families of complexes, C_k^+, $k = 0, \ldots, 2n-2$, and C_k^-, $k = 2, \ldots, 2n$, consisting of the symplectic twistor operators T_{ij}^+ and T_{ij}^-, respectively, and prove that these complexes are elliptic. We note that the symplectic twistor operators are examples of generalized gradients, whose ellipticity is investigated intensively. See, e.g., [4], [12], [18], [32], and [34]. Let us recall, that by an elliptic complex, a complex of differential operators is meant, whose principal symbol sequence is exact for all non-zero cotangent vectors of the base manifold.

Complexes $C_0^+ = (T_{jj}^+)_{j=-1}^{n}$ and $C_{2n}^- = (T_{n,n-j}^-)_{j=0}^{n+1}$ are proven to be elliptic in [23]. Since the formulas for symbols of the differential operators already in these complexes are too complicated, we search rather for relations among the symbols. Let m be a point in M. If the symbols are considered as maps defined on the tensor products $T_m^*M \otimes (\mathcal{E}_{ij})_m$, they can be viewed as $\widetilde{G}$-equivariant. (See Section 3.3 for details.) Thus it seems natural to use the Schur lemma for irreducible representations, to gain the mentioned relations. Let us note that these relations contain complex constants, whose precise determination is not necessary for the proof of the ellipticity. However, we have to specify when these constants are non-zero. This is done using two $\widetilde{G}$-equivariant operators $F^+ : \bigoplus_{i\in\mathbb{Z}} E_i \to \bigoplus_{i\in\mathbb{Z}} E_i$ and $F^- : \bigoplus_{i\in\mathbb{Z}} E_i \to \bigoplus_{i\in\mathbb{Z}} E_i$, which "rise" or "lower" the form-degree of the elements in $\bigoplus_{i\in\mathbb{Z}} E_i$, respectively.

In Section 2 and in the introductory part of Section 3, we set definitions and recall known results, as is the decomposition of E_i into irreducible $\widetilde{G}$-subrepresentations, together with a characterization of restrictions of the maps $F^{\pm}$ to the $\widetilde{G}$-subrepresentations E_{ij} (Lemma 1). In Section 3.1 we present a summary on symplectic manifolds that admit symplectic Weyl-flat connections. This summary contains a description of those manifolds that are, in addition, geodesically complete and posses a compatible Kähler structure. In Section 3.2 we examine, using characteristic classes, which of these manifolds admit a metaplectic structure. In Section 3.3 the mentioned families, $(C_k^+)_{k=0}^{2n-2}$ and $(C_k^-)_{k=2}^{2n}$, are defined. Their ellipticity (Theorem 2) is proved with the help of Lemma 2 on the relations between the symbols. Let us note that in [8], the ellipticity of a specific class of complexes is proved using representation theory and homological algebra.

In the future we would like to study properties of the cohomology groups of complexes of symplectic twistor operators in a connection to the homology of the base manifold (M, ω). See [26], where the de Rham differential tensored by a product connection on the bundle of symplectic spinors is considered. Let us remark that the class of symplectic manifolds equipped with a flat Fedosov connection is used in the BRST- and in the deformation quantization. See, e.g., Cattaneo et al. [9], Kontsevich [20], and the recent paper of Chan et al. [10]. The broader class of symplectic Weyl-flat connections thus seem to be of interest for these quantization procedures.

2. Symplectic spinor representation and symplectic spinor-valued forms

For a real symplectic vector space (V, ω_0) of dimension $2n$, we denote the symplectic group $Sp(V, \omega_0)$ by G, its connected double cover - the metaplectic group $Mp(V, \omega_0)$ - by $\widetilde{G}$, and the appropriate covering homomorphism of the symplectic group by the metaplectic group by λ. In particular, $\lambda : \widetilde{G} \to G$ is a $2:1$ covering map and a Lie group homomorphism. See [3, 33]. Since G is a subgroup of the group $GL(V)$, λ is also a representation of $\widetilde{G}$ on V.

Let us choose a Lagrangian (i.e., maximal isotropic) subspace L of the symplectic vector space (V, ω_0) and a compatible complex structure $J_0 : V \to V$. Thus J_0 is a symplectic map such that the bilinear form $g_0(u,v) = \omega_0(J_0 u, v)$, $u, v \in V$, is positive definite. We denote the Hilbert space $L^2(L)$ of complex valued square Lebesgue integrable functions defined on the space $(L, g_{0|L\times L})$ considered modulo the equality of almost everywhere by E, and the group of linear unitary maps of E by $U(E)$. The *symplectic spinor* representation is a faithful unitary representation $\rho : \widetilde{G} \to U(E)$ of the metaplectic group $\widetilde{G}$ on E. It is known that this representation is a direct sum of two inequivalent irreducible representations ρ_+ and ρ_- of $\widetilde{G}$, which are representation restrictions of ρ to the subspaces E_+ and E_- of E, that consist of elements represented by even and odd functions, respectively. See [3]. The elements of E are called symplectic spinors.

The symplectic Clifford multiplication is defined as a bilinear map $\cdot : V \times E \to E$ in [17]. In [6, 17] it is proved that this bilinear map is $\widetilde{G}$-equivariant in the sense that $\rho(g)(v \cdot s) = \lambda(g)v \cdot \rho(g)s$ for $g \in \widetilde{G}$, $v \in V$ and $s \in S$.

2.1. Symplectic spinor-valued forms - representation theoretic background

Since λ is a representation of $\widetilde{G}$ on the vector space V, it induces the dual representation λ^* of $\widetilde{G}$ on V^*. Using the symplectic form ω_0, it is easy to see that the dual representation is equivalent to λ and therefore we may identify them. For $i \in \mathbb{Z}$, let us denote the i-th antisymmetric power of λ by $\lambda^{\wedge i} : \widetilde{G} \to GL(\bigwedge^i V)$, where we consider $\bigwedge^i V = 0$ if $i \geq 2n+1$ or $i \leq -1$. Let us set $E_{i,\pm} = \bigwedge^i V \otimes E_\pm$ and $E_i = \bigwedge^i V \otimes E$ ($\simeq E_{i,+} \oplus E_{i,-}$) and equip these vector spaces with the tensor product norms. Elements of E_i are called symplectic spinor-valued forms. The complete topological vector spaces E_i are equipped with the tensor product representations $\rho_i : \widetilde{G} \to GL(E_i)$ determined by the prescription $\rho_i(g)(w \otimes s) = \lambda^{\wedge i}(g)w \otimes \rho(g)(s)$, where $g \in \widetilde{G}$, $w \in \bigwedge^i V$ and $s \in E$.

A decomposition of the representations (ρ_i, E_i) into irreducible representations is known. See Krýsl [24, 25]. Namely, if we set $k_{n,i} = n - |n-i|$ for

$$
\begin{array}{ccccccccc}
E_0 & E_1 & E_2 & E_3 & E_4 & E_5 & E_6 & E_7 & E_8 \\
E_{00} & E_{10} & E_{20} & E_{30} & E_{40} & E_{50} & E_{60} & E_{70} & E_{80} \\
 & E_{11} & E_{21} & E_{31} & E_{41} & E_{51} & E_{61} & E_{71} & \\
 & & E_{22} & E_{32} & E_{42} & E_{52} & E_{62} & & \\
 & & & E_{33} & E_{43} & E_{53} & & & \\
 & & & & E_{44} & & & &
\end{array}
$$

Figure 1: Decomposition structure for $n = 4$

$i = 0, \ldots, 2n$, this decomposition is

$$E_{i,\pm} \simeq \bigoplus_{j=0}^{k_{n,i}} E_{ij,\pm}$$

where $E_{ij,\pm}$ are non-zero irreducible $\widetilde{G}$-modules for each $j = 0, \ldots, k_{n,i}$. We set also $E_{ij} = E_{ij,+} \oplus E_{ij,-}$. See Figure 1 where the $\widetilde{G}$-subrepresentations E_{ij} forming E_i appear in the columns. For instance, we have $E_0 \simeq E_{00}$ and $E_6 \simeq E_{60} \oplus E_{61} \oplus E_{62}$ if $n = 4$.

Let us note that for each $i = 0, \ldots, 2n$, the $\widetilde{G}$-representation (ρ_i, E_i) is multiplicity-free, i.e., if τ and τ' are different irreducible subrepresentations in E_i, they are not equivalent as representations. See [24, 25].

We set $K = \{(i,j) | \, i = 0, \ldots, 2n, j = 0, \ldots, k_{n,i}\}$. Note that this set equals to $\{(i,j) \in \mathbb{Z} \times \mathbb{Z} | \, i, j \geq 0, j \leq i \text{ and } i + j \leq 2n\}$, that we call the equivalent description of K. For $(i,j) \in K$, we define $p_{ij,\pm}$ to be the unique $\widetilde{G}$-equivariant projection of E_i onto $E_{ij,\pm}$ and set $p_{ij} = p_{ij,+} + p_{ij,-} : E_i \to E_{ij}$. If $(i,j) \in (\mathbb{Z} \times \mathbb{Z}) \setminus K$, we consider $E_{ij,\pm}$ and E_{ij}, and $p_{ij,\pm}$ and p_{ij} to be the trivial spaces and the trivial homomorphisms, respectively. For $(i,j) \in \mathbb{Z} \times \mathbb{Z}$, let us denote the representations of $\widetilde{G}$ restricted to $E_{ij,\pm}$ by $\rho_{ij,\pm}$ and the ones restricted to $E_{ij} = E_{ij,+} \oplus E_{ij,-}$ by ρ_{ij}.

For $v \in V$, $w \in \bigwedge^i V$ and $s \in E$, let us consider the homogeneous element $\phi = w \otimes s \in E_i$, set $v \wedge \phi = (v \wedge w) \otimes s \in E_{i+1}$, and extend this operation linearly to all elements in E_i. We define the wedge product

$$V \wedge E_{ij,\pm} = \langle \{ v \wedge \phi | \, v \in V, \phi \in E_{ij,\pm} \} \rangle$$

where the brackets $\langle \, , \rangle$ denote the complex linear span. Let us set $V \wedge E_{ij} = (V \wedge E_{ij,+}) \oplus (V \wedge E_{ij,-})$. Thus for instance, $V \wedge E_{00} = V \otimes E_{00} = V \otimes E$.

We note that $V \wedge E_{ij} \simeq (V \otimes E_{ij}) \cap (\bigwedge^{i+1} V \otimes E)$. Further if $B : V \to V$ and $C : E_k \to E_l$ are linear maps, $v \in V$ and $\phi \in E_k$, we set $(B \wedge C)(v \wedge \phi) = B(v) \wedge C(\phi)$, and extend this prescription to $V \wedge E_k$ linearly. In this way, we obtain a linear map $B \wedge C : V \wedge E_k \to V \wedge E_l$.

Let $(e_i)_{i=1}^{2n}$ be a symplectic basis of (V, ω_0) ([17]) and let us consider the real numbers ω^{ij}, $i, j = 1, \dots, 2n$, that are uniquely determined by the formula $\sum_{k=1}^{2n} \omega_{ik}\omega^{jk} = \delta_i^j$, where $\omega_{ij} = \omega_0(e_i, e_j)$, i.e., $(\omega^{ij})_{i,j=1,\dots,2n}$ parametrizes the inverse transpose matrix to the matrix $(\omega_{ij})_{i,j=1,\dots,2n}$. We introduce the following form-degree rising and form-degree lowering operators $F^+ : \bigoplus_{i\in\mathbb{Z}} E_i \to \bigoplus_{i\in\mathbb{Z}} E_i$ and $F^- : \bigoplus_{i\in\mathbb{Z}} E_i \to \bigoplus_{i\in\mathbb{Z}} E_i$ by setting

$$F^+(w \otimes s) = \sum_{j=1}^{2n} \epsilon^j \wedge w \otimes e_j \cdot s \text{ and}$$

$$F^-(w \otimes s) = \sum_{j,k=1}^{2n} \omega^{jk} \iota_{e_j} w \otimes e_k \cdot s$$

for a homogeneous element $w \otimes s \in \bigwedge^i V \otimes E$, where ι_{e_j} denotes the insertion of vector e_j. The above formulas for $F^\pm$ are extended linearly to $\bigoplus_{i\in\mathbb{Z}} E_i$. In the second formula, it is assumed that we first dualize the antisymmetric multivector w by ω_0, getting an antisymmetric form $w^\flat$, insert e_j, and then dualize the wedge form $\iota_{e_j} w^\flat$ back, getting the multivector $(\iota_{e_j} w^\flat)^\sharp$ in $\bigwedge^{i-1} V$.[1] Further we define $F^\pm_{ij,\pm}$ to be the restriction to $E_{ij,\pm}$ of $F^\pm$ (each combination of $+$ and $-$ is considered), and set $F^\pm_{ij} = F^\pm_{ij,+} + F^\pm_{ij,-} : E_{ij} \to E_{i\pm1,j}$.

In the next lemma, we recall a decomposition of the wedge product $V \wedge E_{ij,\pm}$ into irreducible subrepresentations and the $\widetilde{G}$-isomorphism structure of the modules $E_{ij,\pm}$ with respect to a varying form-degree index i and a fixed index j. See Fig. 1.

Lemma 1. *For each couple of integers i, j,*

$$V \wedge E_{ij,\pm} \simeq E_{i+1,j-1,\pm} \oplus E_{i+1,j,\pm} \oplus E_{i+1,j+1,\pm}.$$

If $(i, j) \in K \setminus \{(k,l) | \, k + l = 2n\}$, the map $F^+_{ij,\pm} : E_{ij,\pm} \to E_{i+1,j,\mp}$ is a $\widetilde{G}$-equivariant isomorphism; and if $(i, j) \in K \setminus \{(k,k) | \, k = 0, \dots, n\}$, the map $F^-_{ij,\pm} : E_{ij,\pm} \to E_{i-1,j,\mp}$ is also a $\widetilde{G}$-equivariant isomorphism.

[1] For $w \in V$ and $\alpha \in V^*$, we set $w^\flat(v) = \omega_0(w, v)$ and $\omega_0(\alpha^\sharp, v) = \alpha(v)$ for all $v \in V$ and extend it multilinearly to wedge forms and antisymmetric multivectors.

Proof. The $\widetilde{G}$-equivariance follows from the $\widetilde{G}$-equivariance of the symplectic Clifford multiplication. See Krýsl [24, 25] for the remaining statements. □

The two sets subtracted from the set K in the above lemma correspond to the right-hand and left-hand edges of the triangle in Figure 1 ($n = 4$).

3. Complexes of symplectic twistor operators

Let (M, ω) be a symplectic manifold and ∇ be a symplectic connection, i.e., an affine connection on M whose induced covariant derivative on exterior differential 2-forms maps the symplectic 2-form ω to zero. Moreover, if ∇ is torsion-free, it is called a *Fedosov connection* and the symplectic manifold with such a connection is called a *Fedosov manifold*. Let us remark that a Fedesov connection exists for any symplectic manifold and that the affine space of Fedosov connections is modeled on the vector space $\Gamma(S^3TM)$, where S^3TM denotes the third symmetric power of the tangent bundle TM. See Tondeur [37] and Gelfand, Retakh and Shubin [16].

Any symplectic connection induces, in particular, the curvature tensor field, denoted by R^∇, and the symplectic Weyl curvature tensor field. See Vaisman [39].

Definition 1. *A Fedosov manifold (M, ω, ∇) is called Weyl-flat if the symplectic Weyl curvature tensor field of ∇ vanishes.*

If (M, ω, ∇) is a Weyl-flat Fedosov manifold, ∇ is called *symplectic Weyl-flat*. Let us note that such connections are also called symplectic curvature reducible (Vaisman [39]) or symplectic Ricci-type (Bieliavsky et al. [2] and Cahen et al. [7]).

3.1. Sources of Weyl-flat Fedosov manifolds

1) It is known that each Weyl-flat Fedosov manifold (M^{2n}, ω, ∇) arises locally from the canonical symplectic vector space $(\mathbb{R}^{2n+2}, \omega_0)$ equipped with the standard flat connection ∇_0 on $\mathbb{R}^{2n}$ by a reduction procedure. See [2, 7].

2) Conditions for *bipolarized* and *bi-Lagrangian* manifolds with their canonical connections to be symplectic Weyl-flat are expressed by partial differential equations in [39]. Let us note that each bi-Lagrangian manifold is

bipolarized, and that a symplectic manifold is bi-Lagrangian if and only if it is para-Kähler (Alekseevskii et al. [1]). See also [5, 8, 13] for a description and examples of bi-Lagrangian/para-Kähler manifolds.

3) Let us assume that a Fedosov manifold (M^{2n}, ω, ∇) admits a compatible Kähler structure, i.e., there exists a complex structure J on M, which together with ω induce a hermitian metric on the complexification of TM that is preserved by ∇ and whose fundamental form is equal to ω. In this case, it is known that ∇ is symplectic Weyl-flat if and only if its holomorphic sectional curvature is constant. See [39]. If, in addition, (M, ω, ∇) is geodesically complete, a classification of such manifolds is well known. (See Thm. 8.7 in [28] or Sect. 7, Ch. IX in [19].) Namely, they are covered by the complex projective space $\mathbb{C}P^n$ with a positive multiple of the Fubini–Study metric, by the complex plane $\mathbb{C}^n$ with the canonical Euclidean metric or by the open unit ball $B^{2n} \subseteq \mathbb{C}^n$ with a positive multiple of the complex hyperbolic metric. Moreover, the covering map is a holomorphic isometry.

4) Let us remark that there exists a flat ($R^\nabla = 0$) Fedosov connection ∇ on the Kodaira–Thurston manifold. See Fox [15]. This gives an example of a Weyl-flat Fedosov manifold out of the category of Kähler manifolds.

3.2. Metaplectic structures

For a symplectic manifold (M, ω), let us consider the set of linear maps

$$\mathcal{Q} = \{A : V \to T_m M \,|\, \omega_m(Au, Av) = \omega_0(u, v) \text{ for } u, v \in V, m \in M\}$$

which can be thought of as the set of symplectic frames on M. We equip it with the right action of $G = Sp(V, \omega_0)$ given by $(A \cdot g)(v) = A(g(v))$, where $v \in V$ and $g \in G$, and with a natural induced topology and a bundle atlas. (See, e.g., [35]). Setting $\pi_Q(A) = m$ if and only if $A : V \to T_m M$ for $A \in \mathcal{Q}$, makes $\pi_Q : \mathcal{Q} \to M$ a principal G-bundle on M.

Definition 2. *Let $\pi_P : \mathcal{P} \to M$ be a principal $\widetilde{G}$-bundle and $\Lambda : \mathcal{P} \to \mathcal{Q}$ be a morphism of fibre bundles such that $\Lambda(B \cdot g) = \Lambda(B) \cdot \lambda(g)$ for all $B \in \mathcal{P}$ and $g \in \widetilde{G}$. Then $(\pi_P : \mathcal{P} \to M, \Lambda)$ is called a metaplectic structure.*

It is known that (M, ω) admits a metaplectic structure if and only if the second Stiefel–Whitney class $w_2(TM)$ of the tangent bundle to M vanishes, that holds if and only if the first Chern class of TM equipped with a compatible almost complex structure is even, i.e., divisible by 2 in $H^2(M,\mathbb{Z})$. See Forger, Hess [14] or Robinson, Rawnsley [33].

Examples: We examine the examples of Weyl-flat Fedosov manifolds mentioned in the Section 3.1 concerning the existence of a metaplectic structure.

1) If M is contractible, the second cohomology group $H^2(M,\mathbb{Z})$ vanishes and consequently, (M,ω) admits a metaplectic structure.

2) The second Stiefel–Whitney class of $\mathbb{C}P^{2k+1}$ is known to be zero and thus $\mathbb{C}P^{2k+1}$ admits a metaplectic structure. If M is covered by $Y \in \{\mathbb{C}P^{2k+1}, \mathbb{C}^n, B^{2n}\}$, the second Stiefel–Whitney class of TM has to be zero as well. Indeed, we have $p^*(w_2(TM)) = w_2(TY) = 0$ by the naturality of characteristic classes, where $p: Y \to M$ is the covering map. Since p^* is injective, $w_2(TM)$ is zero. However, the second Stiefel–Whitney class of $\mathbb{C}P^{2k}$ is non-zero. (See, e.g., [29].) If $p: \mathbb{C}P^{2k} \to M$ is a covering map for M, we have $p^*(w_2(TM)) = w_2(T\mathbb{C}P^{2k})$ and thus $w_2(TM) \neq 0$. Consequently, a manifold M covered by $\mathbb{C}P^{2k}$ does not admit any metaplectic structure, though it admits a symplectic Weyl-flat connection.

3) Since the Kodaira–Thurston manifold is diffeomorphic to the quotient of a Lie group by a discrete subgroup (see [38]), it is parallelizable. Consequently, the second Stiefel–Whitney class and the first Chern class of the Kodaira–Thurston manifold are zero and thus this manifold admits a metaplectic structure.

3.3. Complexes of symplectic twistor operators and their ellipticity

In order to define symplectic twistor operators for a Fedosov manifold (M,ω,∇), let us suppose that (M,ω) admits a metaplectic structure and consider the associated vector bundles $\mathcal{E}_{ij,\pm} = \mathcal{P} \times_{\rho_{ij,\pm}} E_{ij,\pm}$, $\mathcal{E}_{ij} = \mathcal{E}_{ij,+} \oplus \mathcal{E}_{ij,-}$, and $\mathcal{E}_i = \mathcal{P} \times_{\rho_i} E_i \simeq \bigoplus_{j=0}^{k_{n,i}} \mathcal{E}_{ij}$ for $i,j \in \mathbb{Z}$. Let us recall that $\mathcal{E}_{ij,\pm} = 0$ if $(i,j) \notin K$. Note that the bundles $\mathcal{E}_i$ are continuous but not smooth in general. See Habermann, Habermann [17] and Krýsl [26]. Thus, when we speak about smooth sections of these bundles, we mean the vector spaces $C^\infty(\mathcal{P}, E_i)^{\widetilde{G}}$ of smooth $\widetilde{G}$-equivariant maps with values in the normed space $E_i = \bigwedge^i V \otimes E$ considered with the tensor product norm. We denote them by $\Gamma(\mathcal{E}_i)$ and keep calling them the smooth sections spaces. A similar remark applies to $\mathcal{E}_{ij}$ as well.

Any symplectic connection ∇ induces the exterior covariant derivatives $\nabla_i : \Gamma(\mathcal{E}_i) \to \Gamma(\mathcal{E}_{i+1})$, where $i \in \mathbb{Z}$. For each pair of integers i, j, we define ∇_{ij} to be the restriction to $\Gamma(\mathcal{E}_{ij})$ of ∇_i. The image of ∇_{ij} is a vector subspace of $\Gamma(\mathcal{E}_{i+1,j-1} \oplus \mathcal{E}_{i+1,j} \oplus \mathcal{E}_{i+1,j+1})$. See [24]. The unique $\widetilde{G}$-equivariant projections p_{ij} and $p_{ij,\pm}$ of E_i onto E_{ij} and onto $E_{ij,\pm}$ induce projections of $\mathcal{E}_i$ onto $\mathcal{E}_{ij}$ and onto $\mathcal{E}_{ij,\pm}$, and projections of $\Gamma(\mathcal{E}_i)$ onto $\Gamma(\mathcal{E}_{ij})$ and onto $\Gamma(\mathcal{E}_{ij,\pm})$, respectively. We denote of all these projections also by p_{ij} and $p_{ij,\pm}$, respectively.

In a parallel to the Riemannian and Lorentzian twistor operators, we set the following definition. For uniformity, all couples (i, j) of integers are considered.

Definition 3. *The operators $T^{\pm}_{ij} = p_{i+1,j\pm1} \circ \nabla_{ij} : \Gamma(\mathcal{E}_{ij}) \to \Gamma(\mathcal{E}_{i+1,j\pm1})$ are called the symplectic twistor operators.*

We have the following

Theorem 1. *For a Weyl-flat Fedosov manifold (M, ω, ∇) equipped with a metaplectic structure and for each couple of integers i, j, the sequence $\left(\Gamma(\mathcal{E}_{i+k,j\pm k}), T^{\pm}_{i+k,j\pm k}\right)_{k\in\mathbb{Z}}$ is a complex.*

Proof. See Krýsl [27]. □

Remark: To get an example of a complex described in the above theorem, we advise the reader to choose a couple (i, j) from the set K for $n = 3$ and follow Fig. 2.

For a vector bundle $\mathcal{F} \to M$, the fibre $p^{-1}(\{m\})$ of $\mathcal{F}$ in $m \in M$ is denoted by $\mathcal{F}_m$. It is easy to realize that the principal symbol map $\sigma^{\pm}_{ij}(\xi) : (\mathcal{E}_{ij})_m \to (\mathcal{E}_{i+1,j\pm1})_m$ in the covector $\xi \in T^*_m M \simeq T_m M$ of the operator $T^{\pm}_{ij}$ is given by $\sigma^{\pm}_{ij}(\xi)(\phi) = p_{i+1,j\pm1}(\xi \wedge \phi)$, where $\phi \in (\mathcal{E}_{ij})_m$ and $(i, j) \in \mathbb{Z} \times \mathbb{Z}$. Choosing a bundle and a base manifold charts around $m \in M$, this symbol map can be viewed as a linear map of E_{ij} into $E_{i+1,j\pm1}$, i.e., $\sigma^{\pm}_{ij}(v)(s) = p_{i+1,j\pm1}(v \wedge s)$, where $v \in V$ and $s \in E_{ij,\pm}$. However, this map is not $\widetilde{G}$-equivariant unless

$v = 0$ or the projection $p_{i+1,j\pm 1}$ is the trivial map. To prove the ellipticity, it is convenient to consider the maps

$$\sigma_{ij}^{\pm} : V \wedge E_{ij} \to E_{i+1,j\pm 1}$$

defined as the restrictions to $V \wedge E_{ij} \subseteq E_{i+1}$ of the projections $p_{i+1,j\pm 1} : E_{i+1} \to E_{i+1,j\pm 1}$, i.e., as maps determined by the formula

$$\sigma_{ij}^{\pm}(v \wedge s) = p_{i+1,j\pm 1}(v \wedge s)$$

for homogeneous elements in $V \wedge E_{ij}$, which we extend to $V \wedge E_{ij}$ linearly. These maps already are $\widetilde{G}$-equivariant and fulfill $\sigma_{ij}^{\pm}(v)(s) = \sigma_{ij}^{\pm}(v \wedge s)$. We hope that this does not cause confusion since the meaning of the maps will be clear from the context. We shall often write 'symbol' and 'symbol map' meaning always 'principal symbol' and 'principal symbol map', respectively.

Let us introduce the families $\{C_i^+ \,|\, i = 0, \ldots, 2n\}$ and $\{C_i^- \,|\, i = 0, \ldots, 2n\}$ of sequences of symplectic twistor operators, both of which consist of $2n+1$ sequences. They are subsequences of the sequences considered in the Theorem 1. The floor of a real number q is denoted by $\lfloor q \rfloor$.

Definition 4. *For $i = 0, \ldots, 2n$, let us set*

$$C_i^+ = \left(\Gamma(\mathcal{E}_{i+j,j}), T_{i+j,j}^+\right)_{-1 \le j \le \lfloor \frac{2n-i}{2} \rfloor}$$

and

$$C_i^- = \left(\Gamma(\mathcal{E}_{\lfloor \frac{i+1}{2} \rfloor + j, \lfloor \frac{i}{2} \rfloor - j}), T^-_{\lfloor \frac{i+1}{2} \rfloor + j, \lfloor \frac{i}{2} \rfloor - j}\right)_{0 \le j \le \lfloor \frac{i}{2} \rfloor + 1},$$

and call them the symplectic twistor complexes.

Remark: 1) Setting $j = -1$ in the above definition, we see that the first member of C_i^+ i.e., $(\Gamma(\mathcal{E}_{i-1,-1}), T_{i-1,-1}^+)$, is the null space and the trivial operator. Setting $j = \lfloor \frac{i}{2} \rfloor + 1$, we get that the last member of C_i^- is the null space and the trivial operator as well.

2) Complexes C_{2n-1}^+, C_{2n}^+, C_0^-, and C_1^- consist of sequences with two elements only and one of these elements is always the null space. Consequently, the differentials in these complexes are trivial and hence we do not consider their ellipticity.

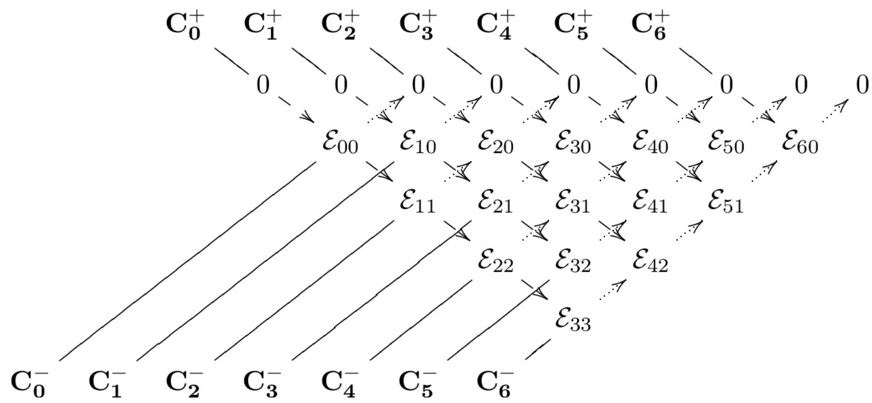

Figure 2: Elliptic complexes for $n = 3$

3) Note that $\Gamma(\mathcal{E}_{ij})$ belongs to $C_k^+ \cup C_k^-$ for some k if and only if $(i,j) \in K$ or $i = -1, \ldots, 2n+1$ and $j = -1$, that holds if and only if $(i,j) \in \{(r,s) \in \mathbb{Z} \times \mathbb{Z} |\, r, s \geq -1, s \leq r \text{ and } r + s \leq 2n\}$. See Figure 2.

4) Let $(i,j) \in \{(r,s) \in \mathbb{Z} \times \mathbb{Z} |\, r, s \geq -1, s \leq r \text{ and } r + s \leq 2n\}$. If $\Gamma(\mathcal{E}_{ij})$ belongs to C_k^+ or to C_k^-, k equals to $i - j$ or to $i + j$, respectively.

5) In Figure 2, complexes C_k^+ (downwards-right, dashed lines) and C_k^- (upwards-right, dotted lines) are depicted for $n = 3$. The lines without arrows connect each complex to its label.

In the next lemma we describe a relation between the symbols of operators in the complex C_i^+ and the symbols of the appropriate operators in C_{i+1}^+, as well as a similar relation for C_i^- and C_{i-1}^-. Restrictions to $V \wedge E_{ij,\pm}$ of the symbol maps $\sigma_{ij}^{\pm} : V \wedge E_{ij} \to E_{i+1,j\pm 1}$ are denoted by $\sigma_{ij,\pm}^{\pm}$ (each combination of $\pm$ allowed). We consider two subsets of K that are subtractions from K of the three right-hand edges (the first subset) and of the left-hand edge and of the top edge (the second subset). See the diagram in Figure 2, where the nodes $\mathcal{E}_{ij}$ correspond to the elements of the set K. The segment with the null spaces is not considered.

Lemma 2. *For $(i,j) \in K \setminus \{(k,l) |\, k + l \geq 2n - 2\}$, there exists a non-zero complex number $\lambda_{\pm}^{+} = \lambda_{\pm}^{+}(i,j)$ such that*

$$\sigma_{i+1,j,\mp}^{+} \circ (Id_V \wedge F_{ij,\pm}^{+}) = \lambda_{\pm}^{+} F^{+} \circ \sigma_{ij,\pm}^{+}; \tag{1}$$

and for $(i,j) \in K \setminus \{(k,0),(l,l) |\, 0 \leq k \leq 2n, 0 \leq l \leq n\}$, there exists a *non-zero complex number* $\lambda^-_{\pm} = \lambda^-_{\pm}(i,j)$ *such that*

$$\sigma^-_{i-1,j,\mp} \circ (Id_V \wedge F^-_{ij,\pm}) = \lambda^-_{\pm} F^- \circ \sigma^-_{ij,\pm} \qquad (2)$$

Proof. Notice that $\sigma^{\pm}_{rs,\pm}$ (all combinations of $\pm$ allowed) and Id_V are $\widetilde{G}$-equivariant, and that F^+ and F^- are $\widetilde{G}$-equivariant by Lemma 1.

We consider the case of C^+_i, i.e., the equation (1), and we only comment the case of C^-_i briefly at the end. Since $V \wedge E_{ij,\pm} \simeq E_{i+1,j-1,\pm} \oplus E_{i+1,j,\pm} \oplus E_{i+1,j+1,\pm} \simeq E_{i+2,j-1,\mp} \oplus E_{i+2,j,\mp} \oplus E_{i+2,j+1,\mp}$ (by Lemma 1) and since $E_{i+2,j-1,\mp} \oplus E_{i+2,j,\mp} \oplus E_{i+2,j+1,\mp}$ is multiplicity-free (Section 2.1), the vector space of $\widetilde{G}$-equivariant maps of $V \wedge E_{ij,\pm}$ into $E_{i+2,j+1,\mp}$ is one dimensional by the Schur lemma ([11], p. 87). Let us choose a generator of this vector space and denote it by $A_{\pm}$. Because the $\widetilde{G}$-equivariant operators $R_{\pm} = F^+_{i+1,j+1,\pm} \circ \sigma^+_{ij,\pm}$ and $L_{\pm} = \sigma^+_{i+1,j,\mp} \circ (\mathrm{Id}_V \wedge F^+_{ij,\pm})$ map $V \wedge E_{ij,\pm}$ into $E_{i+2,j+1,\mp}$, they both are complex multiples of the generator $A_{\pm}$.

Now let us assume that $(i,j) \in K \setminus \{(k,l)|\, k+l \geq 2n-2\}$, i.e., $(i,j) \in K$ and $i+j < 2n-2$, and let us prove the existence of the non-zero constants $\lambda^+_{\pm}$. Note that if (i,j) belongs to K and $i+j < 2n-2$ ($2n-1$ is sufficient), the couple $(i+1, j+1) \in K$ by the equivalent description of the set K in Section 2.1.

i) Since $(i+1, j+1) \in K$ and $(i+1)+(j+1) = i+j+2 < 2n-2+2 = 2n$, the map $F^+_{i+1,j+1,\pm}$ is an isomorphism of $E_{i+1,j+1,\pm}$ onto $E_{i+2,j+1,\mp}$ by Lemma 1.

ii) The map $\sigma^+_{ij,\pm} : V \wedge E_{ij,\pm} \to E_{i+1,j+1,\pm}$ is non-trivial because $(i+1, j+1) \in K$ as shown above. Since F^+ is an isomorphism when restricted to the space $E_{i+1,j+1,\pm}$ by i), the composition $R_{\pm} = F^+_{i+1,j+1,\pm} \circ \sigma^+_{ij,\pm}$ is non-trivial as well.

iii) The homomorphism $\mathrm{Id}_V \wedge F^+_{ij,\pm}$ maps $V \wedge E_{ij,\pm}$ onto the space $V \wedge E_{i+1,j+1,\mp}$, which contains $E_{i+2,j+1,\mp}$ (Lemma 1). Since $(i+2)+(j+1) = i+j+3 \leq 2n-3+3 = 2n$ and since $j+1 \leq i+1 \leq i+2$, the element $(i+2, j+1)$ belongs to K by the equivalent description of K. In particular, the space $E_{i+2,j+1,\pm}$ is non-zero. Since $\sigma^+_{i+1,j,\mp}$ projects onto this space, the composed operator $L_{\pm} = \sigma^+_{i+1,j,\mp} \circ (\mathrm{Id}_V \wedge F^+_{ij,\pm})$ is non-trivial.

Thus the $\widetilde{G}$-equivariant maps $R_\pm$ and $L_\pm$ are non-trivial for all couples $(i,j) \in K \setminus \{(k,l)|\, k+l \geq 2n-2\}$ by ii) and iii). Moreover, since $R_\pm$ and $L_\pm$ are both multiples of $A_\pm$, they have to be non-zero multiples of each other.

In the case of equation (2), we proceed similarly as in i) - iii) using the properties of F^-. $\square$

As already mentioned, the ellipticity of the complexes C_0^+ and C_{2n}^- is known. Using this fact, we prove the ellipticity of the remaining ones by induction.

Theorem 2. *Let (M,ω) be a symplectic manifold which admits a metaplectic structure and ∇ be a symplectic Weyl-flat connection. Then for $k = 0,\ldots,2n-2$, the complex C_k^+ is elliptic and for $k = 2,\ldots,2n$, the complex C_k^- is elliptic as well.*

Proof. We show that the symbol map $\sigma^\pm_{i+j-1,j-1}(v)$ of the symplectic twistor operator $T^\pm_{i+j-1,j-1}$ evaluated in a vector $v \neq 0$ fulfills

$$\operatorname{Im}\sigma^\pm_{i+j-1,j\mp1}(v) = \operatorname{Ker}\sigma^\pm_{i+j,j}(v) \tag{3}$$

for i and j within the ranges set by the definition of the complexes C_k^+ for $k = i+j-1-(j-1) = i$, and of C_k^- for $k = i+j-1+j+1 = i+2j$. (See the item 4 of the last Remark.)

1) The inclusion $\operatorname{Im}\sigma^\pm_{i+j-1,j\mp1}(v) \subseteq \operatorname{Ker}\sigma^\pm_{i+j,j}(v)$ holds because the principal symbol of the composition of differential operators is the composition of the principal symbols of these operators (Solovyov, Troitsky [36]) and because $C_k^\pm$ are complexes due to the assumption on the Weyl-flatness of ∇ (Theorem 1).

2) For the opposite inclusion ($\supseteq$), we proceed by induction on i.

2.1) We prove the ellipticity for complexes C_i^+. I) The ellipticity of C_0^+ is proven in Krýsl [23]. II) Let us suppose that C_{i-1}^+, $i \geq 1$, is an elliptic complex and show the ellipticity of C_i. Consider an element $s \in \operatorname{Ker}\sigma^+_{i+j,j}(v) \subseteq E_{i+j,j}$, i.e., $\sigma^+_{i+j,j}(v \wedge s) = 0$. By the definition of C_i^+, we have $(i+j-1)+j = i+2j-1 \leq i+2\lfloor\frac{2n-i}{2}\rfloor - 1 < 2n$. Thus $F^+_{i+j-1,j}$ has an inverse by Lemma 1. For $s' = (F^+_{i+j-1,j})^{-1}s \in E_{i+j-1,j}$ we obtain that $\sigma^+_{i+j,j}(v \wedge F^+_{i+j-1,j}s') = 0$. Let us write $s' = s'_+ + s'_-$, where $s'_\pm \in E_{i+j-1,j,\pm}$. From $\sigma^+_{i+j,j}(v \wedge F^+_{i+j-1,j}s') = 0$, we obtain that also $\sigma^+_{i+j,j,\mp}(v \wedge F^+_{i+j-1,j}s'_\pm) = 0$ by Lemma 1 because the multiplicity-free structure of $(V \wedge E_{i+j,j,+}) \oplus (V \wedge E_{i+j,j,-}) \simeq V \wedge E_{i+j,j}$

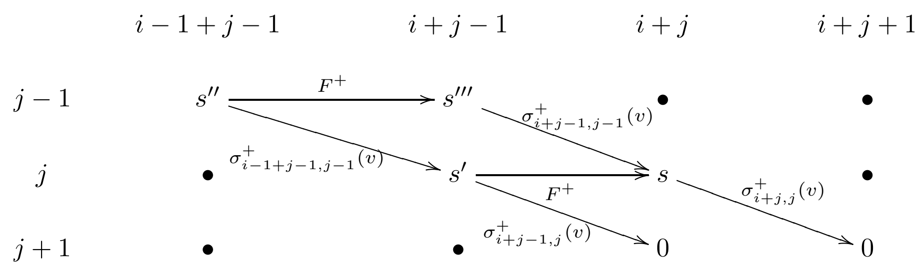

Figure 3: Diagram chasing for proof of Theorem 2.

implies that the images of $\sigma^+_{i+j,j,-}$ and $\sigma^+_{i+j,j,+}$ have the trivial intersection. By Lemma 2, Eq. (1), there exist complex numbers $\lambda^+_\pm = \lambda^+_\pm(i+j-1,j)$ such that

$$\lambda^+_\pm(F^+_{i+j,j+1,\pm} \circ \sigma^+_{i+j-1,j,\pm})(v \wedge s'_\pm) = \sigma^+_{i+j,j,\mp}(v \wedge F^+ s'_\pm) = 0 \qquad (4)$$

Let us recall that we prove the ellipticity of C^+_i at the place $(i+j,j)$, and therefore not only $T^+_{i+j,j}$, but also its target space $\Gamma(\mathcal{E}_{i+j+1,j+1})$ has also to belong to the complex C^+_i and thus $(i+j+1,j+1) \in K$. Consequently, $i+j+1+j+1 \leq 2n$ and $(i+j)+(j+1) < 2n$. Therefore $F^+_{i+j,j+1,\pm}$ is an isomorphism by Lemma 1. The constant $\lambda^+_\pm(i+j-1,j) \neq 0$ by Lemma 2 since $i+2j-1 < 2n-2$. Thus $\sigma^+_{i+j-1,j,\pm}(v \wedge s'_\pm) = 0$ by (4), and the element $s'_\pm$ has to belong to $\operatorname{Ker} \sigma^+_{i+j-1,j,\pm}(v)$, which equals to the kernel of $\sigma^+_{(i-1)+j,j,\pm}(v)$. Since C^+_{i-1} is elliptic by the induction hypotheses, there exist elements $s''_\pm \in E_{(i-1)+(j-1),j-1,\pm}$ such that $s'_\pm = \sigma^+_{(i-1)+(j-1),j-1,\pm}(v)(s''_\pm)$. First, let us consider $j=0$. In this case, $s''_\pm$ belongs to the space $E_{i-2,-1,\pm}$ which is trivial by definition, and consequently $s' = s'_+ + s'_- = 0$ and $s=0$. Thus we get the injectivity of the symbol of the second operator (i.e., the first non-zero operator) in C^+_i. Now, we assume that $j \geq 1$. In this case $\mu^+_\pm = \lambda^+_\pm(i-1+j-1,j-1)$ is non-zero by Lemma 2 and we may set $s'''_\mp = (\mu^+_\pm)^{-1} F^+ s''_\pm \in E_{i+j-1,j-1,\mp}$, which is in the preimage of s by $\sigma^+_{i+j-1,j-1}(v)$. Indeed, we have

$$\begin{aligned} \left(\sigma^+_{i+j-1,j-1,\mp}(v)\right) s'''_\mp &= \sigma^+_{i+j-1,j-1,\mp}(v \wedge (\mu^+_\pm)^{-1} F^+ s''_\pm) = \\ &= \left(F^+ \circ \sigma^+_{i-1+j-1,j-1,\pm}\right)(v \wedge s''_\pm) = \\ &= F^+\left(\sigma^+_{i-1+j-1,j-1,\pm}(v \wedge s''_\pm)\right) = \\ &= F^+\left(\sigma^+_{i-1+j-1,j-1,\pm}(v)(s''_\pm)\right) = F^+ s'_\pm \end{aligned}$$

where Eqn. (1) of Lemma 2 is used in the second step and the definition of $s''_\pm$ is used in the last one. Consequently $(\sigma^+_{i+j-1,j-1}(v))(s'''_+ + s'''_-) = F^+(s'_+ +$

$s'_-) = F^+ s' = s$ and thus, the element $s''' = s'''_+ + s'''_-$ belongs to the preimage of s by the appropriate symbol map.

2.2) Now we prove the ellipticity for the complexes $C^-_k = C^-_{i+2j}$ using the backwards induction on k. Since the proof of this fact is similar to the proof in 2.1, we skip some of the steps. I) The ellipticity of C^-_{2n} is proven in [23]. II) Let us suppose that $C^-_{k+1} = C^-_{i+2j+1}$ is elliptic for an element k in $\{2, \ldots, 2n-1\}$ and prove the ellipticity of $C^-_k = C^-_{i+2j}$. If $s \in \operatorname{Ker} \sigma^-_{i+j,j}(v) \subseteq E_{i+j,j}$, there is an element $s' \in E_{i+j+1,j}$ such that $s = F^-_{i+j+1,j} s'$ (Lemma 1). Let us write $s' = s'_+ + s'_-$ according to $E_{i+j+1,j} \simeq E_{i+j+1,j,+} \oplus E_{i+j+1,j,-}$. If $(i+j+1, j) = (l, l)$ for an element $l = 0, \ldots, n$, we get $i = -1$ and thus $s \in E_{j-1,j}$. Since $j > j-1$, this situation cannot occur by the definition of C^-_k. If $(i+j+1, j) = (l, 0)$ for some l, we have $(i+j+1, j) = (i+1, 0)$. In this case $\sigma^-_{i+1,0,\pm}(v) = 0$ and thus $s'_\pm \in \operatorname{Ker} \sigma^-_{i+j+1,j,\pm}(v)$ for trivial reasons. For the remaining values of i and j, there are constants $\lambda^-_\pm(i+j+1, j) \neq 0$ such that $\lambda^-_\pm(i+j+1,j) F^-_{i+j+2,j-1} \sigma^-_{i+j+1,j,\pm}(v \wedge s'_\pm) = \sigma^-_{i+j,j,\mp}(v \wedge F^- s'_\pm) = 0$ by Lemma 2 and since $s = F^- s'$ belongs to $\operatorname{Ker} \sigma^-_{i+j,j}(v)$. Consequently, $F^-_{i+j+2,j-1} \sigma^-_{i+j+1,j,\mp}(v \wedge s'_\mp) = 0$. For the considered i and j, the map $F^-_{i+j+2,j-1}$ is an isomorphism by Lemma 1. Thus we get that $s'_\pm \in \operatorname{Ker} \sigma^-_{i+j+1,j,\pm}(v)$ as well. By the induction hypothesis, there are elements $s''_\pm \in E_{i+j,j+1,\pm}$ such that $s'_\pm = \sigma^-_{i+j,j+1,\pm}(v)(s''_\pm)$. Since $\mu^-_\pm = \lambda^-_\pm(i+j, j+1) \neq 0$, we may set $s'''_\mp = (\mu^-_\pm)^{-1} F^-_{i+j,j+1,\pm} s''_\pm$ and $s''' = s'''_+ + s'''_-$. Using Eq. (2) from Lemma 2, we obtain that $s = \sigma^-_{i+j-1,j+1}(v)(s''')$ in a similar way as in the appropriate part of 2.1 of this proof. Thus s is in the image of the symbol map $\sigma_{i+j-1,j+1}(v)$ of $T^-_{i+j-1,j+1}$ in the vector v. □

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