Images of elliptic operators on Hilbert bundles on compact manifolds are closed

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Abstract

We prove that the image of an elliptic operator on a smooth separable Hilbert fibre bundle on compact manifolds is closed with respect to the natural pre-Hilbert topology. We consider a tensor product of the operator, which is invariant with respect to an action of the C^*-algebra of compact operators, and compare the image of this tensor product with the image of the original operator. This establishes a ground for the Hodge theory for these structures.

Keywords: Elliptic operators, Hilbert bundles, Hilbert C^*-modules, pseudodifferential operators

1 Introduction

Elliptic operators on smooth *finite rank* vector bundles on a smooth closed manifold M have finite dimensional kernels and closed images with respect to the so-called pre-Hilbert topology generated by an inner product on the space of smooth sections of the vector bundle. (The images are closed also with respect to a natural Fréchet topology.) Moreover, the codimension of the image of such an operator is finite. The total space of the bundle can always be equipped with a hermitian bundle metric h and the base manifold admits a Riemannian metric g. The inner product on the space of the bundle's smooth sections is given by the integral (determined by a density form $|\mathrm{vol}_g|$ induced by g) of the hermitian metric composed with the bundles' sections. Thus for two bundle's sections s, t one sets $(s,t) = \int_{m\in M} h(s(m), t(m))|\mathrm{vol}_g|(m)$, defining the pre-Hilbert topology. These results go back to the work of W. Hodge [20] on the so-called harmonic integrals. See also Palais [39] and Wells [57]. The closedness of the image of an elliptic operator is an important fact which is used to prove several assertions of the Hodge theory for elliptic complexes. In this case, it is connected to the well known fact that extensions of elliptic operator are Fredholm. See [57].

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The aim of this article is to prove the closed image property for elliptic operators defined on smooth sections of infinite rank Hilbert bundles on compact manifolds. In the infinite rank case, neither the kernel nor the cokernel of an elliptic operator have to be finite dimensional vector spaces. For the kernel, it is sufficient to consider the product bundle of a circle with an infinite dimensional Hilbert space V and the operator of the directional (Gateaux) derivative with respect to an angle-coordinate parametrizing the circle. The operator acts on smooth V-valued functions on the circle. Its kernel consists of constant V-valued functions and therefore its dimension equals to the infinite dimension of V.

Let us notice that the closedness of images of elliptic operators on Hilbert bundles is important not only for the Hodge theory of these structures, but it plays a distinguished role in the representation theory of Lie groups. See, e.g., Chapter 3 in Schmid [47]. There is also an extensive literature devoted to consequences of a topological complementability in the Fréchet topology of kernels of elliptic operators (often on finite rank bundles), which is connected to the Schwartz kernel theorem. See, e.g., Tréves [52, 53], Grothendieck (Appendix C.1 in Tréves [53]), Poly [40], and Vogt [54]. The sheaf cohomology of Banach or Fréchet bundles was studied already by Illusie [21] and Röhrler [42]. For holomorphic Banach bundles we refer to Lempert [32] and Kim [23]. Let us notice that Erat in [12] found an example of a holomorphic Banach bundle over the two dimensional sphere whose first sheaf cohomology is non-Hausdorff, i.e., the image of the codifferential is not closed. Regarding global analysis on Banach bundles concerning K-theory and non-commutative geometry, let us mention papers of Maeda and Rosenberg [33], Freed and Lott [16], Larraín-Hubach [31], Krýsl [27], and Fathizadeh and Gabriel [13].

Besides the purely topological question on the closedness of the images of elliptic operators in the realm of global analysis, our motivation comes from quantum field theory ([3, 7, 44]) and, especially, from the so-called Becchi–Rouet–Stora–Tiyutin (BRST) theory for *constrained quantum systems* (see, e.g., Henneaux, Teitelboim [19]). According to the BRST-theory, the state space of a constrained quantum system is the quotient of an inner product or of a normed space associated with the unconstrained system by the image of a pseudodifferential operator defined on certain operator-valued functions or distributions, that represents the constraints. (See Schwartz [49] or the survey in Warner [55].) Since the quotient shall be a Hausdorff topological space, the topology of the images of pseudodifferential operators plays an indispensable role also in the mathematically oriented quantum theory. Last but not least, we are motivated by an investigation of the so called symplectic Dirac operators introduced by Habermann (see Habermann, Habermann [18] and [28]), which are first order differential operators defined on sections of induced Hilbert bundles on a class of symplectic manifolds. This list of references shall not be considered as complete.

The (Rellich–Kondrachov) compact embedding theorem for Sobolev spaces of sections of finite rank vector bundles on a compact manifold is used to prove that continuous extensions of elliptic operators to the Sobolev spaces are Fredholm and, consequently their images are closed. See, e.g., Seeley [45], Wells [57], or Palais [39]. However, it is easy to see that a straight-forward generalization of this theorem does not hold for vector-valued Sobolev spaces with values in a Hilbert space V, denoted by $W^k(M,V)$ in our paper, if V is infinite dimensional. By a straight-forward generalization, we mean the assertion that the inclusion map $W^{k+1}(M,V) \subseteq W^k(M,V)$ is compact for any (separable) Hilbert space V. To see that this generalization does not hold, it is enough to assume that M is a singleton and to recall that the identity on an infinite dimensional Hilbert space V is not compact. It is also possible to consider the following more general situation. Let $J^0_{k+1} : W^{k+1}(M) \to W^k(M)$ denote the inclusion map of Sobolev spaces. It is known that there is a unitary isomorphism $\Phi_k : W^k(M,V) \to W^k(M)\widehat{\otimes}_{HS}V$, where $\widehat{\otimes}_{HS}$ denotes the the Hilbert–Schmidt tensor product completion. (See Wloka [58].) Since 1) the inclusion map $J_{k+1} : W^{k+1}(M,V) \to W^k(M,V)$ equals to $\Phi_k^{-1} \circ (J^0_k \widehat{\otimes}_{HS} \mathrm{Id}_{V*}) \circ \Phi_{k+1}$, 2) Φ_k are norm preserving, and 3) Id_{V*} is not a compact operator, the operator J_{k+1} is not compact as well.

In a connection to the Atiyah–Singer index theorem, Fomenko and Mishchenko in [36] developed a theory for a class of pseudodifferential operators on smooth vector bundles with specific C^*-algebra modules as fibres. The operators have to be C^*-linear and the fibres of the bundles are so-called Hilbert C^*-modules, that are generalizations of Hilbert spaces and also each C^*-algebra is an example of such a module. (See, e.g., Lance [30], Wegge-Olsen [56], and Blackadar [6].) The bundles are usually called Hilbert C^*-bundles (Fomenko and Mishchenko [36], Solovyov and Troitsky [50], and Schick [46]). Notice that there is also a different notion of the so-called 'bundles of C^*-algebras' (Dixmier [10] and Fell [14]). To avoid a misunderstanding, we give a definition of Hilbert C^*-bundles in the paper (Definition 3). Introductory notions related to Hilbert C^*-modules and Hilbert C^*-bundles are recalled in Sections 2 and 3 mostly for to set an unambiguous terminology.

Let us suppose that D is an elliptic operator defined on smooth sections of a separable Hilbert bundle $\mathcal{V}$ on a compact manifold M with the fibre a Hilbert space V, whose image we want to prove closed in the pre-Hilbert topology. The first step we undertake is a transfer to the C^*-framework, i.e., to Hilbert C^*-bundles and C^*-pseudodifferential operators. As a consequence of results of Burghelea and Kuiper [8] and Moulis [37], the space of smooth sections of any smooth infinite rank Hilbert bundle is linearly homeomorphic to the space $C^\infty(M,V)$ if both of these spaces are equipped with the natural Fréchet topology (or if both of them are equipped with the pre-Hilbert topology).

We tensor multiply the space $C^\infty(M,V)$, equipped with a Fréchet topology, by the continuous dual V^* of V, equipped with the dual norm, and we consider the so-called injective completion of the tensor product, denoting the resulting Fréchet space by $C^\infty(M,V)\widehat{\otimes}_\epsilon V^*$ (see Tréves [52] or Grothendieck [17]). Let us consider the Banach space CV of compact operators on V equipped with the operator norm. This space with the adjoint of operators on the Hilbert space V is also a C^*-algebra with respect to the composition, addition and scalar multiplication of operators. It is well known that the injective completion $C^\infty(M,V)\widehat{\otimes}_\epsilon V^*$ and the space $C^\infty(M,CV)$ are linearly homeomorphic when the smooth function spaces are equipped with the Fréchet topologies. Moreover, the vector space CV is a right CV-module and also a Hilbert CV-module, whose induced C^*-norm coincides with the operator norm. We call this Hilbert module over the C^*-algebra CV of compact operators the *compact Hilbert CV-module*.

We tensor multiply the elliptic operator D by the identity on V^*, and extend it continuously to $C^\infty(M,V)\widehat{\otimes}_\epsilon V^* \cong C^\infty(M,CV)$, denoting the resulting operator by $\widehat{D}^\epsilon$. It is a pseudodifferential operator (Lemma 9), that acts on the smooth sections of the product Hilbert CV-bundle $M \times CV \to M$. Moreover, $\widehat{D}^\epsilon$ is right CV-linear (Lemma 8) and CV-elliptic (Lemma 10). Occasionally, we call the Hilbert CV-bundle structure on $M \times CV \to M$ with the compact Hilbert CV-module as its fibre, the *compact Hilbert CV-bundle*.

A *version* of the Rellich–Kondrachov compact embedding theorem is proved for *finitely generated projective* Hilbert C^*-bundles in Fomenko, Mishchenko [36]. Namely, the inclusion maps between appropriate Sobolev-type completions of the bundles' smooth section spaces are so-called C^*-compact. Saying that a Hilbert A-bundle is finitely generated projective means that its fibres are finitely generated and projective as modules over the C^*-algebra A. Let us notice that operators that are C^*-compact need not be compact. Unfortunately, the compact Hilbert CV-bundle $M \times CV \to M$ is easily seen not to be finitely generated. However, we prove a CV-compact embedding of Sobolev completions also in this case, i.e., for any compact Hilbert CV-bundle on a compact manifold (first part of Theorem 7). Using the derived C^*-compact embedding, we get that continuous extensions of elliptic operators to the Sobolev-type completions are so-called CV-Fredholm homomorphisms (part b) of Theorem 7). By Lemma 1, a CV-Fredholm homomorphism has a closed image. Lemma 1 is a simple generalization of a result obtained by Bakič, Guljaš in [5] on the closedness of images of CV-Fredholm endomorphisms. This fact in [5] is based on the density of Hilbert–Schmidt operators in the space of compact operators on this Hilbert space, and the fact that the space of Hilbert–Schmidt operators is naturally a Hilbert space (see Bakić, Guljaš [4] for more details if needed). In Theorem 7, we assert that elliptic CV-pseudodifferential operators, and in particular the operator $\widehat{D}^\epsilon$, have closed images in $C^\infty(M,CV)$ if this space is equipped with the pre-Hilbert or with the Fréchet topology.

Since we want to prove the closed image property for the original pseudodifferential operator D defined on smooth sections of a separable Hilbert bundle $\mathcal{V} \to M$ (Theorem 11), we shall come back from the image of $\widehat{D}^\epsilon$ to the image of D. This is done in the proof of Theorem 11, where we use a lemma on a representation of elements in $C^\infty(M,V)\widehat{\otimes}_\epsilon V^*$ (Lemma 10). In the proof of Theorem 11, we use a theorem on the interchange of the limiting and summation processes for certain vector valued double sequences (Scholium 3), which is due to Antosik [1] and which is a generalization of a similar result of Schur (see Pap [38]).

The paper is organized as follows: In the second section, we summarize basic information on Hilbert C^*-modules and their morphisms and prove assertions on CV-Fredholm operators on Hilbert and pre-Hilbert modules that concern their images. At the beginning of the third section, we recall the notion of a smooth

Hilbert C^*-bundle and pre-Hilbert and Fréchet topologies on the section spaces of these bundles, and we prove two basic lemmas on the topology of section spaces and on symbols. In the main part of the third section, we deal with CV-linear pre-Hilbert morphisms. We prove an assertion on the C^*-compact embedding for the compact Hilbert CV-bundle on a torus (Scholium 1) and its generalization to arbitrary compact manifolds (Theorem 7). The fourth chapter is devoted to elliptic operators on Hilbert bundles, to the proof of the theorem on the closed image (Theorem 11), and to a formulation of a question concerning a generalization of our result. In this chapter, we also introduce the completed injective tensor product and show that the multiplied operators are CV-linear and also how the ellipticity transfers to them.

1.1 Preamble

a) The continuous dual of a topological vector space W is denoted by W^*.

b) Fréchet topological vector spaces are considered to be Hausdorff. We suppose that they are defined by a countable family of separating seminorms, which induce the canonical translationally invariant metric.

c) For a topological vector space W, the symbol $\mathrm{End}(W)$ denotes the vector space of continuous linear endomorphisms of W and $\mathrm{Aut}(W)$ denotes the group of all linear homeomorphisms of W onto W.

d) The field $\mathbb{R}$ of real numbers is supposed to be a subset of the field $\mathbb{C}$ of complex numbers using the linear embedding $r \in \mathbb{R} \mapsto r + 0\imath \in \mathbb{C}$. Inner product spaces are considered over real or complex numbers. The inner product is complex anti-linear in the first variable and complex linear in the second one (convention of "physicists"). For a C^*-algebra A and a pre-Hilbert A-module, we assume the same behaviour of the so-called A-product. The topology of an inner product space is generated by the norm induced by the inner product.

   All Hilbert spaces are supposed to be *separable*.

e) Continuous maps between topological vector spaces are called C^k-differentiable if their lth order Fréchet differential is continuous for $l = 1, \ldots, k$. We call them C^∞-differentiable or smooth if they are C^k-differentiable for all $k \in \mathbb{N}$. C^∞-diffeomorphisms are smooth maps whose inversions are smooth.

f) Any manifold is considered to be a C^∞-differentiable Banach manifold without boundary, i.e., a Hausdorff and second countable topological space which is locally homeomorphic to a fixed Banach space E, and equipped with a maximal C^∞-differentiable manifold atlas. By our convention, elements of the atlas, called manifold charts, are homeomorphisms of open subsets of the manifold into open subsets in E. A C^∞-differentiable atlas means that its transition maps are smooth maps of open sets in the Banach space E.

We do not consider manifolds as equivalence classes with respect to C^∞-diffeomorphisms. Therefore a manifold (in our sense) is what is usually called a C^∞-differentiable structure.

g) Let $\mathcal{W}$ be a manifold, M a finite dimensional manifold, and F a Banach space. A fibre bundle $p : \mathcal{W} \to M$ with a typical fibre the Banach space F is a smooth submersion of manifolds equipped with a maximal C^∞-differentiable bundle atlas whose charts are isometric isomorphisms of Banach spaces when restricted to any fibre $p^{-1}(m)$, $m \in M$. By 'C^∞-differentiable' we mean that the charts of the atlas are supposed to C^∞-diffeomorphisms of $U \times F$ onto the set $p^{-1}(U) \subseteq \mathcal{W}$, where $U \subseteq M$ is an open set. (Notice that transition functions of such an atlas are automatically smooth maps of open sets in M into the locally convex vector space $\mathrm{End}(F)$ equipped with the strong operator topology, and they are also continuous maps into the group $G = \mathrm{Aut}^0(F)$ of isometric isomorphisms that is equipped with the subset topology of the strong operator topology on $\mathrm{End}(F)$.) In the case of Hilbert C^*-bundles we demand that G is the group of unitary C^*-automorphisms of the fibre.

   Comparison to other concepts: We do not consider fibre bundles as equivalence classes of maximal C^∞-differentiable atlases determined by bundle C^∞-diffeomorphisms. Thus fibre bundles are smooth analogues of the so-called coordinate bundles as defined in Steenrod [51]. Notice that more generally, we can consider a topological subgroup G of $\mathrm{Aut}^0(F)$ as the codomain of the transition maps. Sometimes, it is demanded that the transition maps are continuous maps into a topological subgroup G of $\mathrm{Aut}^0(F)$ with the operator norm topology, and that they are smooth maps into $\mathrm{End}(F)$ equipped also with the operator norm topology. We do not follow this more restrictive concept.

h) Pseudodifferential operators on a fibre bundle are, especially, linear maps defined on the vector space of *smooth sections* of the fibre bundle.

2 Images of pre-Hilbert C^*-modules morphisms

Let $(A, \cdot, |\,|_A, {}^*)$ be a C^*-algebra. The closed half-cone of hermitian-symmetric elements of A whose spectra are contained in $\mathbb{R}^{\geqslant 0}$ is denoted by A^+. (See Dixmier [10].) A *pre-Hilbert A-module* is a complex vector space W on which A acts from the right compatibly with the scalar multiplication and which is equipped with a hermitian-symmetric map $(,)_W : W \times W \to A$ that is

i) sesquilinear with respect to the action of the field $\mathbb{C}$ and to the right action of the C^*-algebra A, and

ii) positive definite in the sense that for each $w \in W$, element $(w, w)_W$ belongs to A^+, and $(w, w)_W = 0$ only if $w = 0$.

We denote such a Hilbert A-module by $(W, (,)_W)$, and call the hermitian-symmetric map $(,)_W$ the *C^*-product* or the A-product. Let $c \in \mathbb{C}$, $a \in A$, and $v, w \in W$. The compatibility of the C^*-algebra action with the scalar multiplication means that $v \cdot (ca) = (cv) \cdot a = c(v \cdot a)$. By sesquilinearity we mean that the C^*-product $(,)$ obeys $(v \cdot a, w) = (v, w)a^*, (v, w \cdot a) = (v, w)a$ and that a similar rule holds for the multiplication by complex numbers. The C^*-product $(,)$ is hermitian-symmetric in the sense that $(v, w) = (w, v)^*$ for any $v, w \in W$. See Lance [30] or Wegge-Olsen [56].

The *(induced) C^*-norm*, called also the (induced) A-norm, $| \, |_W : W \to \mathbb{R}$ is defined by $|w|_W = \sqrt{|(w, w)_W|_A}$, where $w \in W$. Any pre-Hilbert A-module is considered as an A-module and as a topological vector space with the topology generated by the C^*-norm.

A map $L : W \to W'$ of pre-Hilbert A-modules $(W, (,)_W)$ and $(W', (,)_{W'})$ is adjointable if there exists a map $L^* : W' \to W$ such that $(Lw, w')_{W'} = (w, L^*w')_W$ for all $w \in W$ and $w' \in W'$. (The map L^* is unique if it exists. See, e.g., [56].) It is easy to see that any adjointable map is complex linear and A-linear. (We say also A-equivariant or A-invariant.) A morphism of pre-Hilbert A-module morphisms W and W' is any adjointable map of W into W'. We denote the set of all pre-Hilbert A-module morphisms by $\mathrm{Hom}_A^*(W, W')$ and by $\mathrm{End}_A^*(W)$ if $W' = W$. Any adjointable pre-Hilbert A-module morphism $L : W \to W'$ such that $L^* \circ L = \mathrm{Id}_W$ and $L \circ L^* = \mathrm{Id}_{W'}$ is called unitary. If $L = L^*$, L is called self-adjoint. An isomorphism of pre-Hilbert A-modules is an isomorphism in the sense of category theory, i.e., it is a morphism of pre-Hilbert A-modules that is right and left invertible by a pre-Hilbert A-module morphism. As usual, the symbol $\mathrm{Aut}_A^*(W)$ denotes the set of isomorphisms of the pre-Hilbert A-module W onto itself.

A pre-Hilbert *A-submodule* of $(W, (,)_W)$ is an algebraic A-submodule W' of W equipped with the restriction of $(,)_W$ to $W' \times W'$. A pre-Hilbert A-submodule is not demanded to be closed.

Subsets $W_1, W_2 \subseteq W$ are called orthogonal if $(w_1, w_2)_W = 0$ for each $w_1 \in W_1$ and $w_2 \in W_2$. If $(W_1, (,)_{W_1})$ and $(W_2, (,)_{W_2})$ are pre-Hilbert A-modules with a trivial intersection, we define their outer direct sum as the direct sum of A-modules and complex vector spaces $W = W_1 \oplus W_2$, and equip it with the A-product defined by $(w_1 + w_2, w_1' + w_2')_W = (w_1, w_1')_{W_1} + (w_2, w_2')_{W_2}$, where $w_1, w_1' \in W_1$ and $w_2, w_2' \in W_2$. This makes $W_1 \oplus W_2$ a pre-Hilbert A-module. If W_1 and W_2 are pre-Hilbert A-submodules and W is isomorphic to $W_1 \oplus W_2$, we write $W = W_1 \oplus W_2$. The pre-Hilbert A-submodules W_1 and W_2 are necessarily closed in W. We do not demand the isomorphism to be unitary. See, e.g., [56].

If N is a subset of a pre-Hilbert A-module W, we denote the closed pre-Hilbert A-submodule $\{w \in W | \, (w, w')_W = 0 \text{ for each } w' \in N\}$ by $N^\perp$, and call it the *orthogonal set* of N. A pre-Hilbert A-submodule W' of a pre-Hilbert module W is called orthogonally complemented if there exists a pre-Hilbert A-module W'' of W such that W is isomorphic to $W' \oplus W''$. In this case $W'' = W'^\perp$.

A pre-Hilbert A-module $(E, (,)_E)$ is called a *Hilbert A-module* if it is complete with respect to the induced C^*-norm $| \, |_E$ on E. The category of Hilbert A-modules is the full subcategory of the category of pre-Hilbert A-modules whose objects are Hilbert A-modules. Notice that any morphism of Hilbert A-modules is not only complex and A-linear, but also continuous. We denote the set of all Hilbert A-module morphisms by $\mathrm{Hom}_A^*(W, W')$ as well. If A is the algebra of complex numbers, Hilbert A-modules are the same as complex Hilbert spaces. A Hilbert A-submodule N of a Hilbert A-module $(M, (,)_M)$ is an A-submodule of M equipped with the restriction of $(,)_M$ to N such that $(N, (,)_{M|N\times N})$ is a Hilbert A-module. Though closed, it is well known that a Hilbert A-submodule needn't be orthogonally complemented in contrary to the case of Hilbert spaces. Notice that Hilbert A-submodules need not be even topologically complemented. (See Lance [30].)

The next definitions might seem to be contingent. However, they are frequently used in the theory of Hilbert C^*-modules. A Hilbert A-module E is called topologically finitely generated if there exists a dense subspace of E which is generated by finite A-linear combinations by a finite subset of E. It is called algebraically finitely generated, or only finitely generated, if E is generated over A by a finite subset of E as an A-module. These definitions are according to [36] and [50]. In Wegge-Olsen [56] topologically finitely generated modules are called finitely generated and (algebraically) finitely generated modules are called algebraically finitely generated. For a positive integer p, let us consider $A^p = \underbrace{A \oplus \ldots \oplus A}_{p-\text{times}}$ with the right diagonal action of A and the A-product $((a_1, \ldots, a_p), (b_1, \ldots, b_p)) = \sum_{i=1}^p a_i^* b_i$, where $a_i, b_i \in A$ for $i = 1 \ldots, p$. We denote the set of all continuous A-linear a complex-linear maps between Hilbert A-modules $(E, (,)_E)$ and $(F, (,)_F)$ by $\mathrm{Hom}_A(E, F)$, i.e., $T \in \mathrm{Hom}_A(E, F)$ iff for each $e \in E$ and $a \in A$, $T(e \cdot a) = T(e) \cdot a$ and T is continuous. A Hilbert A-module M is called *projective* if for each Hilbert A-modules B and C and for every surjective $b \in \mathrm{Hom}_A(B, M)$ and every $c \in \mathrm{Hom}_A(C, M)$ there exists $d \in \mathrm{Hom}_A(C, B)$ such that $c = b \circ d$. Notice that in this definition we consider all continuous A-linear maps between Hilbert A-modules. One calls a Hilbert A-module finitely generated projective shortly if it is finitely generated and projective. In the case of a unital A, the module is finitely generated projective if it is a complementable Hilbert A-submodule of A^p for an integer $p \geqslant 1$. See eventually Frank, Paulsen [15] regarding these definitions and assertions.

For a complex Hilbert space (V, h), we consider the C^*-algebra consisting of all compact operators on V, which is equipped with the multiplication by complex numbers, the addition and composition of operators, the adjoint (with respect to the inner product h), and the operator norm. We call such an algebra *the C^*-algebra of compact operators* and denote it by CV. We recall the following natural definition.

Definition 1: We call a C^*-algebra A *a C^*-algebra of compact operators* if it is a C^*-subalgebra of the C^*-algebra CV of compact operators on a complex Hilbert space V.

If A is a C^*-algebra, and E' and E'' are Hilbert A-modules, an A-compact operator of E' into E'' is the limit in the operator norm topology on $\mathrm{Hom}_A(E', E'')$ of the so-called A-finite rank operators. See Lance [30], or Kasparov [22].

Remark: Let E, F and H be Hilbert A-modules. If $K \in \mathrm{Hom}^*_A(E, F)$ is an A-compact operator and $T \in \mathrm{Hom}^*_A(F, H)$, the operator $T \circ K$ is an A-compact operators and similarly for the composition with an A-compact operator from the left. This is an easy generalization of the well known ideal property of A-compact operators formulated for the case $E = F$, i.e., for endomorphisms ([30], p. 9–10). The proof proceeds as in the endomorphism case. We refer to this property as to the ideal property as well.

Definition 2: A morphism $D : E \to F$ of Hilbert A-modules is called an *A-Fredholm operator* if there is a Hilbert A-module morphism $\breve{D} : F \to E$ such that $K_1 = \breve{D}D - \mathrm{Id}_E$ and $K_2 = D\breve{D} - \mathrm{Id}_F$ are A-compact operators.

We call any operator $\breve{D}$ fulfilling the above equations (called parametrix equations) a partial inverse of D. The next lemma is a straightforward generalization of a theorem of Bakić, Guljaš from [5], p. 268, stated for A-Fredholm endomorphisms.

Lemma 1: Let A be a C^*-algebra of compact operators, E and F be Hilbert A-modules, and $D \in \mathrm{Hom}^*_A(E, F)$. If D is an A-Fredholm operator, its image is closed.

Proof. Let $D : E \to F$ be an A-Fredholm operator and $\breve{D}, K_1$, and K_2 be the corresponding partial inverse and A-compact operators. Thus $\breve{D}D = \mathrm{Id}_E + K_1$ and $D\breve{D} = \mathrm{Id}_F + K_2$.

Let us consider the following block-wise anti-diagonal element $\mathfrak{D} = \left(\begin{smallmatrix} 0 & D^* \\ D & 0 \end{smallmatrix}\right) \in \mathrm{End}^*_A(E \oplus F)$. For this element

$$\begin{pmatrix} 0 & \breve{D} \\ \breve{D}^* & 0 \end{pmatrix} \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{Id}_E + K_1 & 0 \\ 0 & \mathrm{Id}_F + K_2^* \end{pmatrix}$$
$$= \begin{pmatrix} \mathrm{Id}_E & 0 \\ 0 & \mathrm{Id}_F \end{pmatrix} + \begin{pmatrix} K_1 & 0 \\ 0 & K_2^* \end{pmatrix}.$$

Since the last written matrix is an A-compact operator in $\mathrm{End}^*_A(E \oplus F)$, operator $\mathfrak{D}$ is left invertible up to an A-compact operator on $E \oplus F$. The right invertibility (up to A-compact operators) of $\mathfrak{D}$ is proved in a similar way. Thus we see that $\mathfrak{D}$ is an A-Fredholm endomorphism. According to [5], p. 268, its image is closed. This implies that D itself has a closed image as well because E and F are mutually orthogonal sets in $E \oplus F$ and $\mathrm{Im}\,\mathfrak{D} = \mathrm{Im}\, D^* \oplus \mathrm{Im}\, D$. $\square$

Remark:

1) Let A be a C^*-algebra of compact operators. Then a Hilbert A-module morphism $D : E \to F$ is A-Fredholm if and only if the image of D is closed

and the A-dimensions $\dim_A \mathrm{Ker}\, D$ and $\dim_A (\mathrm{Im}\, D)^\perp$ are finite by Corollary 5 in [26]. A definition of the A-dimension is given in [5].

2) The result in [5] on the image cited in the above proof is based on a the result in [4] (Theorem 2.22), where H^*-modules are investigated regarding their relation to Hilbert–Schmidt and to compact operators on a Hilbert space.

In the next theorem, $\widetilde{\Delta}^*$ denotes the adjoint of a continuous extension $\widetilde{\Delta}$ of a pre-Hilbert A-module morphism Δ, and not vice versa. The theorem is a generalization of a procedure, which is used to deduce the closedness of the image of an elliptic operator from the closedness of the image of the operator's continuous extension to appropriately completed domain and codomain spaces. (We use it in the proof of Theorem 7.) Condition ii) is connected to the regularity of an elliptic operator on a Hilbert bundle on a compact manifold, that we prove in Section 3.

Theorem 2 (on smooth parametrix): Let A be a C^*-algebra of compact operators, $(E, (,)_E)$ and $(F, (,)_F)$ be Hilbert A-modules, and $(W, (,)_W)$ be a pre-Hilbert A-submodule of F and a vector subspace of E. Let us consider a self-adjoint map $\Delta \in \mathrm{End}^*_A(W)$ with a continuous extension $\widetilde{\Delta} \in \mathrm{Hom}^*_A(E, F)$ that satisfies

i) $\widetilde{\Delta}$ is A-Fredholm and

ii) $\widetilde{\Delta}^{-1}(W), \widetilde{\Delta}^{*-1}(W) \subseteq W$.

Then the image of Δ is closed in W and $W = \mathrm{Ker}\,\Delta \oplus \mathrm{Im}\,\Delta$. Moreover, there are self-adjoint pre-Hilbert A-module morphisms $\check{\Delta} : W \to W$ and $K : W \to W$ such that $\Delta\check{\Delta} = \check{\Delta}\Delta = K - \mathrm{Id}_W$ and $\Delta K = 0$.

Proof. Assumption ii) implies that $\mathrm{Ker}\,\widetilde{\Delta}, \mathrm{Ker}\,\widetilde{\Delta}^* \subseteq W$.

1) Since $\widetilde{\Delta}$ is A-Fredholm, it has a closed image by Lemma 1. By Mishchenko's kernel-image theorem (Theorem 3.2 in [30]), the image of $\widetilde{\Delta}^* : F \to E$ is closed as well, and the following decompositions

$$E = \mathrm{Ker}\,\widetilde{\Delta} \oplus \mathrm{Im}\,\widetilde{\Delta}^* \text{ and } F = \mathrm{Ker}\,\widetilde{\Delta}^* \oplus \mathrm{Im}\,\widetilde{\Delta}$$

hold.

2) For $w \in \mathrm{Ker}\,\widetilde{\Delta}^* \subseteq F$, we have $w \in W$ by assumption ii). Thus for each $w' \in W$, we can write $(\Delta w, w')_W = (w, \Delta w')_W = (w, \Delta w')_F = (w, \widetilde{\Delta} w')_F = (\widetilde{\Delta}^* w, w')_E = 0$ for all $w' \in W$. This implies that $\Delta w = 0$. Thus $\mathrm{Ker}\,\widetilde{\Delta}^* \subseteq \mathrm{Ker}\,\Delta$.

We show that the decomposition $F = \mathrm{Ker}\,\widetilde{\Delta}^* \oplus \mathrm{Im}\,\widetilde{\Delta}$ restricts to W in the sense that $W = \mathrm{Ker}\,\Delta \oplus (\mathrm{Im}\,\widetilde{\Delta} \cap W)$. If $w \in W$, then $w \in F$. Thus $w = w_1 + w_2$, where $w_1 \in \mathrm{Ker}\,\widetilde{\Delta}^*$ and $w_2 \in \mathrm{Im}\,\widetilde{\Delta}$ by the decomposition of F in item 1. In particular, $w_1 \in \mathrm{Ker}\,\Delta$ since $\mathrm{Ker}\,\widetilde{\Delta}^* \subseteq \mathrm{Ker}\,\Delta$ as proven in the previous paragraph. Since $w, w_1 \in W$, element $w_2 = w - w_1 \in W$. Consequently $W \subseteq \operatorname{Ker}\Delta \oplus (\operatorname{Im}\widetilde{\Delta} \cap W)$. The opposite inclusion is obvious.

If $\Delta w = 0$ and if $w = \widetilde{\Delta} w'$ is in W for $w' \in E$, we get $w' \in W$ by the assumption ii). Thus $0 = (\Delta w, w')_W = (w, \Delta w')_W = (w, w)_W$ and consequently $w = 0$, i.e., the sum is orthogonal. We conclude that $W = \operatorname{Ker}\Delta \oplus (\operatorname{Im}\widetilde{\Delta} \cap W)$.

3) We prove that $\operatorname{Im}\widetilde{\Delta} \cap W = \operatorname{Im}\Delta$. If for $e \in E$, $f = \widetilde{\Delta} e$ is in W, we have that $e \in W$ by ii). Thus $f = \Delta e$ and $\operatorname{Im}\widetilde{\Delta} \cap W \subseteq \operatorname{Im}\Delta$. The opposite inclusion is obvious. Using 2) and 3), we obtain $W = \operatorname{Ker}\Delta \oplus \operatorname{Im}\Delta$. Since the decomposition is orthogonal, the image of Δ is closed.

4) We define $K : W \to W$ as the projection on the kernel of Δ according to the orthogonal decomposition $W = \operatorname{Ker}\Delta \oplus \operatorname{Im}\Delta$. In particular, K is self-adjoint. Since the kernel and the image of Δ are pre-Hilbert A-submodules of W, K is a pre-Hilbert A-module morphism.

5) Let us define a map $\check{\Delta} : W \to W$ by

$$\check{\Delta} = \begin{cases} (\Delta_{|\operatorname{Im}\Delta})^{-1} & \text{on } \operatorname{Im}\Delta \\ 0 & \text{on } \operatorname{Ker}\Delta. \end{cases}$$

By this definition, this operator satisfies $\Delta\check{\Delta} = \check{\Delta}\Delta = \mathrm{Id}_W - K$. Obviously, $\check{\Delta}$ is a self-adjoint pre-Hilbert A-module morphism of W since Δ is self-adjoint.

□

Remark: In applications, E and F play the role of the appropriate Sobolev completions of the space W of smooth sections of a considered bundle.

Let us notice that a topological vector space $\overline{H}$ denotes neither the complex conjugate vector space of a complex topological vector space H, nor a closure of H in an other topological vector space. Thus when not specified otherwise, $\overline{H}$ denotes an arbitrary topological vector space.

Corollary 3: Let A be a C^*-algebra of compact operators, E, F and W be as in Theorem 2, and the same holds for $\overline{E}, \overline{F}$ and $\overline{W}$. Let $D : W \to \overline{W}$ be a pre-Hilbert A-module morphism such that $\Delta = D^*D : W \to W$ and $\overline{\Delta} = DD^* : \overline{W} \to \overline{W}$ have continuous adjointable extensions $\widetilde{\Delta} \in \operatorname{Hom}_A^*(E, F)$ and $\widetilde{\overline{\Delta}} \in \operatorname{Hom}_A^*(\overline{E}, \overline{F})$, respectively, that satisfy assumptions i) and ii) of the previous theorem (Theorem 2). Then the images of D and D^* are closed.

Proof. Since Δ and $\overline{\Delta}$ are self-adjoint and satisfy assumptions i) and ii) of Theorem 2, the decompositions $W = \operatorname{Ker}\Delta \oplus \operatorname{Im}\Delta$ and $\overline{W} = \operatorname{Ker}\overline{\Delta} \oplus \operatorname{Im}\overline{\Delta}$ hold.

Since $\widetilde{\Delta}$ and $\widetilde{\overline{\Delta}}$ are A-Fredholm by assumption ii), we have the pre-Hilbert A-module morphisms $\check{\Delta}, \check{\overline{\Delta}}, K$ and $\overline{K}$ at our disposal by Theorem 2. If $x \in \operatorname{Ker}\Delta$, then $0 = (x, \Delta x)_W = (x, D^*Dx)_W = (Dx, Dx)_W$ which implies that $Dx = 0$. The opposite inclusion $\operatorname{Ker} D \subseteq \operatorname{Ker}\Delta$ is obvious. Thus $\operatorname{Ker} D = \operatorname{Ker}\Delta$. Similarly, we obtain $\operatorname{Ker} D^* = \operatorname{Ker}\overline{\Delta}$.

We prove parallel identities for the images. For $b \in \operatorname{Im} D^*$, there is an element $a \in \overline{W}$ such that $b = D^*a$. For $a' = \check{\overline{\Delta}}a \in \overline{W}$, let us set $b' = D^*a'$. We claim that b' is in the Δ-preimage of b. Indeed $\Delta b' = \Delta D^*a' = D^*DD^*a' = D^*DD^*\check{\overline{\Delta}}a = D^*\overline{\Delta}\,\check{\overline{\Delta}}a = D^*(\mathrm{Id}_{\overline{W}} - \overline{K})a$ by Theorem 2. This expression equals to $D^*a - D^*\overline{K}a = D^*a = b$ because $\overline{K}$ maps into $\operatorname{Ker}\overline{\Delta}$ that is a subspace of $\operatorname{Ker} D^*$ (by the previous paragraph). Thus $\operatorname{Im} D^* \subseteq \operatorname{Im}\Delta$. Since also $\operatorname{Im}\Delta \subseteq \operatorname{Im} D^*$, we get $\operatorname{Im} D^* = \operatorname{Im}\Delta$. Similarly, we prove that $\operatorname{Im} D = \operatorname{Im}\overline{\Delta}$.

Thus, we have the orthogonal decompositions $W = \operatorname{Ker} D \oplus \operatorname{Im} D^*$ and $\overline{W} = \operatorname{Ker} D^* \oplus \operatorname{Im} D$. In particular, $\operatorname{Im} D$ and $\operatorname{Im} D^*$ are closed with respect to the topology generated by the C^*-norm on $\overline{W}$ and W, respectively. □

3 Images of elliptic operators on compact Hilbert CV-bundles

Let $p : \mathcal{W} \to M^n$ be a Banach fibre bundle on an n-dimensional manifold M with the fibre a Banach space $(W, |\,|_W)$ and a maximal C^∞-differentiable bundle atlas $\mathcal{A}$. We denote the fibre $p^{-1}(m)$ of $\mathcal{W}$ in m by $\mathcal{W}_m$. In particular, for any $m \in M$

- the fibre $\mathcal{W}_m$ is a Banach space equipped with a norm, denoted by $|\,|_m$, and
- the topology on $\mathcal{W}_m$ generated by $|\,|_m$ coincides with the subset topology on $\mathcal{W}_m \subseteq \mathcal{W}$.

The space of C^∞-differentiable sections of p is denoted by $\Gamma^\infty(M, \mathcal{W})$ or by $\Gamma^\infty(\mathcal{W})$ when the manifold is known from the context. See also the Preamble e), f), g) for notions of the C^∞-differentiability, fibre bundles, bundle atlases, and charts. We suppose that bundle charts of Banach bundles are fibre-wise isometric isomorphisms.

Let g be a Riemannian metric on a compact manifold M and $\nabla^{\mathcal{W}}$ be a covariant derivative on a Banach fibre bundle $p : \mathcal{W} \to M$. Notice that the existence of a covariant derivative is proven by a choice of a partition of unity on M (regardless the infinite dimension of the fibre) similarly as, e.g., in Kolář, Michor, Slovák [24] (finite rank case). We set for any $l \geqslant 0$ and $s \in \Gamma^\infty(\mathcal{W})$

$$|s|_l^F = \sup\{\big|(\nabla^{\mathcal{W}}_{X_1} \dots \nabla^{\mathcal{W}}_{X_k} s_{|U})(m)\big|_m : X_i \in \Gamma^\infty(M, TU),\ g(X_i, X_i) = 1,$$
$$i = 1, \dots, k,\ m \in U,\ U \text{ open in } M,\ 1 \leqslant k \leqslant l\},$$

which are norms on $\Gamma^\infty(\mathcal{W})$, usually called the Fréchet norms. For $k = 0$, the expression $|\nabla^{\mathcal{W}}_{X_1} \dots \nabla^{\mathcal{W}}_{X_k} s(m)|_m$ denotes $|s(m)|_m$. We call the topology on $\Gamma^\infty(\mathcal{W})$ generated by these norms the *Fréchet topology*. It is obvious that with this topology the space $\Gamma^\infty(M, \mathcal{W})$ is a Fréchet space.

For a Banach bundle $p : \mathcal{W} \to M$, let $\mathcal{W} \times_M \mathcal{W} = \{(w, w') \in \mathcal{W} \times \mathcal{W} | \, p(w) = p(w')\} \subseteq \mathcal{W} \times \mathcal{W}$ be the fibred product (also called the Whitney sum) of p with itself considered with the subset topology, where $\mathcal{W} \times \mathcal{W}$ is equipped with the product topology. The atlas of the fibred product consists of charts which are fibred products of the charts of $\mathcal{W}$. (See, e.g., [24].)

A Banach fibre bundle is called a Hilbert fibre bundle if there is a smooth map $(,)_h : \mathcal{W} \times_M \mathcal{W} \to \mathbb{C}$ which is an inner product in each fibre and such that the induced norm $\sqrt{(w, w)_h}$ equals to the Banach norm $|w|_m$ for each $w \in \mathcal{W}_m$ and $m \in M$. Let us mention that we do not consider Hilbert space bundles in the more general sense of Fell [14] or Dupré [11].

One-point Hilbert A-module

Let A be a C^*-algebra, $(E, (,)_E)$ a Hilbert A-module, M a finite dimensional manifold, and $U \subseteq M$ an embedded submanifold of M. We consider $\underline{E}_U = U \times E$ with the product topology, projection $p(m, e) = m$, norm $|(m, e)|_m = |e|_E$, fibre-wise addition $(m, e') + (m, e'') = (m, e' + e'')$, and scalar multiplication $c(m, e) = (m, ce)$, where $m \in U$, $e, e', e'' \in E$, $a \in A$, and $c \in \mathbb{C}$. Obviously the resulting structure is a (product) Banach fibre bundle when it is equipped with a maximal smooth atlas that contains, e.g., the identity chart $U \times E \to \underline{E}_U$. Moreover, we define the fibre-wise C^*-product by $((m, e'), (m, e''))_m = (e', e'')_E$ and the right action of A by $(m, e) \cdot a = (m, e \cdot a)$, where at the right-hand side the action of A on E is understood. If $U = \{m\} \subseteq M$ is a singleton, the introduced structures on $\underline{E}_{\{m\}}$ make it a Hilbert A-module, that we call a *one-point* Hilbert A-module. In this case, $\underline{E}_{\{m\}}$ is both a Banach bundle and a Hilbert A-module.

Definition 3: We call a Banach fibre bundle $p : \mathcal{E} \to M$ a *Hilbert A-bundle* with the typical fibre a Hilbert A-module $(E, (,)_E)$ and a C^∞-differentiable atlas $\mathcal{A}$ if a C^∞-differentiable right action $\cdot_{\mathcal{E}} : \mathcal{E} \times A \to \mathcal{E}$ of A on $\mathcal{E}$ and a C^∞-differentiable mapping $(,)_{\mathcal{E}} : \mathcal{E} \times_M \mathcal{E} \to A$ are given such that for each point $m \in M$

i) the action $\cdot_{\mathcal{E}}$ and the map $(,)_{\mathcal{E}}$ restricted to $\mathcal{E}_m \times A$ and to $\mathcal{E}_m \times \mathcal{E}_m$, respectively, make the fibre $\mathcal{E}_m$ a Hilbert A-module;

ii) for any atlas chart $\phi : U \times E \to p^{-1}(U)$ such that $m \in U$, the restriction $\phi(m, -) : \{m\} \times E \to \mathcal{E}_m$ is a unitary Hilbert A-module isomorphism of the one-point Hilbert A-module $\underline{E}_{\{m\}} = \{m\} \times E$ with the Hilbert A-module $\mathcal{E}_m$; and

iii) $\mathcal{A}$ is a subset of the atlas of the Banach fibre bundle p and it is a maximal smooth atlas having these properties.

Transition maps of the atlas of a Hilbert A-bundle are necessarily continuous maps of open subsets of M into the group $G = \mathrm{Aut}^0(E) \cap \mathrm{Aut}_A^*(E)$ of unitary elements in $\mathrm{Aut}_A^*(E)$ considered with the topology given by the inclusion $\mathrm{Aut}_A^*(E) \hookrightarrow \mathrm{End}_A^*(E)$, where the space of Hilbert A-module morphisms is equipped with the strong operator topology. The smoothness of the charts implies that transition maps are smooth as maps of open subsets of M into $\mathrm{End}_A^*(E)$ equipped with the strong operator topology as well. For a use of the strong operator and norm operator topologies in fibre bundles and also in the Lie group representation theory, we refer the interested reader to Schottenloher [48].

Notice also that the Banach bundle $\underline{E}_U$ introduced above need not be a Hilbert A-bundle. However, when we want to consider it as a Hilbert A-bundle, we equip it with the maximal smooth bundle atlas that contains the identity chart $\mathrm{Id} : U \times E \to U \times E$ and such that the conditions in Definition 3 are satisfied.

It is immediate to see that a Hilbert bundle is the same as a Hilbert A-bundle for the C^*-algebra A of complex numbers.

Let $\mathcal{E} \to M$ be a Hilbert A-bundle. Let us define an action of A from the right on the complex vector space of smooth sections $\Gamma^\infty(\mathcal{E})$ by the formula $(f \cdot a)(m) = f(m) \cdot_{\mathcal{E}} a$, where $f \in \Gamma^\infty(\mathcal{E})$, $a \in A$ and $m \in M$. This makes $\Gamma^\infty(\mathcal{E})$ a right A-module. The action of A restricts to the set $\Gamma_c^\infty(\mathcal{E})$ of compactly supported elements of $\Gamma^\infty(\mathcal{E})$. We define a map $(,)^\sim : \Gamma^\infty(\mathcal{E}) \times \Gamma^\infty(\mathcal{E}) \to C^\infty(M, A)$ by $(f,h)^\sim(m) = (f(m), h(m))_{\mathcal{E}}$ for $f, h \in \Gamma^\infty(\mathcal{E})$ and $m \in M$. Fixing a Riemannian metric g on M, we choose a density form $|\mathrm{vol}_g|$ on M adapted to g, which induces a Radon measure μ on the Borel σ-algebra on M. We define an A-valued A-sesquilinear map on $\Gamma_c^\infty(\mathcal{E})$ by $(f,h) = \int_{m \in M} (f,h)^\sim d\mu$, where $f, h \in \Gamma_c^\infty(\mathcal{E})$ and for our purposes, the integral sign denotes the Bochner integral of A-valued μ-measurable maps on M. By basic properties of the Bochner integral, the pair $(\Gamma_c^\infty(\mathcal{E}), (,))$ is a pre-Hilbert A-module. The induced C^*-norm is denoted by $||$ in the rest of the paper. We call the topology on $\Gamma_c^\infty(\mathcal{E})$ induced by this norm the *pre-Hilbert topology*. (In general, it is not induced by an inner product.)

Let us recall that a Hilbert A-bundle is called *finitely generated* or *finitely generated projective* if its fibres are finitely generated or finitely generated projective, respectively, as Hilbert A-modules.

We summarize our notation concerning (pre-)Hilbert A-modules and Hilbert A-bundles.

1) The norm on a C^*-algebra A is denoted by $||_A$. The C^*-product on a general (pre-)Hilbert A-module W is denoted by $(,)_W$ and the induced C^*-norm is denoted by $||_W$.

2) The norm on a fixed fibre of a Banach bundle has the "foot-point" of the fibre as the lower index. Thus $||_m$ denotes the norm in the fibre $p^{-1}(m)$.

The Fréchet norms on compactly supported bundle sections are indexed by non-negative integers and denoted by $| |_l^F$.

3) The action of a C^*-algebra A on the total space $\mathcal{E}$ of a Hilbert A-bundle is denoted by $\cdot_{\mathcal{E}}$ and the distinguished A-valued map on the fibred product $\mathcal{E} \times_M \mathcal{E}$ is denoted by $(,)_{\mathcal{E}}$. The right action of A, the A-product, and the induced C^*-norm defined on $\Gamma_c^\infty(\mathcal{E})$ do not have any index.

In the next lemma, the Fréchet and the pre-Hilbert topologies are compared. If τ_1 and τ_2 are topologies defined on the same space, τ_1 is said to be finer than τ_2 if τ_1 contains τ_2, i.e., $\tau_1 \supseteq \tau_2$.

Lemma 4: Let M be a compact manifold and $p : \mathcal{E} \to M$ be a Hilbert C^*-bundle. Then the Fréchet topology on $\Gamma^\infty(\mathcal{E})$ is finer than the pre-Hilbert topology.

Proof. Since the considered topological spaces are metrisable, they are sequential (see [43]). By the triangle inequality, it is sufficient to consider a sequence $(f_n)_{n\in\mathbb{N}} \subseteq \Gamma^\infty(\mathcal{E})$ that converges in the Fréchet topology (i.e., in all Fréchet norms) to the zero section. By the definition of $| |_0^F$, the sequence converges also uniformly to the zero section on M. Especially for any $\epsilon > 0$ there is a positive integer n_0 such that for each $n > n_0$ and all $m \in M$ we have $|f_n(m)|_m < \epsilon$. The constant function ϵ defined on M has a finite integral over the compact manifold M. Consequently, if n approaches infinity, $\int_M (f_n, f_n)^{\sim} d\mu \to 0$ by the dominant convergence for the Bochner integral. Thus $(f_n)_{n\in\mathbb{N}}$ converges to the zero section also in the pre-Hilbert topology. □

Pseudodifferential operators on Banach fibre bundles

Let $p' : \mathcal{W}' \to M$ and $p'' : \mathcal{W}'' \to M$ be Banach bundles and let $D : \Gamma^\infty(\mathcal{W}') \to \Gamma^\infty(\mathcal{W}'')$ be a pseudodifferential operator on M of order $d \geqslant 0$. The principal symbol $\sigma_d(D, \xi) : \mathcal{W}'_m \to \mathcal{W}''_m$ of D in the cotangent vector $\xi \in T_m^* M$, $m \in M$, is defined as in the finite rank case. Thus, let $U \ni m$ be open in M and $g : U \to \mathbb{R}$ be a smooth function whose first de Rham differential in m is ξ, $s : U \to p'^{-1}(U) \subseteq \mathcal{W}'$ be a local section of the considered Banach bundle $\mathcal{W}'$ and let us set $w = s(m)$. Then $\sigma_d(D, \xi)(w) = [D((g - g(m))^d s)](m)$. The principal symbol does not depend on the choice of g. (We omit the usual factors $\frac{(-i)^d}{d!}$ or $\frac{i^d}{d!}$ in its definition.) For a general $d \in \mathbb{Z}$, A-pseudodifferential operators are defined in Solovyov, Troitsky [50] by elements of the so-called symbol algebras. (See, e.g., the formula at p. 104 in [50].) We consider the following natural generalization for Banach fibre bundles. 1) On fibres of the Banach fibre bundles we use norms instead of the induced C^*-norms on fibres of the Hilbert A-bundles, and 2) we drop the condition on the self-duality of the fibres ([35], p. 78) since we do not assume the fibres to be Hilbert C^*-modules. 3) We do not consider the elements of the symbol algebras to be adjointable, but we demand them to be continuous, which is a consequence of the adjointability in the Hilbert C^*-bundle case. For purposes of our statements on smooth Banach fibre bundles on compact manifolds, we demand the continuity in each fibre only. We follow the symbol algebra approach to pseudodifferential operators, but we add also separate proofs in the case when we deal with symbols of pseudodifferential operators of non-negative orders.

A pseudodifferential operator of order d on Banach bundles $p' : \mathcal{W}' \to M$ and $p'' : \mathcal{W}'' \to M$ on a manifold M of dimension $n > 0$ is called *elliptic* if its principal d-symbol $\sigma_d(D, \xi)$ is a linear homeomorphism of $(\mathcal{W}'_m, |\,|'_m)$ onto $(\mathcal{W}''_m, |\,|''_m)$ for any non-zero element $\xi \in T^*_m M$ and any point $m \in M$. If the dimension of the manifold is zero, the set of non-zero cotangent vectors is empty, and therefore any pseudodifferential operator would be elliptic if we allow $n = 0$. Therefore we exclude zero dimensional manifolds from the definition of ellipticity.

Lemma 5: Let $D : \Gamma^\infty(\mathcal{W}) \to \Gamma^\infty(\mathcal{W})$ be a pseudodifferential operator of order d acting on smooth sections of a Banach bundle $p : \mathcal{W} \to M$, with the fibre a Banach space $(W, |\,|_W)$ and with a maximal C^∞-differentiable atlas $\mathcal{A}$. Then for any $m \in M$ and $\xi \in T^*_m M$, the principal d-symbol $\sigma_d(D, \xi) : \mathcal{W}_m \to \mathcal{W}_m$ is continuous with respect to the topology on $\mathcal{W}_m$ generated by the norm $|\,|_m$.

Proof. Let m be a point on M and let us consider a bundle chart in $\mathcal{A}$ with domain $U \times W$ such that $m \in U$. We identify $\mathcal{W}_m$ with W by an isometric isomorphism of Banach spaces, and consider local sections on U as maps of U into W. For $w \in W$ and $\xi \in T^*_m M$, let g be a smooth function defined on U such that $\xi = (dg)_m$, f be a smooth W-valued function on U such that $w = f(m)$, and $(w_i)_{i \in \mathbb{N}}$ be a sequence converging to w in $\mathcal{W}_m \cong W$ (isometric isomorphism).

Suppose that the order d of D is non-negative. We shall prove that $\lim_i \sigma_d(D, \xi) w_i = \sigma_d(D, \xi) w$. For each positive integer i and $m' \in U$, we set $f_i(m') = f(m') + w_i - w$. Consequently, $(f_i)_{i \in \mathbb{N}}$ is a sequence of smooth W-valued functions defined on U converging to f in the Fréchet norms $|\,|^F_l$ restricted to $C^\infty(U, W)$, $l \geqslant 0$. Since D is continuous in the Fréchet topology, we have that $D\left((g - g(m))^d f_i\right)$ converges to $D\left((g - g(m))^d f\right)$ with respect to this topology and, in particular, with respect to $|\,|^F_0$. Since the convergence in $|\,|^F_0$ implies the point-wise convergence, we have $\left(D((g - g(m))^d f_i)\right)(m)$ converges to $\left(D((g - g(m))^d f)\right)(m)$ in the Banach space $\mathcal{W}_m$ if i goes to infinity. Notice that $f_i(m) = w_i$. Consequently $\lim_i \sigma_d(D, \xi) w_i = \lim_i [\sigma_d(D, \xi)(f_i(m))] = \lim_i [D\left((g - g(m))^d f_i\right)(m)] = D\left((g - g(m))^d f\right)(m) = \sigma(D, \xi)(w)$. Thus the symbol of D in direction ξ is continuous with respect to the Banach space topology on $\mathcal{W}_m \cong W$.

If the order of D is negative, its symbol is supposed to be continuous by our conventions. □

Remark: The above lemma can be generalized easily to pseudodifferential operators $D : \Gamma^\infty(M, \mathcal{W}') \to \Gamma^\infty(M, \mathcal{W}'')$ between Banach bundles $\mathcal{W}' \to M$ and $\mathcal{W}'' \to M$.

Agreement: For a C^*-algebra A, let A^e denote the unitalization of A. In particular, $A^e = A \oplus \mathbb{C}$ as a complex vector space. See, e.g., [10] where the unitalization is defined. Any right (pre-)Hilbert A-module W is made a (pre-)Hilbert A^e-module by setting $w \cdot (B, c) = w \cdot B + cw$, where $B \in A$, $w \in W$ and $c \in \mathbb{C}$, and by keeping the A-product unchanged. We call the resulting (pre-)Hilbert A^e-module the *extended (pre-)Hilbert A-module.* Any morphism of (pre-)Hilbert A-modules is also a morphism of the corresponding extended pre-Hilbert A^e-modules. Notice that a (pre-)Hilbert A-module and its extension are equal as topological vector spaces.

Conventions and results regarding A-pseudodifferential operators

We recall results from Mishchenko and Fomenko [36] and Solovyov and Troitsky [50], which we use later. Let A be a C^*-algebra and let $\mathcal{E}', \mathcal{E}'' \to M$ be Hilbert A-bundles on a compact manifold M. We suppose that A-pseudodifferential operators are defined by symbol algebras as in [50] with minor modifications that we mention below. In any case, A-pseudodifferential operators are pre-Hilbert A-module homomorphisms of smooth section spaces of Hilbert A-bundles. Notice that in [50] the fibres are considered to be so-called self-dual Hilbert A-modules, but we shall consider the Hilbert CV-module CV which is not self-dual.

Let $\mathcal{E} \to M$ be a Hilbert A-bundle. Using a partition of unity on M, one defines the so-called *Sobolev-type completion* $(W^{k,2}(M, \mathcal{E}), (,)_k^S)$, $k \in \mathbb{Z}$, of the pre-Hilbert A-module $(\Gamma^\infty(\mathcal{E}), (,))$ of smooth sections. We shall often denote them by $W^k(\mathcal{E})$ instead of $W^{k,2}(M, \mathcal{E})$ if the manifold is known from the context. The definition of Sobolev-type completions in [50] is parallel to the definition of the scalar Sobolev spaces. However, for $k < 0$, we still define $W^k(\mathcal{E})$ by the Fourier transform and not as the A-dual space of $W^{-k}(\mathcal{E})$ due to the lack of the self-dual property of the fibres . The C^*-norm on $W^k(\mathcal{E})$ induced by $(,)_k^S$ is denoted by $| \, |_k^S$. The Sobolev-type completions are Hilbert A-modules. As a Hilbert A-module, $W^k(\mathcal{E})$ depends on the partition of unity. However, its homeomorphism equivalence class does not. If the base manifold is compact, the restriction of $| \, |_0^S$ to $\Gamma^\infty(\mathcal{E})$ is equivalent to the C^*-norm on $\Gamma^\infty(\mathcal{E})$ induced by $(,)$. Let us notice that when we inspect appropriate proofs in [36], we see that it is not needed to assume that A is unital for the construction of the Sobolev-type completions.

A further modification which we adopt concerns the fact that we consider arbitrary Hilbert A-modules as fibres, but we demand that a symbol as a map $\Gamma^\infty(T^*M, \mathcal{E}') \to \Gamma^\infty(T^*M, \mathcal{E}'')$ has a continuous extension to any Sobolev-type completion $W^k(T^*M, \mathcal{E}')$ that is *adjointable* as a map of Hilbert A-modules. When we consider that a symbol map assigns to each cotangent vector $\xi \in T_m^*M$ an element from the algebra $\mathrm{Hom}_A(\mathcal{E}'_m, \mathcal{E}''_m)$, we suppose that this element is adjointable, i.e., in $\mathrm{Hom}_A^*(\mathcal{E}'_m, \mathcal{E}''_m)$ (point-wise adjointability). It is easy to see that this property ensures that an A-pseudodifferential operator $D : \Gamma^\infty(M, \mathcal{E}') \to \Gamma^\infty(M, \mathcal{E}'')$ has the adjoint as a pre-Hilbert A-module morphism, i.e., with respect to the A-products on the smooth section spaces. The

A-differential operators have continuous extensions to the Sobolev-type completions and these extensions are adjointable thanks to the supposed adjointability of the continuous extensions of symbols.

Notice that we shall consider CV-pseudodifferential operators. Their continuous extensions to Sobolev-type completions are continuous CV-module maps. The same is true for the extensions of symbols. By the result of Magajna (see Remark 5, Bakić, Guljaš [5], p. 263), all CV-linear continuous maps between Hilbert CV-modules are adjointable. Especially, our assumptions on the symbols will be satisfied in this case.

Let us describe the situation of pseudodifferential operators of non-negative order using more familiar terms. Let $p' : \mathcal{E}' \to M$ and $p'' : \mathcal{E}'' \to M$ be Hilbert A-bundles with fibres E' and E'', respectively and let $D : \Gamma^\infty(\mathcal{E}') \to \Gamma^\infty(\mathcal{E}'')$ be an A-pseudodifferential operator on M^n of order $d \geqslant 0$. Thus for any $m \in M$ there are bundle charts $U \times E' \to {p'}^{-1}(U)$ and $U \times E'' \to {p''}^{-1}(U)$, a manifold chart $\mu = (x^1, \ldots, x^n) : U \to \mathbb{R}^n$ around $m \in U$, and a finite set $K \subseteq \mathbb{N}_0^n$ such that the operator D can be expressed locally with respect to the charts as $\sum_{\beta \in K} c_\beta \partial^\beta$, where $c_\beta : U \to \mathrm{Hom}_A^*(E', E'')$ are smooth, ∂^β denotes $\partial_{x^1}^{\beta_1} \ldots \partial_{x^n}^{\beta_n}$ and ∂_{x^i} is the Gateaux derivative in the direction of the i-the element of the canonical basis of $\mathbb{R}^n$ acting on E'-valued functions on $\mu(U)$. Regarding the smoothness of c_β's, the vector space $\mathrm{Hom}_A^*(E', E'')$ is considered with the strong operator topology, which is well known to be coarser than the operator norm topology on this space. When this topology is fixed, D maps smooth sections of $\mathcal{E}'$ into smooth sections of $\mathcal{E}''$.

According to Fomenko and Mishchenko [36] an elliptic A-pseudodifferential operator is called *A-elliptic*.

Let us suppose that A is a unital C^*-algebra, $p' : \mathcal{E}' \to M$ and $p'' : \mathcal{E}'' \to M$ are *finitely generated projective* Hilbert A-bundles on a compact manifold M, and $D : \Gamma^\infty(\mathcal{E}') \to \Gamma^\infty(\mathcal{E}'')$ is an A-elliptic operator. If $s \in W^k(\mathcal{E}')$ satisfies $D_k s = f$ for a smooth section $f \in \Gamma^\infty(\mathcal{E}'')$, then s is smooth (elliptic regularity). This result is proved in [36]. See also Theorem 2.1.144 in Solovyov, Troitsky [50]. We prove a generalization of the elliptic regularity using neither the assumption that the Hilbert A-bundle is finitely generated projective, nor that A is unital. We use a theorem on the Sobolev-type smooth embedding (Lemma 5 in Krýsl [25]) for Hilbert A-bundles over compact manifolds. The assumption in [25] on the unitality on A is not used in the proof of Lemma 5. Moreover, it can be removed using the Agreement easily. Notice that a Sobolev-type embedding theorem is proved for finitely generated projective Hilbert A-bundles also in [36]. However, the assumptions are inessential for the proof given there, that the bundle is finitely generated projective.

Let us recall that a pseudodifferential A-operator is called *smoothing* if its continuous extension to $W^k(\mathcal{E}')$ is a map into $W^{k+1}(\mathcal{E}'')$ for each k.

Theorem 6 (elliptic regularity): Let A be a C^*-algebra, $p' : \mathcal{E}' \to M$ and $p'' : \mathcal{E}'' \to M$ be Hilbert A-bundles over a compact manifold M, and

$D : \Gamma^\infty(\mathcal{E}') \to \Gamma^\infty(\mathcal{E}'')$ be an A-elliptic operator of order d. If $D_k s \in \Gamma^\infty(\mathcal{E}'')$ for a section $s \in W^k(\mathcal{E}')$ and an integer k, then $s \in \Gamma^\infty(\mathcal{E}')$. Moreover, the adjoint D_k^* fulfils the elliptic regularity as well.

Proof. For $s \in W^k(\mathcal{E}')$, let us set $f = D_k s$. Using the A-ellipticity, we get a Hilbert A-module morphism $\check{D}_{k-d} : W^{k-d}(\mathcal{E}'') \to W^k(\mathcal{E}')$ by inverting the symbol of D out of the zero section of the cotangent bundle of M and using the Fourier transform (subordinated to a partition of unity on M) in the same classical way as, e.g., in the proof of Theorem 3.4 in Fomenko, Mishchenko [36]. (Cf. also Thm. 4 in Chapter XI in Palais [39] for the finite rank case.) We obtain operators $N_k = \check{D}_{k-d} D_k - \mathrm{Id}_{W^k(\mathcal{E}')}$, that are smoothing. For a section s satisfying the assumptions of the theorem, we get $s = \check{D}_{k-d} D_k s - (\check{D}_{k-d} D_k - \mathrm{Id}_{W^k(\mathcal{E}')}) s = \check{D}_{k-d} f - N_k s \in W^{k+1}(\mathcal{E}')$ because $\check{D}_{k-d} f \in \Gamma^\infty(\mathcal{E}')$ and because N_k is smoothing. This implies that $s \in \bigcap_{l=k}^\infty W^l(\mathcal{E}')$ which equals to $\Gamma^\infty(\mathcal{E}')$ by Lemma 5 in [25].

By our conventions, the adjoint of D_k exists. Let us choose a point $m \in U \subseteq M$, where U is open, and a smooth local function $g : U \to \mathbb{R}$ such that $(dg)_m = \xi$ is non-zero. Since $(s, D_k^* t)_k^S = (D_k s, t)_{k-d}^S$ for any smooth sections s, t, we have

$$\begin{aligned}\left(s, \lambda^{-d} e^{-i\lambda g} D_k^*(t e^{i\lambda g})\right)_k^S &= \lambda^{-d}\left(e^{i\lambda g} s, D_k^*(t e^{i\lambda g})\right)_k^S \\ &= \lambda^{-d}\left(D_k(e^{i\lambda g} s), e^{i\lambda g} t\right)_{k-d}^S \\ &= \left(\lambda^{-d} e^{-i\lambda g} D_k(e^{i\lambda g} s), t\right)_{k-d}.\end{aligned}$$

Taking the limit $\lambda \to \infty$, we get $(s, \sigma(D_k^*, \xi) t)_k^S = (\sigma(D_k, \xi) s, t)_{k-d}^S$, i.e., the principal symbol of D_k^* is the adjoint of the principal symbol of D_k with respect to the Sobolev-type C^*-products $(,)_{k-d}^S$ and $(,)_k^S$. Especially, if the principal symbol of D is an isomorphism, the principal symbol of D_k^* restricted to smooth sections is an isomorphism as well. Therefore we can proceed as above to find a smoothing operator N_k, and obtain that a solution s to the equation $D_k^* s = f$ is smooth by the cited lemma in [25]. □

3.1 Compact Hilbert CV-module

Let V^* be the continuous dual of a complex Hilbert space (V, h). By the (Riesz's) representation theorem for Hilbert spaces, there exists a complex anti-linear homeomorphism $\sharp : V^* \to V$ onto V such that $h(\sharp\alpha, v) = \alpha(v)$ holds for any $\alpha \in V^*$ and $v \in V$. This makes us able to define $h^*(\alpha, \beta) = h(\sharp\beta, \sharp\alpha)$, where $\alpha, \beta \in V^*$. It is easy to see that the hermitian-symmetric sesquilinear form h^* is an inner product on V^* and that (V^*, h^*) is a Hilbert space.

The complex vector space CV of compact operators on the Hilbert space (V, h) is a right CV-module with respect to the action $CV \times CV \ni (B, C) \mapsto$

$B \cdot C = B \circ C \in CV$. Let us set $(B, C)_{CV} = B^* \circ C$ for $B, C \in CV$, which is a C^*-product. The induced C^*-norm is the operator norm on the C^*-algebra CV of compact operators on V by the C^*-property of C^*-algebras norms. The action and the C^*-product define a structure of a Hilbert CV-module CV. We call this Hilbert module the *compact Hilbert CV-module*. Let us notice that this construction works for any C^*-algebra and not only for CV. It is well known that the compact Hilbert CV-module is not finitely generated. (See, e.g., Wegge-Olsen [56].)

We recall the following definition ([5], p. 254).

Definition 4: Let $(E, (,)_E)$ be a Hilbert A-module. We call a subset $(v_j)_{j \in J} \subseteq E$ an *orthonormal basis* of $(E, (,)_E)$ if

i) the set $(v_j)_{j \in J}$ generates (by taking finite right A-linear combinations) a dense A-submodule of the Hilbert A-module E;

ii) $(v_j, v_{j'})_E = 0$ whenever $j \neq j'$;

iii) for each $j \in J$, the element $\xi_j = (v_j, v_j)_E \in A$ is an orthogonal projection, i.e., a non-zero hermitian-symmetric idempotent in A; and

iv) $\xi_j A \xi_j = \mathbb{C}\xi_j$ for all $j \in J$ (minimality).

Let us consider a Hilbert basis $(e_j)_{j \in \mathbb{N}}$ of the separable Hilbert space (V, h) (and also an orthonormal basis of the Hilbert $\mathbb{C}$-module V) and denote its dual basis by $(\epsilon^j)_{j \in \mathbb{N}}$. It is immediate to see that it is a Hilbert basis of (V^*, h^*). We set

$$v_i = e_i \otimes \epsilon^1 \in CV$$

for $i \in \mathbb{N}$, where for any $u, w \in V$ and $\alpha \in V^*$, $(u \otimes \alpha)(w) = \alpha(w)u$. In the bra-ket notation, $u \otimes \alpha$ is written as $|u\rangle\langle\alpha|$ or more precisely as $|u\rangle\langle\sharp\alpha|$. It is easy to realize that $(v_i)_{i \in \mathbb{N}}$ is an orthonormal basis of the compact Hilbert CV-module. (A similar Hilbert CV-module basis appears also in [5].)

3.2 Compact Hilbert CV-bundle and a CV-compact embedding for tori

In Fomenko and Mishchenko [36], a version of the Rellich–Kondrachov compact embedding theorem is proved for finitely generated projective Hilbert A-bundles over compact manifolds. The proof in [36] is based substantially on the fact that the bundles are finitely generated. We prove a C^*-compact embedding theorem for certain Hilbert CV-bundles which are not finitely generated but they satisfy the following

Definition 5: Let M be a smooth manifold and CV be the compact Hilbert CV-module. We consider the product Banach bundle $\underline{CV}_M = M \times CV \to M$

equipped with the Hilbert CV-bundle structure introduced below the Definition 3 for $U = M$, and we call this bundle the *compact Hilbert CV-bundle* on M.

Let us consider the n-dimensional torus T^n as the quotient $\mathbb{R}^n/\mathbb{Z}^n$ of the canonical manifold structures on $\mathbb{R}^n$ and $\mathbb{Z}^n$, equip it with the flat Riemannian metric induced by the standard Euclidean inner product $(,)_{\mathbb{R}^n}$ on $\mathbb{R}^n$, and denote the corresponding norm on $\mathbb{R}^n$ by $||_{\mathbb{R}^n}$. We denote the measure induced by volume density form of the Riemannian metric by μ_{T^n}. For $i, j \in \mathbb{N}$, $\vec{m} \in \mathbb{Z}^n$ and $\vec{\theta} \in \mathbb{R}^n$, we define functions ${\phi^{\vec{m}}}_i^j : T^n \to CV$ by setting

$${\phi^{\vec{m}}}_i^j([\vec{\theta}]) = e^{2\pi\imath(\vec{m},\vec{\theta})_{\mathbb{R}^n}} e_i \otimes \epsilon^j,$$

where $[\vec{\theta}]$ denotes the equivalence class of $\vec{\theta} \in \mathbb{R}^n$ in the quotient $\mathbb{R}^n/\mathbb{Z}^n$. Recall that $(e_i)_{i\in\mathbb{N}}$ is a Hilbert basis of V and $(\epsilon^i)_{i\in\mathbb{N}}$ is the dual Hilbert basis.

Let us consider the (positive semi-definite) Laplace operator $\Delta = -\sum_{i=1}^n \partial^2_{x^i}$ for the standard Euclidean space $(\mathbb{R}^n, (,)_{\mathbb{R}^n})$, and the appropriate differential operator Δ^{CV} that acts on smooth CV-valued functions which are defined on the Euclidean space. We consider the partial derivative with respect to the variable x^i as the Gateaux derivative in the direction of the i-th vector of the canonical basis of $\mathbb{R}^n$. By the translation invariance of these derivatives, Δ^{CV} quotients to a differential operator on the smooth CV-valued functions on the torus. The resulting operator is obviously right CV-linear. We denote it by $\Delta^{CV}_{T^n}$. (Equivalently, we can multiply tensorially the negative of the Laplace–Beltrami operator on the flat torus T^n by the identity operator on CV and take its continuous extension to the injective tensor product completion $C^\infty(T^n)\widehat{\otimes}_\epsilon CV \cong C^\infty(T^n, CV)$ (linear homeomorphism), where the spaces of C^∞-functions are equipped with the Fréchet topology. We use this approach below in a more general context.)

In particular $\Delta^{CV}_{T^n} : C^\infty(T^n, CV) \to C^\infty(T^n, CV)$. Sobolev-type CV-products $(,)^S_k$ are computed similarly as in [36], pp. 107 and 108. Namely for $\vec{m}_1, \vec{m}_2 \in \mathbb{Z}^n$ and $i, j, l, p \in \mathbb{N}$,

$$\begin{aligned}({\phi^{\vec{m}_1}}_i^j, {\phi^{\vec{m}_2}}_p^l)^S_k &= \int_{T^n} \left({\phi^{\vec{m}_1}}_i^j\right)^* \circ \left(\left(\mathrm{Id}_{C^\infty(T^n,CV)} + \frac{1}{4\pi^2}\Delta^{CV}_{T^n}\right)^k {\phi^{\vec{m}_2}}_p^l\right) d\mu_{T^n} \\ &= (1 + |\vec{m}_2|^2_{\mathbb{R}^n})^k \delta^{\vec{m}_1,\vec{m}_2} \delta_{ip} e_j \otimes \epsilon^l,\end{aligned}$$

where $\delta^{\vec{m}_1,\vec{m}_2} = 0$ or 1 iff $\vec{m}_1 \neq \vec{m}_2$ or $\vec{m}_1 = \vec{m}_2$, respectively; the Kronecker's δ_{ip} has its classical meaning; the kth power of the operator means the k-folded composition of the operator inside the brackets with itself; and the integral is the Bochner integral of maps on the torus with values in the Banach space CV. (Notice that we use a suitable multiple in front of $\Delta^{CV}_{T^n}$ in order to avoid a further normalization of the set of generators $({\phi^{\vec{m}}}_i^j)$ of $C^\infty(T^n, CV)$ or a change of the Euclidean product on $\mathbb{R}^n$.) The above computation is based on the observation that $-\partial^2_{x^j}(e^{2\pi\imath(\vec{m},\vec{\theta})_{\mathbb{R}^n}} e_p \otimes \epsilon^l) = 4\pi^2 m_j^2 e^{2\pi\imath(\vec{m},\vec{\theta})_{\mathbb{R}^n}} e_p \otimes \epsilon^l$. For $k \in \mathbb{Z}$, let us set ${\psi^{\vec{m},k}}_i^j = (1+|\vec{m}|^2_{\mathbb{R}^n})^{-k/2}{\phi^{\vec{m}}}_i^j$. By the above computation, the set $({\psi^{\vec{m},k}}_i^1)_{i\in\mathbb{N},\vec{m}\in\mathbb{Z}^n}$

is an orthonormal basis of the Hilbert CV-modules $(W^k(T^n,CV),(,)_k^S)$ in the sense of Definition 4.

Let us consider the canonical inclusion

$$I_{k+1}:W^{k+1}(T^n,CV)\to W^k(T^n,CV).$$

For all $n\geqslant 1$ (n is the dimension of the torus), we obtain

$$\begin{aligned}\lim_{(\vec{m},i)\to\infty}|I_{k+1}(\psi^{\vec{m},k+1}{}^1_i)|_k^S &= \lim_{(\vec{m},i)\to\infty}|\psi^{\vec{m},k+1}{}^1_i|_k^S\\ &= \lim_{(\vec{m},i)\to\infty}|(1+|\vec{m}|^2_{\mathbb{R}^n})^{-\frac{k+1}{2}}\phi^{\vec{m}}{}^1_i|_k^S\\ &= \lim_{(\vec{m},i)\to\infty}|(1+|\vec{m}|^2_{\mathbb{R}^n})^{-\frac{1}{2}}(1+|\vec{m}|^2_{\mathbb{R}^n})^{-\frac{k}{2}}\phi^{\vec{m}}{}^1_i|_k^S\\ &= \lim_{(\vec{m},i)\to\infty}(1+|\vec{m}|^2_{\mathbb{R}^n})^{-\frac{1}{2}}|\psi^{\vec{m},k}{}^1_i|_k^S\\ &= \lim_{(\vec{m},i)\to\infty}(1+|\vec{m}|^2_{\mathbb{R}^n})^{-\frac{1}{2}}=0.\end{aligned}$$

The limits considered above are limits of nets whose domain is the lexicographically ordered Cartesian product $\mathbb{Z}^n\times\mathbb{N}$ and $\mathbb{N}$ and $\mathbb{Z}$ have their natural orderings. Notice also that the limit does not depend on i as realized in the second last step of the computation.

We use a result in [5] and the above computation deriving

Scholium 1: If $n\geqslant 1$, the mapping I_{k+1} is CV-compact for each $k\geqslant 0$.

Proof. Since $\lim_{(\vec{m},i)\to\infty}|I_{k+1}(\psi^{\vec{m},k+1}{}^1_i)|_k^S=0$ as computed above, I_{k+1} is CV-compact by Bakić, Guljaš [5], Theorem 9 (ii). □

Remark:

1) Scholium 1 can be proved in a similar way for negative integers k by considering $\phi^{\vec{m}}{}^1_i$ as distributions acting on smooth functions on the torus, that are used to define the Sobolev-type completions.

2) If we consider $n=0$ (e.g., $M=\{1\}$), $\lim_{(\vec{m},i)\to\infty}|I_{k+1}(\psi^{\vec{m},k+1}{}^1_i)|_k^S=1$ because $|\vec{m}|_{\mathbb{R}^n}=0$. In this case, I_{k+1} is the identity map on the Banach space CV, which is not a compact map if the dimension of V is infinite. Moreover, I_{k+1} is not even CV-compact by Theorem 9 in [5].

3.3 Closed images of CV-elliptic operators on compact CV-bundles

Let A be a unital C^*-algebra, M a compact manifold, and $\mathcal{E}\to M$ a Hilbert A-bundle with the fibre a finitely generated projective Hilbert A-module $(E,(,)_E)$.

For each $k \in \mathbb{Z}$, Hilbert A-module homomorphisms $P_{[k]} : \bigoplus_{i=1}^{l} W^k(T^n, E) \to W^k(M, \mathcal{E})$ are defined using a partition of unity on M in Construction 2.1.76 of [50], in which several approaches from Palais [39] are adapted to the Hilbert C^*-bundle setting. The assumption on the unitality of A can be dropped by considering the unitalization of A as it is explained in the Agreement. In Chapter 2.1 in [50] the assumption is not used that the bundle is finitely generated projective. Thus $P_{[k]}$ can be defined for the compact Hilbert CV-bundle on M by the same formulas. Moreover, for $l \leqslant k$, $P_{[l]}$ is an extension of $P_{[k]}$ since all $P_{[k]}$ are defined as extensions to Sobolev-type spaces of a single map $P : C^\infty(T^n, E) \to \Gamma^\infty(M, \mathcal{E})$. See [50].

The existence of a right inverse $\gamma_{[k]}$ to $P_{[k]}$ is stated in Theorem 2.1.77 in [50]. However, its construction does not appear there. It can proceed in the same way as in Palais [39] (Chapter X, Paragraph 4, Corollary 2 of Theorem 2), where it is done for finite rank vector bundles on compact manifolds. The appropriate Hilbert A-bundle analogue of $\gamma_{[k]}$, which we also denote by $\gamma_{[k]}$, is A-linear because it is constructed by scalar valued functions (derived from a partition of unity on M) and by Hilbert A-bundle charts only. It is adjointable since the Hilbert A-bundle charts are such. Consequently, $\gamma_{[k]}$ is a Hilbert A-module morphism. Notice also that for $l \leqslant k$, $\gamma_{[l]}$ is an extension of $\gamma_{[k]}$. The proof of this statement can be done as in Palais [39] (Chapter X).

Let us recall (cf., e.g., 2.1.28 [50]) that a *Rellich A-chain* is a descending chain (with respect to $\subseteq$) of Hilbert A-modules $(X_k)_{k \in \mathbb{Z}}$ such that the inclusion maps $X_{k+1} \hookrightarrow X_k$ are A-compact for each integer k.

Next we prove a theorem, whose first part is an analogue of the C^*-compact embedding theorem from [36] but it concerns Hilbert C^*-bundles that are not finitely generated. For a notational simplicity of the proof of the next theorem, we suppose that the finite sum $\bigoplus_{i=1}^{l} W^k(T^n, E)$ contains one element only, i.e., $l = 1$. In the first paragraph of the proof, we proceed with minor changes as in the proof of a parallel statement in Palais [39] (Chapter X, Par. 4, Thm. 3).

Theorem 7: Let M be a compact manifold of dimension n and

$$D : C^\infty(M, CV) \to C^\infty(M, CV)$$

be a CV-elliptic operator of order d on the compact Hilbert CV-bundle $\underline{CV}_M = M \times CV$ over M. Then

a) the canonical inclusion $J_{k+1} : W^{k+1}(M, CV) \to W^k(M, CV)$ is CV-compact;

b) for all $k \in \mathbb{Z}$, the image of the continuous extension D_k of D to $W^k(M, CV)$ is closed in $W^{k-d}(M, CV)$ in the topology induced by the Sobolev-type norm $| \, |^S_{k-d}$; and

c) the image of D in $C^\infty(M, CV)$ is closed in the pre-Hilbert and also in the Fréchet topology.

Proof. 1) We have sequences $(W^k(T^n, CV))_{k\in\mathbb{Z}}$ and $(W^k(M, CV))_{k\in\mathbb{Z}}$ at our disposal. The sequence $(W^k(T^n, CV))_{k\in\mathbb{Z}}$ is a Rellich CV-chain because the inclusions $I_{k+1} : W^{k+1}(T^n, CV) \to W^k(T^n, CV)$ are CV-compact by Scholium 1. Let us consider the inclusion map $J_{k+1} : W^{k+1}(M, CV) \to W^k(M, CV)$. By the definitions of $P_{[k+1]}$ and $\gamma_{[k+1]}$ and since I_{k+1} and J_{k+1} are inclusions, the diagram

$$\begin{array}{ccc} W^{k+1}(T^n, CV) & \xrightarrow{I_{k+1}} & W^k(T^n, CV) \\ {\scriptstyle \gamma_{[k+1]}}\uparrow & & \downarrow{\scriptstyle P_{[k]}} \\ W^{k+1}(M, CV) & \xrightarrow{J_{k+1}} & W^k(M, CV) \end{array}$$

commutes. Indeed, $P_{[k]} I_{k+1} \gamma_{[k+1]} s = P_{[k]} \gamma_{[k+1]} s = P_{[k]} \gamma_{[k]} s = s = J_{k+1} s$, where $s \in W^{k+1}(M, CV)$. Consequently, since I_{k+1} is CV-compact (Scholium 1), J_{k+1} is CV-compact by the ideal property of C^*-compact operators, recalling the Remark below Definition 1. Consequently, $(W^k(M, CV))_{k\in\mathbb{Z}}$ is a Rellich CV-chain as well, and the inclusions are CV-compact homomorphisms. Thus a) is proved.

2) Let us fix $k \in \mathbb{Z}$. Since D is elliptic, there exists a partial inverse for the extension D_k of D, i.e., a Hilbert CV-module morphism, denoted by $\check{D}_{k-d}$, such that the operator $N_k = \check{D}_{k-d} D_k - \mathrm{Id}_{W^k(M,CV)}$ maps $W^k(M, CV)$ into $W^{k+1}(M, CV) \subseteq W^k(M, CV)$. Thus N_k is smoothing. Operator $\check{D}_{k-d}$ is constructed by inverting the symbol of D out of the zero section of T^*M, taking the Fourier transform, and extending the resulting operator continuously to the Sobolev-type space $W^{k-d}(M, CV)$. See, e.g., the proof of Theorem 3.4 in [36], where this classical procedure is used in the realm of C^*-pseudodifferential operators. Since $N_k = J_{k+1} \circ N_k$ and J_{k+1} is CV-compact (item 1), N_k is CV-compact as a map of $W^k(M, CV)$ into $W^k(M, CV)$ by the ideal property for C^*-compact homomorphisms.

Similarly, we proceed with the composition of D_k and $\check{D}_{k-d}$. Consequently, D_k is CV-Fredholm. By Lemma 1, the image of D_k is closed in the Sobolev-type space $(W^{k-d}(M, CV), (,)^S_{k-d})$. Thus b) follows.

3) We prove the closedness of the range of D in $C^\infty(M, CV)$ with respect to the pre-Hilbert topology. For Corollary 3, we set $A = CV$, $W = \overline{W} = C^\infty(M, CV)$, $\Delta = D^*D$, and $E = \overline{E} = W^{2d}(M, CV)$ and $F = \overline{F} = W^0(M, CV)$. Since D is CV-elliptic, the self-adjoint operator Δ is CV-elliptic as well (Thms. 2.1.24 and 2.1.17 in [50] on the symbol of the composition and the symbol of the adjoint). By the previous paragraph, the operator $\widetilde{\Delta} = \Delta_{2d}$ (continuous extension of Δ to $W^{2d}(M, CV)$) is CV-Fredholm, and thus assumption i) of Corollary 3 is satisfied. Moreover, since the continuous extension $\widetilde{\Delta}$ is a continuous Hilbert CV-module morphism, its adjoint exists by the mentioned result in Bakić, Guljaš in [5], p. 263. By Theorem 6 (elliptic regularity), $\widetilde{\Delta}$ and $\widetilde{\Delta}^*$ satisfy also the assumption ii) of Corollary 3, that is also the assumption ii) in Theorem 2. Therefore, we conclude by this corollary that the image of D is closed in $C^\infty(M, CV)$ with respect to the pre-Hilbert topology.

By Lemma 4, each closed set in the pre-Hilbert topology is closed in the Fréchet topology, which proves the rest of c). □

4 Images of elliptic operators

In this chapter, we investigate topological properties of images of elliptic operators defined on smooth sections of Hilbert bundles on compact manifolds, not assuming the invariance of the operators with respect to the C^*-algebra CV of compact operators.

4.1 Injective completion of tensor products

Let $\widehat{E}$ be the Hausdorff completion of a metric space E, which is constructed by taking all Cauchy sequences in E and considering two of such sequences as equivalent if they differ by a sequence converging to zero in E (null-sequence). The completion $\widehat{E}$ is equipped with a canonical extension of the metric, which is defined on E. With respect to the extension, E is dense in $\widehat{E}$. We consider also the canonical isometric embedding of E into $\widehat{E}$ which maps an element $a \in E$ to the equivalence class containing the constant sequence $(a)_{i \in \mathbb{N}}$. The element of $\widehat{E}$ determined by a Cauchy sequence $(a_i)_{i \in \mathbb{N}} \subseteq E$ is denoted by $[(a_i)_i]$ or by $\lim_i a_i$. If $(a)_{i \in \mathbb{N}}$ is a constant sequence, we denote its image in the completion $\widehat{E}$ by $[a]$. See [52] for this classical construction.

Let E_1 and E_2 be both real or both complex vector spaces. We denote their (algebraic) tensor product over the ring of real or complex numbers, respectively, by $E_1 \otimes E_2$. If E_1 and E_2 are vector spaces equipped with countable families of seminorms, we denote the tensor product $E_1 \otimes E_2$ by $E_1 \otimes_\epsilon E_2$ when we consider it with the so-called injective seminorms. See Tréves [52]. The Hausdorff completion of the tensor product $E_1 \otimes_\epsilon E_2$, equipped with the metric generated by the injective seminorms, is denoted by $E_1 \widehat{\otimes}_\epsilon E_2$ and called the *completed injective tensor product.* If E_1 and E_2 are Fréchet spaces, the completion is a Fréchet topological vector space.

Let E be a Fréchet topological vector space and (V, h) be a Hilbert space. For a homogeneous element $C = f \otimes \alpha \in E \otimes V^*$ and an element $B \in CV$, we define a right action of B on C by $C \cdot B = f \otimes (\alpha \circ B)$ and extend the action linearly to the tensor product $E \otimes V^*$. It is convenient to think of the action on $E \otimes V^*$ by a fixed element $B \in CV$ as of a map $\mathrm{Id}_E \otimes P_B$, where $P_B(\alpha) = \alpha \circ B$ for each $\alpha \in V^*$. The action by B on elements of the injective completion $E \widehat{\otimes}_\epsilon V^*$ is defined to be the continuous extension of the map $\mathrm{Id}_E \otimes P_B$ to $E \widehat{\otimes}_\epsilon V^*$. Notice that the resulting action is continuous as a map $(E \widehat{\otimes}_\epsilon V^*) \times CV \to E \widehat{\otimes}_\epsilon V^*$ if

CV is considered with the operator norm topology or with the strong operator topology.

For any continuous map $D : E \to F$ of Fréchet spaces E and F, we consider the continuous map $D^\epsilon = D \otimes \mathrm{Id}_{V*} : E \otimes_\epsilon V^* \to F \otimes_\epsilon V^*$ and its unique continuous extension $\widehat{D}^\epsilon : E \widehat{\otimes}_\epsilon V^* \to F \widehat{\otimes}_\epsilon V^*$ to the completed injective tensor product.

Lemma 8: If $D : E \to F$ is a continuous linear map between Fréchet spaces E and F, the map $\widehat{D}^\epsilon$ is CV-linear from the right.

Proof.

1) If $C \in E \otimes V^*$, then $C = \sum_{i=1}^k f_i \otimes \alpha_i$, where $f_i \in E$ and $\alpha_i \in V^*$ for $i = 1, \ldots, k$ and $k \in \mathbb{N}$. For $B \in CV$, we have $C \cdot B = \sum_{i=1}^k f_i \otimes (\alpha_i \circ B)$ which implies $D^\epsilon(C \cdot B) = (D \otimes \mathrm{Id}_{V*})(C \cdot B) = \sum_{i=1}^k Df_i \otimes (\alpha_i \circ B) = \sum_{i=1}^k (Df_i \otimes \alpha_i) \cdot B = \Big((D \otimes \mathrm{Id}_{V*}) \sum_{i=1}^k f_i \otimes \alpha_i\Big) \cdot B = ((D \otimes \mathrm{Id}_{V*})(C)) \cdot B = (D^\epsilon C) \cdot B$. Thus D^ϵ is a CV-linear map on $E \otimes V^*$.

2) For $C \in E \widehat{\otimes}_\epsilon V^*$, let us consider a Cauchy sequence $(C_i)_{i \in \mathbb{N}}$ in $E \otimes_\epsilon V^*$ such that $C = \lim_i C_i$ in the completed injective tensor product. For $B \in CV$, we have $C \cdot B = (\lim_i C_i) \cdot B = \lim_i (C_i \cdot B)$ by the continuity of the action. Using the fact that $D \widehat{\otimes}_\epsilon \mathrm{Id}_{V*}$ is continuous and the CV-linearity of $D^\epsilon = D \otimes \mathrm{Id}_{V*}$ on $E \otimes V^*$, we have

$$
\begin{aligned}
(D \widehat{\otimes}_\epsilon \mathrm{Id}_{V*})(C \cdot B) &= (D \widehat{\otimes}_\epsilon \mathrm{Id}_{V*})(\lim_i (C_i \cdot B)) = \lim_i ((D \otimes_\epsilon \mathrm{Id}_{V*})(C_i \cdot B)) \\
&= \lim_i (((D \otimes_\epsilon \mathrm{Id}_{V*}) C_i) \cdot B) = \lim_i ((D \otimes_\epsilon \mathrm{Id}_{V*}) C_i) \cdot B \\
&= \lim_i \big((D \widehat{\otimes}_\epsilon \mathrm{Id}_{V*}) C_i\big) \cdot B = \Big(\big(D \widehat{\otimes}_\epsilon \mathrm{Id}_{V*}\big) \lim_i C_i\Big) \cdot B \\
&= \Big(\big(D \widehat{\otimes}_\epsilon \mathrm{Id}_{V*}\big) C\Big) \cdot B = (\widehat{D}^\epsilon C) \cdot B.
\end{aligned}
$$

Consequently, $\widehat{D}^\epsilon$ is right CV-linear as a map $E \widehat{\otimes}_\epsilon V^* \to F \widehat{\otimes}_\epsilon V^*$. □

Smooth trivialization of Hilbert bundles

Let $p : \mathcal{V} \to M$ be an infinite rank Hilbert bundle on a manifold M^n with a Hilbert space (V, h) as the fibre and with a maximal C^∞-differentiable atlas. It is known that the unitary group of an infinite dimensional Hilbert space V equipped with the strong operator topology is *continuously* contractible. (See Dixmier, Douady [9]. (It is also known that this group is continuously contractible as well when considered with the operator norm topology. See Kuiper [29], but we shall not use this fact.) Consequently, an infinite rank Hilbert bundle is continuously trivializable, i.e., there is a fibre bundle homeomorphism of

$\mathcal{V}$ onto the product Hilbert bundle $\underline{V}_M = M \times V \to M$ on M. (See [41] for a Čech cohomology approach to a possible proof of this 'colloquial' fact.)

However, the continuous situation is not sufficient for our purpose since we shall consider differential and pseudodifferential operators. By results of Burghelea, Kuiper in [8] and Moulis in [37], it is possible to approximate the trivializing homeomorphism by a C^∞-diffeomorphism, i.e., by a C^∞-differentiable bundle homeomorphism whose inverse is C^∞-differentiable. In particular, this has as a consequence that p is also *smoothly trivializable*. Therefore, the space of C^∞-differentiable sections of p and that of the appropriate product bundle are linearly homeomorphic when they are considered with the corresponding topologies, e.g., both with the Fréchet or both with the pre-Hilbert topology. Thus we identify $p : \mathcal{V} \to M$ with the product Hilbert bundle $\underline{V}_M = M \times V \to M$ and the space $\Gamma^\infty(\mathcal{V})$ with $C^\infty(M, V)$.

Let V and V' be Banach spaces and let us consider the operator norm topology on the continuous dual V^* of V. We have the following isomorphism of Fréchet topological vector spaces

$$C^\infty(M, V')\widehat{\otimes}_\epsilon V^* \cong C^\infty(M, V'\widehat{\otimes}_\epsilon V^*),$$

proved in [52] (Theorem 44.1, p. 449). Moreover, if $V \cong V'$ are unitarily isomorphic Hilbert spaces, we have $C^\infty(M, V'\widehat{\otimes}_\epsilon V^*) \cong C^\infty(M, CV)$, where the space CV of compact operators on V is considered with the operator norm topology (Theorem 48.3 in [52]).

We start investigating the ellipticity of $\widehat{D}^\epsilon$. Let us recall that the Fourier transform $\mathfrak{F}^E$ of compactly supported smooth functions (or distributions) defined on $\mathbb{R}^n$ and having values in a Banach space E is the tensor product $\mathfrak{F}\widehat{\otimes}_\epsilon \mathrm{Id}_E$ of the scalar Fourier transform and the identity on E. See, e.g., Schwartz [49].

Lemma 9: Let D be an elliptic operator on the product Hilbert bundle $p : \underline{V}_M \to M^n$ with the fibre a Hilbert space V. Then the operator $\widehat{D}^\epsilon : C^\infty(M, CV) \to C^\infty(M, CV)$ is a CV-elliptic operator on $\underline{CV}_M = M \times CV \to M$ and it has the same order as D.

Proof. By Lemma 8, operator $\widehat{D}^\epsilon$ is CV-linear.

1) Let us suppose that the operator's order d is non-negative. For a given manifold chart $\mu = (x^1, \ldots, x^n)$ on an open set $U \subseteq M$ belonging to the atlas of M, there is a finite set $K \subseteq \mathbb{N}_0^n$ such that $D = \sum_{\beta \in K} c_\beta \partial^\beta$ is a local description of the pseudodifferential operator D on U with respect to the chosen manifold chart. Then it is immediate to realize that $\sum_{\beta \in K}(c_\beta \widehat{\otimes}_\epsilon \mathrm{Id}_{V^*})\partial^\beta$ is a local description for $\widehat{D}^\epsilon$. Obviously, for each $\beta \in K$, $c_\beta \widehat{\otimes}_\epsilon \mathrm{Id}_{V^*}$ is an adjointable pre-Hilbert CV-module morphism with the adjoint $c_\beta^* \widehat{\otimes}_\epsilon \mathrm{Id}_V$. Thus $\widehat{D}^\epsilon$ is a pseudodifferential CV-operator. The order of $\widehat{D}^\epsilon$ is equal to the order of D as follows from the local description of $\widehat{D}^\epsilon$.

In the next two steps, we compute the symbol of $\widehat{D}^\epsilon = D\widehat{\otimes}_\epsilon \mathrm{Id}_{V^*}$. For $C = f \otimes \alpha \in C^\infty(M, V) \otimes V^*$ and $m \in M$, let $C(m)$ denote the element $f(m) \otimes \alpha \in V \otimes V^*$.

i) (Algebraic tensor product.) Let $\xi \in T_m^*M$ and g be a smooth function defined on U satisfying $\xi = (dg)_m$. For the homogeneous element $C = f \otimes \alpha \in C^\infty(M,V) \otimes V^*$, we compute

$$\begin{aligned}[(D \otimes \mathrm{Id}_{V^*})\left((g-g(m))^d C\right)](m) &= \left[D\left((g-g(m))^d f\right)\right](m) \otimes \alpha \\ &= (\sigma_d(D,\xi) f(m)) \otimes \mathrm{Id}_{V^*}(\alpha) \\ &= (\sigma_d(D,\xi) \otimes \mathrm{Id}_{V^*})\,(f(m) \otimes \alpha) \\ &= (\sigma_d(D,\xi) \otimes \mathrm{Id}_{V^*})\,(C(m)).\end{aligned}$$

For a non-homogeneous element $C \in C^\infty(M,V) \otimes V^*$, we obtain $[(D \otimes \mathrm{Id}_{V^*})\left((g-g(m))^d C\right)](m) = (\sigma_d(D,\xi) \otimes \mathrm{Id}_{V^*})(C(m))$ in a similar way as above using the linearity of $D \otimes \mathrm{Id}_{V^*}$.

ii) (Completed tensor product.) Let $C \in C^\infty(M,V)\widehat{\otimes}_\epsilon V^*$, and let $(C_i)_{i\in\mathbb{N}}$ be a Cauchy sequence in $C^\infty(M,V) \otimes_\epsilon V^*$ such that $C = \lim_i C_i$, where the limit is taken in the completed injective tensor product topology on $C^\infty(M,V)\widehat{\otimes}_\epsilon V^*$. Since the symbols are fibre-wise continuous (Lemma 5), we have

$$\begin{aligned}\sigma_d(\widehat{D}^\epsilon,\xi)(\lim_i C_i) &= \lim_i \sigma_d(\widehat{D}^\epsilon,\xi) C_i \\ &= \lim_i \left[\left((D \otimes_\epsilon \mathrm{Id}_{V^*})\,(g-g(m))^d\, C_i\right)(m)\right] \\ &= \lim_i \left[(\sigma_d(D,\xi) \otimes_\epsilon \mathrm{Id}_{V^*})\,(C_i(m))\right]\end{aligned}$$

by item 1 of this proof. By the continuity of symbols, we get that the last expression equals to $(\sigma_d(D,\xi)\widehat{\otimes}_\epsilon \mathrm{Id}_{V^*})(C(m))$. In conclusion, we obtain $\sigma_d(\widehat{D}^\epsilon,\xi) = \sigma_d(D,\xi)\widehat{\otimes}_\epsilon \mathrm{Id}_{V^*}$.

Since Id_{V^*} is a linear homeomorphism and $\sigma_d(D,\xi) : V \to V$ is a linear homeomorphism for any $\xi \neq 0$ by the assumption, the inverse $\sigma_d(D,\xi)^{-1}\widehat{\otimes}_\epsilon \mathrm{Id}_{V^*}$ of the map $\sigma_d(\widehat{D}^\epsilon,\xi) = \sigma_d(D,\xi)\widehat{\otimes}_\epsilon \mathrm{Id}_{V^*}$ is continuous for any $\xi \neq 0$. In particular, $\sigma_d(\widehat{D}^\epsilon,\xi)$ is a homeomorphism. Therefore $\widehat{D}^\epsilon$ is elliptic.

2) Let us suppose that the order of D is negative. In this case, the pseudodifferential operator D is defined by an element σ of the symbol algebra via the Fourier transform, manifold charts, and a finite partition of unity $(U_j,\chi_j)_{j\in K}$, where for each $j \in K$, the set U_j is the domain of a chart in the manifold atlas and $0 \leqslant \chi_j \leqslant 1$ are smooth functions forming the partition. (See, e.g., [50], p. 104.) For each $\xi \in T_m^*M$, $\sigma(\xi)$ is a map of V into V. Let us set $\sigma' = \sigma\widehat{\otimes}_\epsilon \mathrm{Id}_{V^*}$, by which we mean $\sigma'(\xi) = \sigma(\xi)\widehat{\otimes}_\epsilon \mathrm{Id}_{V^*} : V\widehat{\otimes}_\epsilon V^* \cong CV \to V\widehat{\otimes}_\epsilon V^* \cong CV$. Especially, $\sigma' : T^*M \to \mathrm{End}_{CV}(CV)$ and this map is continuous, when CV is equipped with the norm topology.

We denote the coordinate description of σ restricted to T^*U_j by σ_j. It maps an open subset in $\mathbb{R}^{2n}$ into $\mathrm{End}(V)$. The coordinate description of σ' on T^*U_j is denoted by σ'_j. It maps an open set in $\mathbb{R}^{2n}$ into $\mathrm{End}_{CV}(CV)$. The push-forward $\chi_j \circ \mu^{-1}$ of the partition map χ_j is denoted by χ_j^μ.

By the equality $\mathfrak{F}^{CV} = \mathfrak{F}\widehat{\otimes}_\epsilon \mathrm{Id}_{CV} = \mathfrak{F}\widehat{\otimes}_\epsilon(\mathrm{Id}_V\widehat{\otimes}_\epsilon \mathrm{Id}_{V*}) = \mathfrak{F}^V\widehat{\otimes}_\epsilon \mathrm{Id}_{V*}$, we get

$$\begin{aligned}\sum_{j\in K}\mathfrak{F}^{CV}\circ\chi_j^\mu\sigma_j'\circ(\mathfrak{F}^{CV})^{-1} &= \sum_{j\in K}(\mathfrak{F}^V\widehat{\otimes}_\epsilon \mathrm{Id}_{V*})\circ(\chi_j^\mu\sigma_j\widehat{\otimes}_\epsilon \mathrm{Id}_{V*})\circ\left(\mathfrak{F}^V\widehat{\otimes}_\epsilon \mathrm{Id}_{V*}\right)^{-1}\\ &= \sum_{j\in K}\left((\mathfrak{F}^V\circ\chi_j^\mu\sigma_j)\widehat{\otimes}_\epsilon(\mathrm{Id}_{V*}\circ \mathrm{Id}_{V*})\right)\circ\left((\mathfrak{F}^V)^{-1}\widehat{\otimes}_\epsilon \mathrm{Id}_{V*}\right)\\ &= \sum_{j\in K}\left(\mathfrak{F}^V\circ\chi_j^\mu\sigma_j\circ(\mathfrak{F}^V)^{-1}\right)\widehat{\otimes}_\epsilon \mathrm{Id}_{V*}\\ &= \sum_{j\in K}\left(\mathfrak{F}^V\circ\chi_j^\mu\sigma_j\circ(\mathfrak{F}^V)^{-1}\right)\widehat{\otimes}_\epsilon \mathrm{Id}_{V*} = D\widehat{\otimes}_\epsilon \mathrm{Id}_{V*} = \widehat{D}^\epsilon.\end{aligned}$$

Thus $\sigma' = \sigma\widehat{\otimes}_\epsilon \mathrm{Id}_{V*}$ is the symbol of $\widehat{D}^\epsilon$. The adjoint of $\sigma'(\xi)$ is $\sigma(\xi)^*\widehat{\otimes}_\epsilon \mathrm{Id}_{V*}$. Thus $\widehat{D}^\epsilon$ is a CV-pseudodifferential operator according to our convention since it is given by an adjointable symbol.

Similarly as in 1) above, we prove that for each point $m \in M$ and each 'direction' $0 \neq \xi \in T_m^*M$, $\sigma\widehat{\otimes}_\epsilon \mathrm{Id}_{V*}$ is a linear homeomorphism. Thus $\widehat{D}^\epsilon$ is CV-elliptic. Moreover since $\sigma\widehat{\otimes}_\epsilon \mathrm{Id}_{V*}$ is the symbol of $\widehat{D}^\epsilon$, the order of $\widehat{D}^\epsilon$ is the same as the order of D since the identity Id_{V*} has the zero homogeneity.

In both cases 1) and 2), $\widehat{D}^\epsilon$ is a CV-elliptic pseudodifferential operator of the same order as D. □

Remark: The previous lemma has an appropriate generalization for an elliptic operator D acting between smooth sections of Hilbert bundles $p' : \underline{V'}_M \to M$ and $p'' : \underline{V''}_M \to M$ with fibres the Hilbert spaces V' and V'', respectively. Namely, let V be a Hilbert space. In this case, we consider the operator $\widehat{D}^\epsilon = D\widehat{\otimes}_\epsilon \mathrm{Id}_{V*} : C^\infty(M, V'\widehat{\otimes}_\epsilon V^*) \to C^\infty(M, V''\widehat{\otimes}_\epsilon V^*)$, whose principal symbol is $\sigma_d(D,\xi)\widehat{\otimes}_\epsilon \mathrm{Id}_{V*}$. The symbol's inverse is $\sigma_d(D,\xi)^{-1}\widehat{\otimes}_\epsilon \mathrm{Id}_{V*}$. Thus this operator is CV-elliptic as well.

We prove the next convenient lemma on a representation of smooth sections of the compact Hilbert CV-bundle $\underline{CV}_M$ on a compact manifold M. Recall that our Hilbert spaces are supposed to be separable. See the Preamble d). Nevertheless, it is easy to realize that the next lemma holds also in the case of non-separable Hilbert spaces and that it can be proved by using nets instead of sequences in a similar way.

Lemma 10: For any $\widehat{f'}$ in $C^\infty(M,V)\widehat{\otimes}_\epsilon V^*$ and $i,j \in \mathbb{N}$, there exists a smooth function $\phi_{ij} \in C^\infty(M,V)$ such that $\widehat{f'}$ is the equivalence class of the Cauchy sequence in $C^\infty(M,V)\otimes_\epsilon V^*$ with elements $f_i' = \sum_{j=1}^\infty \phi_{ij}\otimes\epsilon^j$, $i \in \mathbb{N}$, i.e., $\widehat{f'} = \lim_i f_i'$ with respect to the Fréchet topology on $C^\infty(M,V)\widehat{\otimes}_\epsilon V^* \cong C^\infty(M,CV)$. The equality holds also with respect to the pre-Hilbert topology on $C^\infty(M,CV)$. Moreover, for each $j \in \mathbb{N}$ the limit $\lim_i \phi_{ij}$ exists in both of these topologies.

Proof. Let $(f_i')_{i\in\mathbb{N}} \subseteq C^\infty(M,V)\otimes_\epsilon V^*$ be a Cauchy sequence representing $\widehat{f'}$, i.e., $\widehat{f'} = \lim_i f_i'$. By the definition of the algebraic tensor product of vector spaces, we have that for each $i \in \mathbb{N}$ there is a positive integer $m_i \in \mathbb{N}$, and for each $k = 1, \ldots, m_i$, there is a function $f_{ik} \in C^\infty(M, V)$ and a continuous functional $\alpha_k \in V^*$ such that $f'_i = \sum_{k=1}^{m_i} f_{ik} \otimes \alpha_k$. For each $j \in \mathbb{N}$, there exist complex numbers θ_{kj}, $k = 1, \ldots, m_i$, such that $\alpha_k = \sum_{j=1}^{\infty} \theta_{kj} \epsilon^j$ because $(\epsilon^j)_{j \in \mathbb{N}}$ is a Hilbert basis of the separable Hilbert space (V, h). Consequently, $f'_i = \sum_{k=1}^{m_i} f_{ik} \otimes \sum_{j=1}^{\infty} \theta_{kj} \epsilon^j$ for each $i \in \mathbb{N}$. Thus $f'_i = \sum_{j=1}^{\infty} \sum_{k=1}^{m_i} f_{ik} \otimes \theta_{kj} \epsilon^j = \sum_{j=1}^{\infty} \phi_{ij} \otimes \epsilon^j$, where $\phi_{ij} = \sum_{k=1}^{m_i} \theta_{kj} f_{ik}$ for each $i, j \in \mathbb{N}$. By the isomorphism $C^\infty(M, V) \widehat{\otimes}_\epsilon V^* \cong C^\infty(M, CV)$ of Fréchet topological vector spaces referred above, we consider $f'_i = \sum_{j=1}^{\infty} \phi_{ij} \otimes \epsilon^j$ as an element of $C^\infty(M, CV)$: $f'_i(m) = \sum_{j=1}^{\infty} \phi_{ij}(m) \otimes \epsilon^j \in CV$, $m \in M$. Since the Fréchet topology on $C^\infty(M, CV)$ is finer than the pre-Hilbert topology on the same space (Lemma 4), the equality $\widehat{f'} = \lim_i f'_i$ holds also in the pre-Hilbert topology on $C^\infty(M, CV)$.

For $v \in V$, let us consider the evaluation map $\mathrm{ev}_v : C^\infty(M, CV) \to C^\infty(M, V)$ defined by $(\mathrm{ev}_v f)(m) = f(m)(v)$, where $f \in C^\infty(M, CV)$ and $m \in M$. It is easy to see that this map is continuous with respect to the Fréchet topologies. Due to $f'_i = \sum_{j=1}^{\infty} \phi_{ij} \otimes \epsilon^j$ and the continuity of ev_v, we have $\phi_{ij} = \mathrm{ev}_{e_j}(f'_i)$. Taking the limit of the expression with respect to i, we get $\lim_i \phi_{ij} = \mathrm{ev}_{e_j} \lim_i (f'_i) = \mathrm{ev}_{e_j} \widehat{f'}$. Since the Fréchet topology is finer than the pre-Hilbert topology, we get an equality in the pre-Hilbert topology as well. □

Scholium 2: Let D be a pseudodifferential operator on the Hilbert bundle $\underline{V}_M$ on a compact manifold M with the fibre a Hilbert space V. Let $(a_j)_{j \in \mathbb{N}} \subseteq C^\infty(M, V) \otimes_\epsilon V^*$ be a sequence such that the series $\sum_{j=1}^{\infty} a_j$ converges with respect to the Fréchet topology on $C^\infty(M, V) \widehat{\otimes}_\epsilon V^*$. Then

$$\widehat{D}^\epsilon \sum_{j=1}^{\infty} a_j = \sum_{j=1}^{\infty} D^\epsilon a_j$$

with respect to the Fréchet topology.

Proof. Since $\widehat{D}^\epsilon$ is a pseudodifferential operator (Lemma 9), $\sum_{j=1}^{\infty} \widehat{D}^\epsilon a_j = \sum_{j=1}^{\infty} D^\epsilon a_j$ converges in the Fréchet topology on the smooth CV-valued functions. Since $\widehat{D}^\epsilon$ is continuous with respect to this topology, we get the equality from the assertion. □

Remark: The equality in the above lemma holds also if we consider the convergence of the series with respect to the pre-Hilbert topology on $C^\infty(M, CV)$. This is proved using appropriate Sobolev-type completions $W^d(M, CV)$ and $W^0(M, CV)$ of the smooth CV-valued functions on the compact manifold M and the continuous extension $\widehat{D}^\epsilon$ to $W^d(M, CV)$, where d is the order of D.

4.2 Closed images of elliptic operators

Let $p' : \mathcal{V}' \to M$ and $p'' : \mathcal{V}'' \to M$ be infinite rank Hilbert fibre bundles on a manifold M with fibres the separable Hilbert spaces (V', h') and (V'', h''), respectively. Let $D : \Gamma^\infty(\mathcal{V}') \to \Gamma^\infty(\mathcal{V}'')$ be a pseudodifferential operator. We consider the pseudodifferential operator D in a global smooth trivialization as described above. Thus we have a map $\widetilde{D} : C^\infty(M, V') \to C^\infty(M, V'')$ defined by $\widetilde{D} = j'' \circ D \circ j'^{-1}$, where j' is an homeomorphism of $\Gamma(M, \mathcal{V}')$ onto $C^\infty(M, V')$ and similarly for j''. Since j' and j'' are homeomorphisms, the image of D is closed if and only if the image $\widetilde{D}$ is closed. This statement is true if we investigate the images of D and $\widetilde{D}$ also with respect to the pre-Hilbert topology considered on both of the smooth section spaces and on both of the smooth function spaces, respectively, although in the pre-Hilbert topology case, the operators D and $\widetilde{D}$ needn't be continuous. For our purposes, we identify $\widetilde{D}$ with D and write $D : C^\infty(M, V') \to C^\infty(M, V'')$.

Since we shall investigate elliptic operators only, we suppose that (V', h') and (V'', h'') are homeomorphic. We identify these spaces and denote by (V, h). We keep denoting the differential operator by the same symbol, i.e., $D : C^\infty(M, V) \to C^\infty(M, V)$.

Recall that we have the continuous operator $D^\epsilon = D \otimes \mathrm{Id}_{V*} : C^\infty(M, V) \otimes V^* \to C^\infty(M, V) \otimes V^*$ and its continuous extension $\widehat{D}^\epsilon : C^\infty(M, V)\widehat{\otimes}_\epsilon V^* \to C^\infty(M, V)\widehat{\otimes}_\epsilon V^*$ to the completed space at our disposal. It can be considered also as a continuous operator $\widehat{D}^\epsilon : C^\infty(M, CV) \to C^\infty(M, CV)$ if the domain and target spaces are both considered with the Fréchet topology. Operator $\widehat{D}^\epsilon$ is CV-linear by Lemma 8.

Scholium 3 (limit and sum interchange): For each $i, j \in \mathbb{N}$, let us consider smooth maps $\phi_{ij} \in C^\infty(M, V)$, and suppose that for each $i \in \mathbb{N}$ the series $L_i = \sum_{j=1}^\infty \phi_{ij} \otimes_\epsilon \epsilon^j$ converges, and that $L = \lim_i L_i$ exists in $C^\infty(M, V)\widehat{\otimes}_\epsilon V^* \cong C^\infty(M, CV)$ with respect to the Fréchet topology. Then $M = \sum_{j=1}^\infty \lim_i \phi_{ij} \otimes_\epsilon \epsilon^j$ exists with respect to this topology, and equals to L.

Proof. Let g be a Riemannian metric on M and ∇' be a covariant derivative on the bundle $\underline{V}_M = M \times V \to M$ (e.g., the one defined by the Cartesian product structure of $\underline{V}_M$). For each vector field X on M, let us consider the operator ∇_X on the section space $\Gamma^\infty(\underline{CV}_M) \cong C^\infty(M, CV) \cong C^\infty(M, V)\widehat{\otimes}_\epsilon V^*$ of $\underline{CV}_M = M \times CV \to M$ defined by $\nabla_X = \nabla'_X \widehat{\otimes}_\epsilon \mathrm{Id}_{V*}$. It is easy to see that ∇ is a covariant derivative on $\underline{CV}_M$. We suppose that the Riemannian metric g and the covariant derivative ∇ determine the Fréchet norms on $C^\infty(M, CV) \cong C^\infty(M, V)\widehat{\otimes}_\epsilon V^*$.

For each $l \geqslant 0$, the space $(C^\infty(M, CV), |\,|_l^F)$ is a normed abelian group and thus by Antosik [1] it satisfies the "FLYUS" convergence conditions in [1], p. 369. By Theorem 2 in [1], $\sum_{j=1}^\infty \lim_i \phi_{ij} \otimes \epsilon^j$ converges with respect to the norm $|\,|_l^F$ and $L = M$ if the limits $\lim_i \phi_{ij} \otimes \epsilon^j$ and $\lim_j \phi_{ij} \otimes \epsilon^j$ exist for all j and i, respectively, and the series $L_i = \sum_{j=1}^\infty \phi_{ij} \otimes \epsilon^j$ is subseries convergent in $(C^\infty(M, CV), |\,|_l^F)$ for each i.

a) Since $\mathrm{ev}_v : C^\infty(M, CV) \to C^\infty(M, V)$ is continuous with respect to the

Fréchet topologies for each $v \in V$, we have

$$\begin{aligned}\mathrm{ev}_{e_j} L &= \mathrm{ev}_{e_j} \lim_i \sum_{k=1}^{\infty} \phi_{ik} \otimes \epsilon^k = \lim_i \sum_{k=1}^{\infty} \mathrm{ev}_{e_j}(\phi_{ik} \otimes \epsilon^k) \\ &= \lim_i \sum_{k=1}^{\infty} \phi_{ik} \mathrm{ev}_{e_j}(\epsilon^k) = \lim_i \sum_{k=1}^{\infty} \phi_{ik} \delta_{kj} = \lim_i \phi_{ij}.\end{aligned}$$

Consequently, $\lim_i(\phi_{ij} \otimes \epsilon^j) = (\mathrm{ev}_{e_j} L) \otimes \epsilon^j$ exists for all j.

b) For the limit with respect to j, it is sufficient to realize that $\phi_{ij} \otimes \epsilon^j = \sum_{k=1}^{j} \phi_{ik} \otimes \epsilon^k - \sum_{k=1}^{j-1} \phi_{ik} \otimes \epsilon^k$ and that if $j \to \infty$, both of the sums at the right-hand side converge by the assumption.

c) In the rest of the proof, we shall investigate the assumption on the subseries convergence in $(C^{\infty}(M, CV), |\,|_l^F)$. Let us consider an increasing sequence of integers $\sigma : \mathbb{N} \to \mathbb{N}$, with help of which we choose the subseries. Let us set $L_i^{\sigma}(k,r) = \sum_{j=k}^{r} \phi_{i\sigma(j)} \otimes \epsilon^{\sigma(j)}$ and $L_i(k,r) = L_i^{\mathrm{Id}}(k,r)$, where Id denotes the identity sequence $\mathrm{Id}(j) = j$, $j \in \mathbb{N}$, and $1 \leqslant k < r$ are arbitrary. We prove that for each $l \in \mathbb{N}_0$, $|L_i^{\sigma}(k,r)|_l^F \leqslant |L_i(k,r')|_l^F$ for some $r' \geqslant k$.

c.i) Let us suppose that $l = 0$. For a fixed $i \in \mathbb{N}$, we define $\Psi^{\sigma}(m,k,r) = \sum_{j=k}^{r} \phi_{i\sigma(j)}(m) \otimes \epsilon^{\sigma(j)}$ and $\Psi(m,k,r) = \Psi^{\mathrm{Id}}(m,k,r)$, where $m \in M$. These series are related to the convergence of the subseries and to the original series, respectively. Let us set $P^{\sigma}(m,k,r) = \sum_{j=k}^{r} e_{\sigma(j)} \otimes \epsilon^{\sigma(j)}$, which is a constant projection-valued map on M. For each $m \in M$, $P^{\sigma}(m,k,r)$ is of finite rank, thus in CV. It is easy to verify that

$$\Psi^{\sigma}(m,k,r) = \Psi(m,k,\sigma(r)) \circ P^{\sigma}(m,k,r).$$

Since $|\,|_{CV}$ is submultiplicative and $|P^{\sigma}(m,k,r)|_{CV} = 1$ for each $m \in M$, we have $|\Psi^{\sigma}(m,k,r)|_{CV} \leqslant |\Psi(m,k,\sigma(r))|_{CV}$. Taking the supremum of the both sides of this inequality over the points $m \in M$, we get that for each $i \in \mathbb{N}$

$$|L_i^{\sigma}(k,r)|_0^F \leqslant |L_i(k,\sigma(r))|_0^F \tag{1}$$

c.ii) Let $l \geqslant 1$ and let us consider local unit length tangent vector fields X_i on M, $g(X_i, X_i) = 1$, $i = 1, \ldots, l$. For deriving the appropriate estimates, we consider the CV-valued maps $\nabla_{X_1} \ldots \nabla_{X_l}(\phi_{ij} \otimes \epsilon^j) = (\nabla'_{X_1} \ldots \nabla'_{X_l} \phi_{ij}) \otimes \epsilon^j$ in item i) instead of the formerly used $\phi_{ij} \otimes \epsilon^j$. The inequality (1) translates to

$$|L_i^{\sigma}(k,r)|_l^F \leqslant |L_i(k,\sigma(r))|_l^F \tag{2}$$

Thus we see that it is sufficient to take $r' = \sigma(r)$.

Now using the derived inequalities (1) and (2), we prove that L_i are subseries convergent. Recall that $C^{\infty}(M, CV)$ is sequentially Cauchy complete with respect to the Fréchet topology. For proving the convergence of the subseries (determined by the sequence σ) in the normed space $(C^\infty(M, CV), | \,|_l^F)$, it is sufficient to prove that it converges in $C^\infty(M, CV)$ with respect to the translationally invariant metric determining the Fréchet topology and for that it is sufficient to prove that it is a Cauchy sequence in this space which means that it Cauchy with respect to all of the Fréchet seminorms. Let us fix $l' \geqslant 0$ and $\epsilon > 0$. Since for each i, the original series L_i is convergent by the assumption, it is Cauchy with respect to all Fréchet norms $| \,|_{l'}^F$. Thus there is $k_0 \geqslant 0$ (dependent on i, ϵ and l') such that for all $k \geqslant k_0$ and all $p' > 0$ we have $|L_i(k, k+p')|_{l'}^F < \epsilon$. Let $k \geqslant k_0$ and let us consider an arbitrary $p > 0$. By inequalities (1) and (2), we obtain $|L_i^\sigma(k, k+p)|_{l'}^F \leqslant |L_i(k, k+(\sigma(k+p)-k))|_{l'}^F$. Thus taking $p' = \sigma(k+p)-k$ in the inequality $|L_i(k, k+p')|_{l'}^F < \epsilon$, we get $|L_i^\sigma(k, k+p)|_{l'}^F < \epsilon$ for all $p > 0$, proving that each L_i is subseries Cauchy with respect to all $| \,|_{l'}^F$. Consequently, each L_i is subseries convergent in the Fréchet topology. Especially, it is subseries convergent in $(C^\infty(M, CV), | \,|_l^F)$. Thus $\sum_{j=1}^\infty \lim_i \phi_{ij} \otimes \epsilon^j \to L$ in $(C^\infty(M, CV), | \,|_l^F)$ by the cited theorem in Antosik [1]. Since l was arbitrary, the sequence $(L_i)_{i \in \mathbb{N}}$ converges also with respect to the Fréchet topology on $C^\infty(M, CV)$. □

Remark: In the above proof, Theorem 2 of Antosik [1] is used, that is a generalization of a theorem on a sum and limit interchange of Schur. See, e.g., [38]. (In [1], pages 371 and 372 shall be swapped and renumbered.) Notice also that absolute convergence criteria cannot be used directly. For it consider, e.g., the series $L_i = \sum_{j=1}^\infty \frac{e_i}{ij} \otimes \epsilon^j$, $i \geqslant 1$, each of which converges and $\lim_i L_i = 0$, but they do not converge absolutely. The cited theorem of Antosik is based on the so-called Antosik–Mikusiński basic matrix theorem in [2].

We prove that the closed image property of $\widehat{D}^\epsilon$ implies that the image of the elliptic operator D is closed as well.

Theorem 11: Let D be an elliptic operator of order d acting between sections of infinite rank Hilbert bundles $\mathcal{V}'$ and $\mathcal{V}''$ on a compact manifold M. Then the image of D is closed in $\Gamma^\infty(\mathcal{V}'')$ with respect to the pre-Hilbert topology and consequently with respect to the Fréchet topology on this section space.

Proof. Since the principal symbol of D is a linear homeomorphism of topological vector spaces for any non-zero cotangent vector on M, we may identify the fibre V' of $\mathcal{V}'$ with the fibre V'' of $\mathcal{V}''$, denote them by V and we consider D as a map of $C^\infty(M, V)$ into $C^\infty(M, V)$ as explained at the beginning of section 4.2.

Let $(g_i)_{i \in \mathbb{N}} \subseteq C^\infty(M, V)$ be a sequence in the image of D that converges to an element g with respect to the pre-Hilbert topology. Let f_i be in the D-preimage of g_i for each $i \in \mathbb{N}$, i.e., $Df_i = g_i$. Since $g_i \otimes \epsilon^1 = Df_i \otimes \epsilon^1 = \widehat{D}^\epsilon(f_i \otimes \epsilon^1)$, the sequence $(g_i \otimes \epsilon^1)_i$ is in $\operatorname{Im} \widehat{D}^\epsilon$. Moreover $\int_M (g_i \otimes \epsilon^1 - g \otimes \epsilon^1) d\mu = \int_M (g_i - g) d\mu \otimes \epsilon^1 \to 0$ since $g_i \to g$ with respect to the pre-Hilbert topology and consequently, $g_i \otimes \epsilon^1$ converges to $g \otimes \epsilon^1$. By Lemma 9, $\widehat{D}^\epsilon$ is CV-elliptic and thus the image of $\widehat{D}^\epsilon$ is closed in $C^\infty(M, CV)$ in the pre-Hilbert topology by Theorem 7. Therefore $g \otimes \epsilon^1 \in \operatorname{Im} \widehat{D}^\epsilon$.

Let us choose a $\widehat{D}^\epsilon$-preimage $\widehat{f}'$ of $g \otimes \epsilon^1 \in C^\infty(M, CV)$, i.e.,

$$\widehat{D}^\epsilon \widehat{f}' = g \otimes \epsilon^1. \tag{3}$$

Element $\widehat{f}'$ (as an equivalence class in $C^\infty(M,V)\widehat{\otimes}_\epsilon V^*$) is representable by a Cauchy sequence $(f_i')_{i\in\mathbb{N}} \subseteq C^\infty(M,V)\otimes_\epsilon V^*$, where $C^\infty(M,V)$ is equipped with the Fréchet topology. By Lemma 10, for each $i \in \mathbb{N}$ there exists a countable family of smooth functions $\phi_{ij} \in C^\infty(M,V)$, $j \in \mathbb{N}$, such that $f_i' = \sum_{j=1}^\infty \phi_{ij} \otimes \epsilon^j$ (convergence with respect to the Fréchet topology on $C^\infty(M,CV)$).

Using the continuity of $\widehat{D}^\epsilon$ with respect to the Fréchet topology, we get $\widehat{D}^\epsilon \widehat{f}' = \widehat{D}^\epsilon[(f_i')_i] = [(D^\epsilon f_i')_i] = [(D^\epsilon(\sum_{j=1}^\infty \phi_{ij} \otimes \epsilon^j))_i] = [(\sum_{j=1}^\infty D\phi_{ij} \otimes \epsilon^j)_i]$, where the last equality follows by Scholium 2. Comparing this result with (3), we obtain that the sequence $(\sum_{j=1}^\infty D\phi_{ij} \otimes \epsilon^j)_{i\in\mathbb{N}}$ differs from the constant sequence $(g\otimes\epsilon^1)_i$ by a null-sequence with respect to the Fréchet topology, i.e., $g \otimes \epsilon^1 = \lim_i \sum_{j=1}^\infty D\phi_{ij} \otimes \epsilon^j$. By Scholium 3 on the sum and limit interchange, we get that $g \otimes \epsilon^1 = \sum_{j=1}^\infty (\lim_i D\phi_{ij}) \otimes \epsilon^j$. Since $(\epsilon^i)_i$ is a Hilbert basis, we obtain from the last equality that $\lim_i D\phi_{ij} = 0$ for all $j \in \mathbb{N}\backslash\{1\}$. The existence of $\lim_i \phi_{i1}$ is one of the assertions of Lemma 10. Setting $\phi = \lim_i \phi_{i1}$, we get

$$\begin{aligned} g \otimes \epsilon^1 &= \sum_{j=1}^\infty (\lim_i D\phi_{ij}) \otimes \epsilon^j = \lim_i (D\phi_{i1}) \otimes \epsilon^1 \\ &= (D \lim_i \phi_{i1}) \otimes \epsilon^1 = D\phi \otimes \epsilon^1 \end{aligned}$$

by the continuity of D with respect to the Fréchet topologies.

We thus have $D\phi = g$ and especially $g \in \operatorname{Im} D$. Consequently, the limit g of any sequence $(g_i)_{i\in\mathbb{N}} \subseteq \operatorname{Im} D$ converging in $C^\infty(M,V)$ with respect to the pre-Hilbert topology belongs to $\operatorname{Im} D$, which is therefore closed in this topology. The image is a closed set also in the Fréchet topology by Lemma 4. □

Remark: By Wloka [58], Sobolev spaces $W^k(M,V)$, $k \in \mathbb{Z}$, are isometrically isomorphic to $W^k(M,\mathbb{C})\widehat{\otimes}_{HS}V$, where $\widehat{\otimes}_{HS}$ denotes the Hilbert–Schmidt completion of the (algebraic) tensor product. In particular they are Hilbert spaces. Note also that V is a self-dual Hilbert $\mathbb{C}$-module. Thus in this case, the adjoint D^* of $D : C^\infty(M,V) \to C^\infty(M,V)$ as a map of pre-Hilbert spaces can be constructed using Hilbert chains as in Palais [39] or Solovyov, Troitsky [50]. Namely, they are restrictions to the smooth V-valued functions of the adjoint of the extension D_k using the identification of the continuous dual of the Hilbert spaces $W^k(M,V)$ with $W^{-k}(M,V)$. Then the adjoint of $\widehat{D}^\epsilon$ is $D^*\widehat{\otimes}_\epsilon \mathrm{Id}_{V^*}$. In particular, we do not need to suppose the adjointability of symbols for its existence.

Question: It seems natural to ask whether the statement on the closedness of images is also true for Hilbert bundles that have non-separable Hilbert spaces as fibres.

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