Families of elliptic complexes in symplectic geometry

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Abstract

For a symplectic manifold admitting a metaplectic structure and equipped with a Weyl-flat torsion-free symplectic connection, we introduce two families of complexes of symplectic twistor operators and prove that these complexes are elliptic using two complexes that are already proved to be elliptic.

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1. Introduction

Twistor operators in the riemannian and pseudo-riemannian geometry are first order differential operators that originate in the twistor geometry of R. Penrose. See, e.g., Penrose [18] or Penrose, Rindler [19]. These operators and their higher spin analogues are examples of so-called generalized Stein–Weiss operators. (See Stein and Weiss [23], Branson [2], Kalina et. al [8], and Pilca [20].) In this article, we study their symplectic counterparts, the so-called *symplectic twistor operators*.

The symplectic spinor operators are defined for symplectic manifolds (M, ω) that admit a metaplectic structure, a symplectic version of a spin structure (Kostant [9]). We recall its definition in this paper briefly. In contrary to the riemannian and pseudo-riemannian twistor operators, the symplectic twistor operators are defined on sections of infinite rank vector bundles. These bundles are associated to a chosen metaplectic structure using the so-called symplectic spinor representation of the double cover of the symplectic group, the metaplectic group $\widetilde{G} = Mp(V, \omega_0)$. This representation, denoted by ρ in the paper, is known to be faithful and unitary. (See

Borel, Wallach [1].) It is a representation on a separable Hilbert space, that we denote by E. Its definition is attributed to Shale [22] and to Weil [28]. In particular, we have a homomorphism $\rho : Mp(V, \omega_0) \to U(E)$ into the group of unitary operators on E. We notice that this representation is also called the oscillator, the metaplectic or the Segal–Shale–Weil representation.

In this article, we consider tensor products of this representation with a natural representation of $Mp(V, \omega_0)$ on the wedge products $\bigwedge^i V^*$, getting representations of $Mp(V, \omega_0)$ on E-valued antisymmetric i-forms, i.e., on $E_i = \bigwedge^i V^* \otimes E$. As a representation of $\widetilde{G}$, each of the spaces E_i decomposes into irreducible $\widetilde{G}$-submodules, denoted by $E_{ij,\pm}$, where (i, j) belongs to a finite subset K of $\mathbb{Z} \times \mathbb{Z}$, that is specified in Section 2. Similarly to the (pseudo-)riemannian spin geometry, we associate the modules $E_i = \bigwedge^i V^* \otimes E$ and $E_{ij} = E_{ij,+} \oplus E_{ij,-}$ to the metaplectic structure, and denote the resulting associated vector bundles by $\mathcal{E}_i$ and $\mathcal{E}_{ij}$, respectively. Sections of $\mathcal{E}_i$ are the so-called symplectic spinor-valued differential antisymmetric i-forms.

If (M^{2n}, ω) is a symplectic manifold of dimension $2n$ equipped with a metaplectic structure and with a symplectic connection ∇, the connection can be induced on the bundles $\mathcal{E}_i$ and $\mathcal{E}_{ij}$. See Cahen et al. [3] or Habermann, Habermann [7] for details. The symplectic twistor operators $T^+_{ij} : \Gamma(\mathcal{E}_{ij}) \to \Gamma(\mathcal{E}_{i+1,j+1})$ and $T^-_{ij} : \Gamma(\mathcal{E}_{ij}) \to \Gamma(\mathcal{E}_{i+1,j-1})$ are defined as compositions of the induced connection on $\mathcal{E}_{ij}$ with specific projections of $\mathcal{E}_i$ onto $\mathcal{E}_{i+1,j+1}$ and of $\mathcal{E}_i$ onto $\mathcal{E}_{i+1,j-1}$, respectively. It is proved in Krýsl [15] that the symplectic twistor operators form complexes if the symplectic connection ∇ is torsion-free and so-called Weyl-flat (also called of symplectic Ricci-type). We define two families C^+_i and C^-_i, $i = 0, \ldots, 2n$, of complexes consisting of the symplectic twistor operators, and prove that C^+_i, $i = 0, \ldots, 2n-2$, and C^-_i, $i = 2, \ldots, 2n$, are elliptic.

Let us emphasize, that by an elliptic complex a complex of differential operators is meant whose symbol sequence is exact for all non-zero cotangent vectors of the base manifold. It is well known that the ellipticity of a complex of differential operators on finite rank hermitian vector bundles is equivalent to the bijectivity of the symbols of Laplacian operators associated to the complex out of the image of the zero section of T^*M. (See, e.g., Wells [29].) This fact is generalized to complexes of Hilbert bundles in Krýsl [11]. Let us remark that in the case of a single differential operator, different notions of ellipticity are considered ([2, 8, 20]), and that in certain cases, elliptic operators on infinite rank vector bundles are known to posses pseudoinverses, the so-called Green's operators (e.g., Solovyov, Trotsky [25]). However, we notice that appropriate results from [25] do not apply straightforwardly in the studied case since the space E is not finitely generated as a module over the complex numbers.

Two complexes, namely $C_0^+ = (T_{jj}^+)_{j=-1}^n$ and $C_{2n}^- = (T_{n,n-j}^-)_{j=0}^{n+1}$, are *proved to be elliptic* in [13]. We use this fact to show the ellipticity of the remaining complexes $C_i^\pm$ (Theorem 2), except the four complexes C_{2n-1}^+, C_{2n}^+, C_0^-, and C_1^-, for which the ellipticity is not defined. To achieve this, we establish a relation for the symbols of the operators in the complexes $C_i^\pm$ and those in $C_{i\pm 1}^\pm$ (Lemma 2) using $\widetilde{G}$-equivariant maps (denoted by F^+ and F^-), which are defined on the space of the symplectic spinor-valued antisymmetric forms $\bigoplus_{i=0}^{2n} E_i$. We recall a characterization on when the restrictions of these maps to the irreducible submodules $E_{ij,\pm}$ are isomorphisms (Lemma 1). Symbols $\sigma_{ij}^\pm(\xi)$ of the symplectic twistor operators $T_{ij}^\pm$ in a cotangent vector $\xi \in T^*M$ are compositions of the wedge multiplication by ξ followed by a projection onto the bundles $\mathcal{E}_{i+1,j\pm 1}$. Our proof of the mentioned relation of the symbols is based on the Schur lemma for infinite dimensional irreducible modules over complex Lie algebras and a decomposition of $(V^* \otimes E_{ij}) \cap (\bigwedge^{i+1} V^* \otimes E) \simeq V^* \wedge E_{ij}$ into irreducible $\widetilde{G}$-modules.

In Section 2 and in the introductory part of Section 3 we set definitions and recall known results, as the mentioned decomposition of the modules E_i and of the wedge product $V^* \wedge E_{ij}$ into irreducible $\widetilde{G}$-modules. The ellipticity of the symplectic twistor operators is proved by a diagram chasing (Theorem 2).

In the future, we would like to investigate properties of the cohomology groups of the mentioned families of elliptic complexes in a connection to the homology of the base manifold. See Krýsl [10] for such an investigation done for the case of a trivial (i.e., product) connection on the bundle $\mathcal{E}$.

2. Symplectic spinor representation and symplectic spinor-valued forms

For a symplectic vector space (V, ω_0) of dimension $2n$, we denote the symplectic group $Sp(V, \omega_0)$ by G, the metaplectic group $Mp(V, \omega_0)$ by $\widetilde{G}$, and the covering homomorphism of G by $\widetilde{G}$ by λ. In particular, $\lambda : \widetilde{G} \to G$ is a $2:1$ covering of G and a Lie group homomorphism. (See Borel and Wallach [1] or Robinson and Rawnsley [21].) Let us denote the group of linear homeomorphisms of a topological vector space W onto itself by $GL(W)$. Since G is a subgroup of the group $GL(V)$, λ is a representation of $\widetilde{G}$ on V.

We choose a Lagrangian subspace L of the symplectic vector space (V, ω_0) and a complex structure $J_0 : V \to V$ ($J_0^2 = -\mathrm{Id}_V$) such that the bilinear form $g_0(v,w) = \omega_0(J_0 v, w)$ defines a scalar product on V, where $v, w \in V$. Let us denote the Hilbert space $L^2(L)$ of square Lebesgue integrable functions on the space $(L, g_{0|L\times L})$ considered modulo the equivalence of the equality almost everywhere by E and the group of linear unitary maps of E by $U(E)$. The *symplectic spinor* representation is a faithful unitary representation $\rho : \widetilde{G} \to U(E)$ of the metaplectic group $\widetilde{G}$ on E. See, e.g., [1] or [21]. It is known that this representation is a direct sum of two inequivalent irreducible representations ρ_+ and ρ_- of $\widetilde{G}$, which are representation restrictions of ρ to subspaces E_+ and E_-, that consists of even and odd functions in E, respectively. Elements of E are called *symplectic spinors*. (See Cahen et al. [3] for a realization of the symplectic spinor representation by L^2-integrable holomorphic functions.)

Let us denote the Schwartz space of rapidly decreasing smooth functions defined on L by S. (See [1] for the role of the Schwartz space in the context of the symplectic spinor representation.) Let $(e_i)_{i=1}^{2n}$ be a symplectic basis of (V, ω_0) such that its first n members $(e_i)_{i=1}^{n}$ belong to L. The symplectic Clifford multiplication is defined on the dense subspace S of E by the formulas

$$(e_i \cdot f)(x) = \imath x^i f(x) \text{ and } (e_{i+n} \cdot f)(x) = \frac{\partial f}{\partial x^i}(x)$$

where $x = \sum_{j=1}^{n} x^j e_j \in L$, $f \in S$, and $i = 1, \ldots, n$, which are extended linearly to all elements $v = \sum_{i=1}^{2n} v^i e_i \in V$. This defines the *symplectic Clifford multiplication* as a bilinear map $\cdot : V \times S \to S$. Let us notice that this multiplication is $\widetilde{G}$-equivariant in the sense that $\rho(g)(v \cdot s) = \lambda(g)v \cdot \rho(g)s$ for $g \in \widetilde{G}$ and $s \in S$. See Habermann, Habermann [7].

2.1. Symplectic spinor-valued forms

Since λ is a representation of $\widetilde{G}$ on the vector space V, it induces the dual representation λ^* of $\widetilde{G}$ on V^*. Using the symplectic form ω_0, it is easy to see that the dual representation is equivalent to the representation λ and therefore we may identify them. For any $i \in \mathbb{Z}$, let us denote the i-th antisymmetric power of λ by $\lambda^{\wedge i} : \widetilde{G} \to GL(\bigwedge^i V)$. We consider $\bigwedge^i V = 0$ if $i \geq 2n+1$ or $i \leq -1$. Let us set $E_{i,\pm} = \bigwedge^i V \otimes E_\pm$ and

$E_i = \bigwedge^i V \otimes E \simeq E_{i,+} \oplus E_{i,-}$ and equip these vector spaces with the tensor product and the direct sum norms, respectively. Elements of E_i are called symplectic spinor-valued forms. The topological vector spaces E_i are equipped with the tensor product representations $\rho_i : \widetilde{G} \to GL(E_i)$ determined by the formula $\rho_i(g)(w \otimes s) = \lambda^{\wedge i}(g)w \otimes \rho(g)(s)$, where $w \in \bigwedge^i V$ and $s \in E$, and it is extended linearly to the tensor product $\bigwedge^i V \otimes E$.

A decomposition of the representations (ρ_i, E_i) into irreducible representations is known. See Krýsl [12] or [16]. We describe some of its properties briefly. Let us set $k_{n,i} = n - |n-i|$, $i = 0, \ldots, 2n$. By the mentioned decomposition

$$E_{i,\pm} \simeq \bigoplus_{j=0}^{k_{n,i}} E_{ij,\pm}$$

for $i = 0, \ldots, 2n$, where $E_{ij,\pm}$ are non-zero irreducible $\widetilde{G}$-modules for all $i = 0, \ldots, 2n$ and $j = 0, \ldots, k_{n,i}$. (See Figure 1.) Let us notice that the spaces $\mathbb{E}_{ij,\pm}$ of smooth vectors in $E_{ij,\pm}$ are irreducible $\widetilde{\mathfrak{g}}^{\mathbb{C}}$-modules as well (Krýsl [12]), where $\widetilde{\mathfrak{g}}^{\mathbb{C}}$ denotes the complexification of the Lie algebra of $\widetilde{G}$. We set $K = \{(i,j) | \, i = 0, \ldots, 2n, j = 0, \ldots, k_{n,i}\}$, $E_{ij,\pm} = 0$ for all $(i,j) \in (\mathbb{Z} \times \mathbb{Z}) \setminus K$, and $E_{ij} = E_{ij,+} \oplus E_{ij,-}$. The set K can be characterized *equivalently* as the set of couples of non-negative integers (i,j) such that $j \leq i$ and $i + j \leq 2n$. We denote the appropriate representations of $\widetilde{G}$ on $E_{ij,\pm}$ by $\rho_{ij,\pm}$ and the ones on $E_{ij} = E_{ij,+} \oplus E_{ij,-}$ by ρ_{ij}. Thus $\rho_{ij} = \rho_{ij,+} \oplus \rho_{ij,-}$.

It is known that for each $i = 0, \ldots, 2n$, the representation (ρ_i, E_i) is multiplicity-free, i.e., if τ and τ' are different irreducible subrepresentations of $\widetilde{G}$ in (ρ_i, E_i), they are not equivalent. This fact transfers to the $\widetilde{\mathfrak{g}}^{\mathbb{C}}$-modules $\mathbb{E}_i$ of smooth vectors in E_i since the highest weights of the $\widetilde{\mathfrak{g}}^{\mathbb{C}}$-modules $\mathbb{E}_{ij,\pm}$ are mutually different. (See [12] or eventually, [16].) The unique projections of E_i onto $E_{ij,\pm}$ are denoted by $p_{ij,\pm}$. Further, we set $p_{ij} = p_{ij,+} + p_{ij,-}$. The spaces E_i are represented by the columns in the Figure 1 (for example $E_0 \simeq E_{00}$ and $E_7 \simeq E_{70} \oplus E_{71}$ if $n = 4$).

For each $v \in V$, $w \in \bigwedge^i V$ and $s \in E$, we consider the element $\phi = w \otimes s \in E_i$, set $v \wedge \phi = (v \wedge w) \otimes s \in E_{i+1}$, and extend this multiplication linearly to all elements of E_i. Finally, we define the wedge product

$$V \wedge E_{ij,\pm} = \langle \{ v \wedge \phi | \, v \in V, \phi \in E_{ij,\pm} \} \rangle$$

where the brackets $\langle \, , \, \rangle$ denote the complex linear span. Similarly we define the wedge product of V with $E_{ij} = E_{ij,+} \oplus E_{ij,-}$. It is not difficult to see

E_0	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8
E_{00}	E_{10}	E_{20}	E_{30}	E_{40}	E_{50}	E_{60}	E_{70}	E_{80}
	E_{11}	E_{21}	E_{31}	E_{41}	E_{51}	E_{61}	E_{71}	
		E_{22}	E_{32}	E_{42}	E_{52}	E_{62}		
			E_{33}	E_{43}	E_{53}			
				E_{44}				

Figure 1: Decomposition structure for $n = 4$

that $V \wedge E_{ij} \simeq (V \otimes E_{ij}) \cap (\bigwedge^{i+1} V \otimes E)$. Obviously, $V \wedge E_{00,\pm} = V \otimes E_{00,\pm}$. Further if $B : V \to V$ and $C : E_i \to E_l$ are linear maps, $v \in V$ and $\phi \in E_k$, we set $(B \wedge C)(v \wedge \phi) = B(v) \wedge C(\phi)$, and extend this prescription linearly. In this way, we get a map $B \wedge C : V \wedge E_i \to V \wedge E_l$.

Let us consider the system ω^{ij}, $i, j = 1, \ldots, 2n$, of real numbers, that satisfy $\sum_{k=1}^{2n} \omega_{ik}\omega^{jk} = \delta_i^j$ for each i, j, where $\omega_{ij} = \omega_0(e_i, e_j)$, i.e., $(\omega^{ij})_{i,j=1,\ldots,2n}$ parametrizes the inverse transpose matrix to the matrix $(\omega_{ij})_{i,j=1,\ldots,2n}$. We introduce the following form-degree rising and form-degree lowering operators $F^+ : E_i \to E_{i+1}$ and $F^- : E_i \to E_{i-1}$, respectively, by setting for an homogeneous element $\alpha \otimes s \in \bigwedge^i V \otimes E$

$$F^+(\alpha \otimes s) = \sum_{j=1}^{2n} \epsilon^j \wedge \alpha \otimes e_j \cdot s \text{ and}$$

$$F^-(\alpha \otimes s) = \sum_{j,k=1}^{2n} \omega^{jk} \iota_{e_j} \alpha \otimes e_k \cdot s.$$

These formulas are extended linearly to $\bigwedge^i V \otimes E$. The operators are easily seen to be $\widetilde{G}$-equivariant. (See Krýsl [15].) Further we define $F^\pm_{ij,\pm}$ to be the restriction to $E_{ij,\pm}$ of $F^\pm$ (each combination of $\{+,-\}$), and set $F^\pm_{ij} = F^\pm_{ij,+} + F^\pm_{ij,-} : E_{ij} \to E_{i\pm1,j}$.

In the next lemma, we describe the wedge product $V \wedge E_{ij,\pm}$ and the $\widetilde{G}$-isomorphism structure of the modules $E_{ij,\pm}$ with respect to a fixed index j and a varying form-degree index i. See Fig. 1.

Lemma 1. *For all couples of integers i, j, we have*

$$V \wedge E_{ij,\pm} \simeq E_{i+1,j-1,\mp} \oplus E_{i+1,j,\mp} \oplus E_{i+1,j+1,\mp}.$$

For $(i,j) \in K \setminus \{(k,l)|\, k+l = 2n\}$, the map $F^+_{ij,\pm} : E_{ij,\pm} \to E_{i+1,j,\mp}$ is a $\widetilde{G}$-equivariant isomorphism; and for $(i,j) \in K \setminus \{(k,k)|\, k = 0,\dots,n\}$, the map $F^-_{ij,\pm} : E_{ij,\pm} \to E_{i-1,j,\mp}$ also is a $\widetilde{G}$-equivariant isomorphism.

Proof. See Krýsl [12] or [16]. □

The two sets subtracted from the set K in the above lemma are represented by the right-hand edge and left-hand edge in the Figure 1 ($n = 4$).

3. Symplectic twistor operators

Let (M, ω) be a symplectic manifold and ∇ be a symplectic connection, i.e., an affine connection on M whose induced covariant derivative on exterior differential 2-forms maps the symplectic 2-form ω to zero. If moreover, ∇ is torsion-free, it is called a *Fedosov connection.* Let us notice that for any symplectic manifold (M, ω) a Fedesov connection exists and if M has a positive dimension, it is not unique. See Tondeur [26] and Gelfand, Retakh and Shubin [6]. Any symplectic connection induces a curvature tensor field, defined by the classical formula $R^\nabla(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, where $X, Y, Z \in \Gamma(TM)$, and the so-called sympletic Ricci and symplectic Weyl curvature tensor fields. (See Vaisman [27].)

Definition 1. *A Fedosov connection is called Weyl-flat if its symplectic Weyl curvature tensor field vanishes.*

Now let us consider the set of linear maps

$$\mathcal{Q} = \{A : V \to T_mM|\, \omega_m(Av, Aw) = \omega_0(v,w) \text{ for all } v, w \in V, m \in M\}$$

which can be thought of as the set of symplectic frames on M. We equip it with the right action of $G = Sp(V, \omega_0)$ given by $(A \cdot g)(v) = A(g(v))$, where $v \in V$ and $g \in G$, and we give it a natural topology and a bundle atlas induced from the base manifold M. (See, e.g., [24].) Setting $\pi_Q(A) = m$ if and only if $A : V \to T_mM$ for $A \in \mathcal{Q}$, makes $\pi_Q : \mathcal{Q} \to M$ a principal G-bundle on M. Let $\pi_P : \mathcal{P} \to M$ be a principal $\widetilde{G}$-bundle and $\Lambda : \mathcal{P} \to \mathcal{Q}$ be a morphism of fibre bundles such that $\Lambda(B \cdot g) = \Lambda(B) \cdot \lambda(g)$ for all $B \in \mathcal{P}$ and $g \in \widetilde{G}$. Each couple $(\pi_P : \mathcal{P} \to M, \Lambda)$ is called a *metaplectic structure.* (See Forger, Hess [5] or Robinson, Rawnsley [21] for a classification of metaplectic and so called complex metaplectic structures.)

If a symplectic manifold (M, ω) admits a metaplectic structure, we consider the associated vector bundles $\mathcal{E}_{ij,\pm} = \mathcal{P} \times_{\rho_{ij,\pm}} E_{ij,\pm}$, $\mathcal{E}_{ij} = \mathcal{E}_{ij,+} \oplus \mathcal{E}_{ij,-}$, and $\mathcal{E}_i = \mathcal{P} \times_{\rho_i} E_i \simeq \bigoplus_{j=0}^{k_{n,i}} \mathcal{E}_{ij}$ for all $i, j \in \mathbb{Z}$. Note that $\mathcal{E}_{ij,\pm} = 0$ whenever $(i, j) \notin K$. Each symplectic connection ∇ induces a covariant derivative $\nabla_i : \Gamma(\mathcal{E}_i) \to \Gamma(\mathcal{E}_{i+1})$ on the space of sections $\Gamma(\mathcal{E}_i)$ of the symplectic spinor bundle $\mathcal{E}_i$. For integers i, j we define ∇_{ij} to be the restriction of ∇_i to $\Gamma(\mathcal{E}_{ij})$. The image of ∇_{ij} is a vector subspace of $\Gamma(\mathcal{E}_{i+1,j-1} \oplus \mathcal{E}_{i+1,j} \oplus \mathcal{E}_{i+1,j+1})$. See Krýsl [12]. The unique $\widetilde{G}$-equivariant projections p_{ij} and $p_{ij,\pm}$ of E_i onto E_{ij} and of E_i onto $E_{ij,\pm}$, respectively, induce projections of $\mathcal{E}_i$ onto $\mathcal{E}_{ij}$ and of $\mathcal{E}_i$ onto $\mathcal{E}_{ij,\pm}$, and also projections of $\Gamma(\mathcal{E}_i)$ onto $\Gamma(\mathcal{E}_{ij})$ and of $\Gamma(\mathcal{E}_i)$ onto $\Gamma(\mathcal{E}_{ij,\pm})$. We denote these projections by p_{ij} and $p_{ij,\pm}$. We hope that this causes no confusion.

In a parallel to riemannian and lorentzian twistor operators, we set the following

Definition 2. *The operators $T^{\pm}_{ij} = p_{i+1,j\pm1} \circ \nabla_{ij} : \Gamma(\mathcal{E}_{ij}) \to \Gamma(\mathcal{E}_{i+1,j\pm1})$ are called symplectic twistor operators.*

Theorem 1. *For a symplectic manifold equipped with a metaplectic structure and with a Weyl-flat Fedosov connection, $\left(\Gamma(\mathcal{E}_{i+k,j\pm k}), T^{\pm}_{i+k,j\pm k}\right)_{k \in \mathbb{Z}}$ is a complex for each couple of integers i, j.*

Proof. See Krýsl [17]. □

Remark: To get an example of a complex described in the above theorem, we advice to choose a couple (i, j) from the set K for $n = 3$ and follow Fig. 2.

3.1. Ellipticity of Symplectic Twistor Operators

It is easy to realize that the symbol $\sigma^{\pm}_{ij}(\xi) : (\mathcal{E}_{ij})_m \to (\mathcal{E}_{i+1,j\pm1})_m$ of the (first order) symplectic twistor operator $T^{\pm}_{ij}$ in the covector $\xi \in T^*_m M \simeq T_m M$ is given by $\sigma^{\pm}_{ij}(\xi)(\phi) = p_{i+1,j\pm1}(\xi \wedge \phi)$ for $\phi \in (\mathcal{E}_{ij})_m$, the fiber of $\mathcal{E}_{ij}$ in $m \in M$. At any point $m \in M$, the symbol map corresponds to a linear map, that we denote by the same symbol, i.e., we have $\sigma^{\pm}_{ij}(v) : E_{ij} \to E_{i+1,j\pm1}$ and

$\sigma^{\pm}_{ij}(v)(s) = p_{i+1,j\pm1}(v \wedge s)$, where $v \in V \simeq V^*$ and $s \in E_{ij,\pm}$. However, this map is not $\widetilde{G}$-equivariant unless $v = 0$ or unless the projection $p_{i+1,j\pm1}$ is the zero map. Instead, it is convenient to study the maps

$$\sigma^{\pm}_{ij} : V \wedge E_{ij} \to E_{i+1,j\pm1}$$

defined as the restrictions to $V \wedge E_{ij} \subseteq E_{i+1}$ of the projections $p_{i+1,j\pm1} : E_{i+1} \to E_{i+1,j\pm1}$, i.e., as maps defined by

$$\sigma^{\pm}_{ij}(v \wedge s) = p_{i+1,j\pm1}(v \wedge s)$$

on homogeneous elements and extended linearly to $V \wedge E_{ij}$. These maps are $\widetilde{G}$-equivariant and the operators' symbols are induced by them if we view the symbols as bundle morphisms with an independent variable $\xi \in T^*M$.

Now we introduce the families $\{C^+_i \,|\, i = 0, \ldots, 2n\}$ and $\{C^-_i \,|\, i = 0, \ldots, 2n\}$ of sequences of symplectic twistor operators, each of which consists of $2n+1$ sequences. They are subsequences of some of the sequences considered in the Theorem 1 above. Let us denote the largest integer that is smaller or equal to the real number q, i.e., the floor of q, by $\lfloor q \rfloor$.

For $i = 0, \ldots, 2n$, we set

$$C^+_i = \big(\Gamma(\mathcal{E}_{i+j,j}), T^+_{i+j,j}\big)_{-1 \leq j \leq \lfloor \frac{2n-i}{2} \rfloor}$$

and

$$C^-_i = \Big(\Gamma(\mathcal{E}_{\lfloor \frac{i+1}{2} \rfloor + j, \lfloor \frac{i}{2} \rfloor - j}), T^-_{\lfloor \frac{i+1}{2} \rfloor + j, \lfloor \frac{i}{2} \rfloor - j}\Big)_{0 \leq j \leq \lfloor \frac{i}{2} \rfloor + 1}.$$

Remark: 1) Setting $j = -1$ in the above definition, we obtain that the first member in each of the complexes C^+_i i.e., $(\Gamma(\mathcal{E}_{i-1,-1}), T^+_{i-1,-1})$, is the null space and the trivial operator. Setting $j = \lfloor \frac{i}{2} \rfloor + 1$, we get that the last member in each of the complexes C^-_i is the null space and the trivial operator as well.

2) The complexes C^+_{2n-1}, C^+_{2n}, C^-_0 and C^-_1 consist of sequences with two spaces only. We do not consider ellipticity of such sequences. In addition, one of the two spaces is the null space.

3) In Figure 2, complexes C^+_i (downwards-right, dashed lines) and C^-_i (upwards-right, dotted lines) are depicted for $n = 3$. The lines without arrows connect the complexes to their denotations.

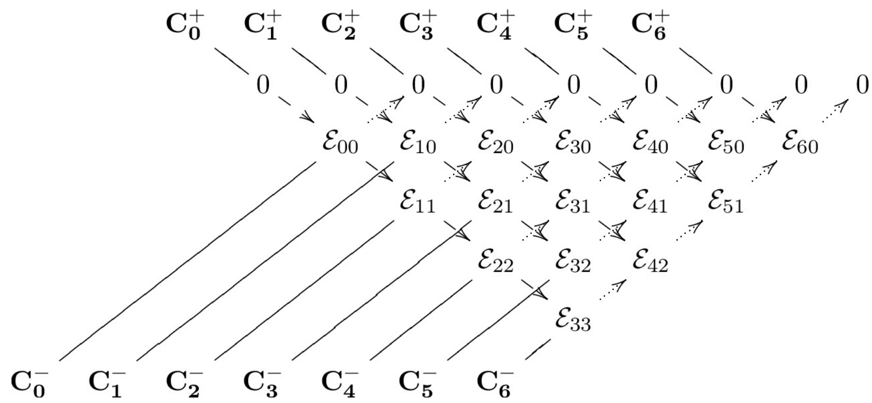

Figure 2: Elliptic complexes for $n = 3$

In the next lemma we describe a relation of the symbols of the operators in the complex C_i^+ with the symbols of the operators in C_{i+1}^+, as well as a similar relation for the case of C_i^- and C_{i-1}^-. Restrictions of $\sigma_{ij}^{\pm}$ to $V \wedge E_{ij,\pm}$ are denoted by $\sigma_{ij,\pm}^{\pm}$ (each combination of $\pm$). We considered two subsets of K that arise as subtractions from K of 1) the three right-hand rim edges in the case of the first set; and of 2) the left-hand rim edge and of the top rim edge in the case of the second one. (See Figures 1 and 2.)

Lemma 2. *For* $(i,j) \in K \setminus \{(k,l)|\, k+l \geq 2n-2\}$, *there exists a non-zero complex number* $\lambda_{\pm}^{+} = \lambda_{\pm}^{+}(i,j)$ *such that*

$$\sigma_{i+1,j,\mp}^{+} \circ (Id_V \wedge F_{ij,\pm}^{+}) = \lambda_{\pm}^{+} F^{+} \circ \sigma_{ij,\pm}^{+}; \tag{1}$$

and for $(i,j) \in K \setminus \{(k,0),(l,l)|\, 0 \leq k \leq 2n, 0 \leq l \leq n\}$, *there exists a non-zero complex number* $\lambda_{\pm}^{-} = \lambda_{\pm}^{-}(i,j)$ *such that*

$$\sigma_{i-1,j,\mp}^{-} \circ (Id_V \wedge F_{ij,\pm}^{-}) = \lambda_{\pm}^{-} F^{-} \circ \sigma_{ij,\pm}^{-} \tag{2}$$

Proof. Note that $\sigma_{rs,\pm}^{\pm}$ (all four combinations of $\pm$) and Id_V are $\widetilde{G}$-equivariant, and that F^+ and F^- are $\widetilde{G}$-equivariant by Lemma 1.

We consider the case of C_i^+, i.e., the equation (1), and only comment the case of C_i^- at the end. Since $V \wedge E_{ij,\pm} \simeq E_{i+1,j-1,\pm} \oplus E_{i+1,j,\pm} \oplus E_{i+1,j+1,\pm} \simeq E_{i+2,j-1,\mp} \oplus E_{i+2,j,\mp} \oplus E_{i+2,j+1,\mp}$ by Lemma 1 and since $E_{i+2,j-1,\mp} \oplus E_{i+2,j,\mp} \oplus E_{i+2,j+1,\mp}$ is multiplicity-free (Section 2.1), the vector space of $\widetilde{G}$-equivariant maps of $V \wedge E_{ij,\pm}$ into $E_{i+2,j+1,\mp}$ is one dimensional by the Schur lemma (e.g., [4], p. 87). Let us choose a generator of this vector space and denote it by $A_\pm$. Because the $\widetilde{G}$-equivariant operators $L_\pm = F^+_{i+1,j+1,\pm} \circ \sigma^+_{ij,\pm}$ and $R_\pm = \sigma^+_{i+1,j,\mp} \circ (\mathrm{Id}_V \wedge F^+_{ij,\pm})$ map $V \wedge E_{ij,\pm}$ into $E_{i+2,j+1,\mp}$, they are complex multiples of the generator $A_\pm$.

Now we assume that $(i,j) \in K \setminus \{(k,l)|\, k+l \geq 2n-2\}$, i.e., $(i,j) \in K$ and $i+j < 2n-2$, and prove the existence of the non-zero constants $\lambda^+_\pm$. Note that if $(i,j) \in K$ and $i+j < 2n-2$ ($2n-1$ is sufficient), then $(i+1, j+1) \in K$ by the equivalent characterization of the set K (Section 2.1).

i) Since $(i+1)+(j+1) = i+j+2 < 2n-2+2 = 2n$ and $(i+1, j+1) \in K$, $F^+_{i+1,j+1,\pm}$ is an isomorphism of $E_{i+1,j+1,\pm}$ onto $E_{i+2,j+1,\mp}$ by Lemma 1.

ii) The map $\sigma^+_{ij,\pm} : V \wedge E_{ij,\pm} \to E_{i+1,j+1,\pm}$ is non-zero since it is a projection onto $E_{i+1,j+1,\pm}$ which is a non-zero space since $i+j < 2n-2$ and $(i,j) \in K$ imply $(i+1, j+1) \in K$. Since F^+ is an isomorphism when restricted to the space $E_{i+1,j+1,\pm}$ by i), the composition $L_\pm = F^+ \circ \sigma^+_{ij,\pm}$ is non-zero.

iii) The homomorphism $\mathrm{Id}_V \wedge F^+_{ij,\pm}$ maps $V \wedge E_{ij,\pm}$ into the space $V \wedge E_{i+1,j+1,\mp}$ which contains $E_{i+2,j+1,\mp}$ by Lemma 1. Since $(i+2)+(j+1) = i+j+3 \leq 2n-3+3 = 2n$ and since $j+1 \leq i+1 \leq i+2$, the element $(i+2, j+1)$ belongs to K by the equivalent characterization of K (Section 2.1). In particular, the space $E_{i+2,j+1,\pm}$ is non-zero. Since $\sigma^+_{i+1,j,\mp}$ projects onto this space, the composed operator $R_\pm = \sigma^+_{i+1,j,\mp} \circ (\mathrm{Id}_V \wedge F^+_{ij,\pm})$ is non-zero as well.

By ii) and iii), the $\widetilde{G}$-equivariant maps $R_\pm$ and $L_\pm$ are both non-zero for the couples $(i,j) \in K \setminus \{(k,l)|\, k+l \geq 2n-2\}$. In addition, since $R_\pm$ and $L_\pm$ are multiples of $A_\pm$, they are non-zero multiples of each other.

For the equation (2), one proceeds similarly as in i) - iii) using the properties of F^- instead of F^+. $\square$

As already mentioned, the ellipticity of the two complexes C^+_0 and C^-_{2n} is known. Using this fact, we prove the ellipticity of the remaining complexes.

Theorem 2. *Let (M,ω) be a symplectic manifold which admits a metaplectic structure and ∇ be a Weyl-flat Fedosov connection. Then for $k = 0, \ldots, 2n-2$, the complex C^+_k is elliptic and for $k = 2, \ldots, 2n$, the complex C^-_k is elliptic as well.*

Proof. We have to show that the symbol map $\sigma^{\pm}_{i+j-1,j-1}(v)$ of the symplectic twistor operator $T^{\pm}_{i+j-1,j-1}$ in the vector $v \neq 0$ fulfills

$$\operatorname{Im} \sigma^{\pm}_{i+j-1,j\mp 1}(v) = \operatorname{Ker} \sigma^{\pm}_{i+j,j}(v)$$

for i and j in ranges given in the definition of the complex C^+_k for $k = i$, and of C^-_k for $k = i + 2j$.

1) The inclusion $\operatorname{Im} \sigma^{\pm}_{i+j-1,j\mp 1}(v) \subseteq \operatorname{Ker} \sigma^{\pm}_{i+j,j}(v)$ follows since the symbol of a composition of differential operators is the composition of their symbols (Solovyov, Troitsky [25], or Wells [29]) and since $C^{\pm}_k$ are complexes due to the assumption of the Weyl-flatness of ∇ (Theorem 1).

2) For the opposite inclusion ($\supseteq$), we proceed by induction on i. We prove the ellipticity for the complexes $C^+_k = C^+_i$, and only comment the case of $C^-_k = C^-_{i-2j}$ at the end. I) The ellipticity of C^+_0 is proved in [13]. II) Suppose that the inclusion ($\supseteq$) holds for the complex C^+_{i-1} and consider an element $\phi \in \operatorname{Ker} \sigma^+_{i+j,j}(v) \subseteq E_{i+j,j}$, i.e., $\sigma^+_{i+j,j}(v \wedge \phi) = 0$. By the definition of the complex C^+_i, we have $(i+j-1)+j = i+2j-1 \leq i+2\lfloor \frac{2n-i}{2} \rfloor -1 < 2n$. Thus the inverse of $F^+_{i+j-1,j}$ exists by Lemma 1. For $\phi' = (F^+_{i+j-1,j})^{-1}\phi \in E_{i+j-1,j}$, we get $\sigma^+_{i+j,j}(v \wedge F^+_{i+j-1,j}\phi') = 0$. Let us write $\phi' = \phi'_+ + \phi'_-$, where $\phi'_{\pm} \in E_{i+j-1,j,\pm}$. From $\sigma^+_{i+j,j}(v \wedge F^+_{i+j-1,j}\phi') = 0$, we obtain by Lemmma 1 (property of F^+ and the multiplicity-free structure of $V \wedge E_{i+j,j,\mp}$) that $\sigma^+_{i+j,j,\mp}(v \wedge F^+_{i+j-1,j}\phi'_{\pm}) = 0$. By Lemma 2, Eq. (1), there exist complex numbers $\lambda^+_{\pm} = \lambda^+_{\pm}(i + j - 1, j)$ such that

$$\lambda^+_{\pm}(F^+_{i+j,j+1,\pm} \circ \sigma^+_{i+j-1,j,\pm})(v \wedge \phi'_{\pm}) = \sigma^+_{i+j,j,\mp}(v \wedge F^+\phi'_{\pm}) = 0. \tag{3}$$

Recall that we are proving the ellipticity of C^+_i at the place $(i + j, j)$, and therefore, in particular, not only $T^+_{i+j,j}$, but also the operator $T^+_{i+j+1,j+1}$ has to belong to the complex C^+_i and thus $(i+j+1, j+1) \in K$. Consequently $i+j+1+j+1 \leq 2n$ and $(i+j)+(j+1) < 2n$. Therefore $F^+_{i+j,j+1,\pm}$ is an isomorphism by Lemma 1 and $\lambda^+_{\pm} \neq 0$ by Lemma 2. Thus $\sigma^+_{i+j-1,j,\pm}(v \wedge \phi'_{\pm}) = 0$ by (3) and $\phi'_{\pm} \in \operatorname{Ker} \sigma^+_{i+j-1,j,\pm}(v) = \operatorname{Ker} \sigma^+_{(i-1)+j,j,\pm}(v)$. Because C^+_{i-1} is elliptic by the induction hypothesis, there exist elements $\phi''_{\pm} \in E_{(i-1)+(j-1),j-1,\pm}$ such that $\phi'_{\pm}$ are the images of $\phi''_{\pm}$ by the maps $\sigma^+_{(i-1)+(j-1),j-1,\pm}(v)$. (See the Figure 3 for a position of $\phi' = \phi'_+ + \phi'_-$ and $\phi'' = \phi''_+ + \phi''_-$ in the diagram.) Setting $\phi'''_{\pm} = (\mu^+_{\pm})^{-1}F^+\phi''_{\pm} \in E_{i+j-1,j-1,\pm}$ for $\mu^+_{\pm} = \lambda^+_{\pm}(i - 1 + j - 1, j - 1)$, that is non-zero by Lemma 2, we get a preimage of ϕ by $\sigma^+_{i+j-1,j-1}(v)$. Indeed, we have

$$\left(\sigma^+_{i+j-1,j-1,\pm}(v)\right) \phi'''_{\pm} \;\; = \;\; \sigma^+_{i+j-1,j-1,\pm}(v \wedge (1/\mu^+_{\pm})F^+\phi''_{\pm})$$

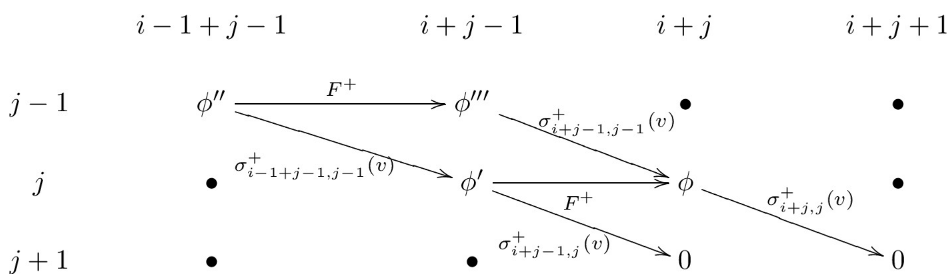

Figure 3: Diagram chasing for proof of Theorem 2.

$$\begin{aligned} &= (\mu^+_\pm/\mu^+_\pm)\left(F^+ \circ \sigma^+_{i-1+j-1,j-1,\mp}\right)(v \wedge \phi''_\pm) \\ &= F^+\left(\sigma^+_{i-1+j-1,j-1,\pm}(v \wedge \phi''_\pm)\right) \\ &= F^+\left(\sigma^+_{i-1+j-1,j-1,\pm}(v)(\phi''_\pm)\right) = F^+\phi'_\pm \end{aligned}$$

where the equation (1) is used in the second step and the definition of $\phi''_\pm$ is used in the last step. Consequently $(\sigma^+_{i+j-1,j-1}(v))(\phi'''_+ + \phi'''_-) = F^+(\phi'_+ + \phi'_-) = F^+\phi' = \phi$ and the element $\phi''' = \phi'''_+ + \phi'''_-$ belongs to the preimage of ϕ by the appropriate symbol map.

For complexes C^-_k, the proof proceeds similarly using the map F^- and the equation (2) instead of F^+ and (1) starting the induction from the complex C^-_{2n} and proceeding back up to the complex C^-_2. □

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