Lefschetz map on symplectic spinor-valued forms

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1) Symplectic manifolds and symplectic connections

Let (M, ω) be a symplectic manifold, i.e., ω point-wise non-degenerate antisymmetric differential 2-form and closed ($d\omega = 0$)

Let ∇ be symplectic ($\nabla\omega = 0$) and torsion-free connection: Fedosov connection
Non-unique in contrary to Riemannian geometry. Form affine space modeled on smooth sections $\Gamma(Sym^3(T^*M))$ of the bundle $Sym^3(T^*M)$ (Libermann; Tondeur; see also Gelfand, Retakh, Shubin)

Used for *deformation quantization* (of Poisson algebra of smooth functions on M, an L_∞-morphism)

Curvature R of ∇ : $R = \Sigma + W$, no scalar curvature, Σ constructed only by ω and Ric.

Symplectic Weyl-flat manifolds (M, ω, ∇)

We always suppose that $T^{\nabla} = 0$ so we speak about Fedosov connections

Definition: **Symplectic Weyl-flat** $\iff W = 0$

(called also symplectic Ricci type)

Examples of symplectic Weyl-flat manifolds:

1) Kähler with constant holomorphic sectional curvature: If geodesically complete, they are covered by $\mathbb{C}P^n$, open balls B^n, or Euclidean $\mathbb{C}^n$ with their standard Riemannian structures and their constant multiples [Igusa].

2) Bipolarized, bi-Lagrangian (=para-Kähler by [Alexeevskii, Medori, Tomassini], see also [Etayo et al.]) both satisfying specific PDEs [Vais]

3) Local models: symplectic Weyl-flat arise locally by descent from $(\mathbb{R}^{2n+2}, \omega_0)$: Bieliavsky, Cahen, Gutt, Schwachhöfer

4) also the Kodaira–Thurston manifold with a flat symplectic connection [Fox]

Symplectic spinors

(V, ω_0) real symplectic vector space of dimension $2n$, G the symmetry group - symplectic group $Sp(V, \omega_0)$ of maps of V preserving the bilinear form ω_0.
Can choose $V = \mathbb{R}^{2n}$ for simplicity.

G is non-compact; maximal compact in G :
$K = G \cap SO(2n, R) \simeq U(n)$. Fundamental group
$\pi_1(G) \simeq \pi_1(U(n)) \simeq \mathbb{Z}$.

$\Longrightarrow \exists\ \lambda : \widetilde{G} \to G \subseteq \mathrm{Aut}(V)$, connected Lie group double cover of G; $\widetilde{G} = Mp(V, \omega_0)$ - the **metaplectic group**: non-matrix Lie group, $2:1$ covering as $Spin(m) \to SO(m)$
λ is also a representation of $\widetilde{G}$ on V

Symplectic spinors - properties

U be a maximal ω_0-isotropic subspace of (V, ω_0), $U \simeq \mathbb{R}^n$
Let $L : \widetilde{G} \to \mathcal{U}(L^2(U))$ be the so called **symplectic spinor representation**.
Hilbert space $S = L^2(U)$ called space of **symplectic spinors**

Unitary, faith-full, infinite dimensional; decomposes into two non-equivalent irreducible representations; $S = S_+ \oplus S_-$
Its 'infinitesimal structure' (i.e., Harish-Chandra module) is $\bigoplus_{i=0}^{\infty} Sym^i(U) \simeq Pol(x_1, \ldots, x_n)$ [Kirillov]

Also known as Segal–Shale–Weil, metaplectic, oscillator representation: [Shale], [Weil], [Howe]

Discovered by quantization of Klein–Gordon fields (David Shale and Irving Segal), symmetries of ϑ-functions (Weil)

2) Model for the symplectic spinor complex

$E^i = \bigwedge^i V^* \otimes S$ - symplectic spinor-valued wedge i-forms
$E = \bigoplus_{i=0}^{2n} \bigwedge^i V^* \otimes S$ - **symplectic spinor-valued wedge forms**

$$\rho(g)(\alpha \otimes s) = \lambda(g)^* \alpha \otimes L(g)s, \alpha \otimes s \in E^i, g \in \widetilde{G}$$

Remark: Similar - model for Dolbeault complex.
$U(T_x M, J_x, g_x)$-module $\bigoplus_{0 \leq p+q \leq 2n} \bigwedge^p (T_x M^{1,0})^* \otimes \bigwedge^q (T_x M^{0,1})^*$, $(T_x M, J_x, g_x)$ hermitian vector space and (M, J, g) is a complex hermitian manifold.

Theorem 1 [KryJLieThy]: The module E decomposes as a $\widetilde{G}$-module into a finite direct sum

$$\bigoplus_{(i,j) \in P} E^{ij}, \text{where } P \text{ is a finite subset of } \mathbb{Z} \times \mathbb{Z}.$$

$E^{ij} = E^{ij,+} \oplus E^{ij,-} \subseteq E^i$ and $E^{ij,\pm}$ are non-equivalent irreducible $\widetilde{G}$-modules. (E^{ij} not irreducible.)

Decomposition of symplectic spinor-valued wedge forms

Dim $M = 6$

$\mathbf{E}^0$	$\mathbf{E}^1$	$\mathbf{E}^2$	$\mathbf{E}^3$	$\mathbf{E}^4$	$\mathbf{E}^5$	$\mathbf{E}^6$
E^{00}	E^{10}	E^{20}	E^{30}	E^{40}	E^{50}	E^{60}
	E^{11}	E^{21}	E^{31}	E^{41}	E^{51}	
		E^{22}	E^{32}	E^{42}		
			E^{33}			

$p^{ij} : E^i \to E^{ij}$ the unique projection according to the splitting above

Inducing the model to the metaplectic structures

If a symplectic manifold (M, ω) admits a metaplectic structure (symplectic analogue of the riemannian or pseudoriemannian spin structure, specific principal bundle that double-covers the bundle of symplectic frames), denoted by $\mathcal{P}$, $\Longrightarrow$ form

associated bundles $\mathcal{E} = \mathcal{P} \times_\rho E$ - bundle of **symplectic spinor valued wedge forms**

and associated bundles $\mathcal{E}^{ij} = \mathcal{P} \times_\rho E^{ij}$

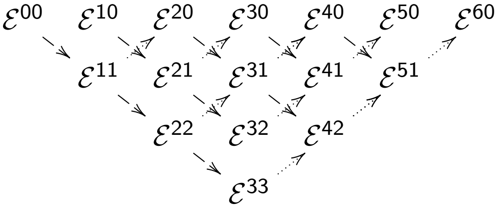
For a symplectic connection, construct the exterior covariant derivatives $d_i^\nabla : \Gamma(\mathcal{E}^i) = \Omega^i(M)\widehat{\otimes}_\epsilon S \to \Gamma(\mathcal{E}^{i+1}) = \Omega^{i+1}(M)\widehat{\otimes}_\epsilon S$

Sequences of symplectic twistor operators

Definition: Let ∇ be a symplectic connection on (M, ω). Then $T_{\pm}^{ij} = p^{i+1, j\pm 1} d^{\nabla}_{|\Gamma(\mathcal{E}^{ij})}$ is called the $(\pm)$-**symplectic twistor operator**, or (i,j)-th $(\pm)$-symplectic twistor operator.

Symplectic Dirac operators introduced by Habermann [KHMathNachr].

Dim $M = 4$



Complexes for (M, ω, ∇) with a metaplectic structure

Theorem 2: If ∇ is symplectic, torsion-free and symplectic Weyl flat, then for any i, j, the sequence $\left(\Gamma(E^{i+k,j\pm k}), T_{\pm}^{i+k,j\pm k}\right)_{k\in\mathbb{Z}}$ is a complex, i.e., $T_{\pm}^{i+k+1,j\pm k\pm 1} T_{\pm}^{i+k,j\pm k} = 0$.

Proof. [KryCliffAlg].

3) Primitive forms and symplectic twistor cohomology

Let $(e_i)_{i=1}^{2n}$ be a symplectic basis of (V, ω_0) such that $(e_i)_{i=1}^{n} \subseteq U$. For $s \in \mathcal{S}(U)$ (Schwartz functions on $U, \mathcal{S}(U) \subseteq_{dense} S = L^2(U)$), set

$$(e_i \cdot s)(x) := \imath x^i s(x) \text{ and } (e_{i+n} \cdot s)(x) := \frac{\partial s}{\partial x^i}(x),$$

where $U \ni x = \sum_{i=1}^n x^i e_i$. Extend linearly to V, getting $v \cdot s$ ($v = \sum_{i=1}^{2n} x^i e_i$). It is the **canonical quantization prescription** up a constant.

$$\text{Set } Y(\alpha \otimes s) := \sum_{i,j=1}^{2n} \omega^{ij} \iota_{e_i} \alpha \otimes e_j \cdot s,$$

where ι denotes insertion. Extend linearly. (Motivation [Slupinski].)

Remark: Extension $\cdot : V \times S \to S$ to the map $V \otimes S \to S$ is $\widetilde{G}$-equivariant w.r.t. representations $\lambda \otimes L$ and L.

Primitive forms

Definition: Symplectic spinor-valued i-form $\phi = \sum_k \alpha_k \otimes s_k$ is called **primitive** if it is an element of the kernel of Y.

Set $X(\alpha \otimes s) = \sum_{i=1}^{2n} \epsilon^i \wedge \alpha \otimes e_i \cdot s$ (extend linearly), where α is a differential form and s is a symplectic spinor field.

Lemma 3 (Rep-thy-Lemma): Let $0 \leq i \leq n$. Symplectic spinor-valued i-form is primitive if and only if it is a section of $\mathcal{E}^{ij}$ for $i = j$. It is primitive if it is in the kernel of $X^{2n-2i+1}$.

Proof. Follows from [KrJLieThy].

4) Decomposition into primitive forms and map $[X]$

Theorem 4 (Lefschetz type decomposition): For a symplectic manifold (M, ω, ∇) with symplectic Weyl-flat connection and $0 \leq i \leq n$

$$\mathcal{E}^i = \bigoplus_{j=0}^{i} X^{i-j} \mathcal{E}^{jj}$$

and also $\Gamma(\mathcal{E}^i) = \bigoplus_{j=0}^{i} X^{i-j} \Gamma(\mathcal{E}^{jj})$.

Proof. Schur lemma, $\widetilde{G}$-equivariance of X and decomposition structure of E (see also Lemma 1).

Definition: The $(+)$-**twistor cohomology group** is the quotient

$$H_T^{i,j}(M) = \mathrm{Ker}\ T_+^{i,j} / \mathrm{Im}\ T_+^{i-1,j-1}.$$

The $+$-case is for simplicity.

Theorem 5: If (M, ω, ∇) is a symplectic manifold with a symplectic Weyl-flat connection, then X descends to the twistor cohomology groups, i.e., $[X] : H_T^{i,j}(M) \to H_T^{i+1,j}(M)$, $[X][\phi] := [X(\phi)]$, is a well defined linear map.

Proof. $[\psi] = 0 \implies \psi \in \mathrm{Im}\, T_+^{i-1,j-1} \implies \psi = p^{i,j} d^\nabla \phi \implies$
$X\psi = Xp^{i,j} d^\nabla \phi$.
Since X is G-**equivariant**. By Schur lemma for intertwining operators: $Xp^{i,j} = -\mu p^{i+1,j} X$ for a constant μ, possibly zero. Thus $X\psi = -\mu p^{i+1,j} X d^\nabla \phi$.
It is easy to compute that $Xd^\nabla = -d^\nabla X$ using the torsion-free property.
Conclude: $X\psi = -\mu p^{i+1,j} X d^\nabla \phi = p^{i+1,j} d^\nabla X(\mu\phi) = T_+^{i,j-1}(\mu\phi)$, thus it is in the image of $T_+^{i,j-1}$. □

Lefschetz map and hard Lefschetz property

Assumptions:

Let ∇ be Fedosov (torsion-free and symplectic) and flat. Then we have $d^{\nabla}_{i+1} d^{\nabla}_i = 0$. Thus $(\Gamma(\mathcal{E}^i), d^{\nabla}_i)_i$ is a complex.
Form **symplectic spinor cohomology**:
$H^i_{sys}(M,S) = \mathrm{Ker}\, d^{\nabla}_i / \mathrm{Im}\, d^{\nabla}_{i-1}$

Easy to derive action of $\omega = X \circ X$.

Definition: Set $[\omega^{\wedge k}] = [X^{2k} \wedge] : H^{n-k}_{sys}(M,S) \to H^{n+k}_{sys}(M,S)$ is called the (symplectic spinor) **Lefschetz map**.

Lefschetz property for symplectic spinors

Theorem 6: Let (M, ω, ∇) be compact symplectic and flat, then $[\omega^{\wedge k}] : H^{n-k}_{sys}(M, S) \to H^{n+k}_{sys}(M, S)$ is an isomorphism for each $0 \leq k \leq n$.

Idea of proof: $H^{n-k}_{sys}(M, S) \simeq K_{harm} := \mathrm{Ker}\Delta_{n-k}$ (by harmonic theory), where $\Delta_i = (d^{\nabla}_i)^* d^{\nabla}_i + d^{\nabla}_{i-1}(d^{\nabla}_{i-1})^*$, where the adjoints are with respect to a hermitian metric compatible with ω. ω commutes with Δ_i, and moreover with $\delta^{\nabla}_i = (d^{\nabla}_i)^*$. Problems: Commuting $\omega^{\wedge k}\wedge$ with the adjoints of derivatives is difficult. (Codifferentials do not have an easy Leibniz property). Escape by divergence formula:

$$d^{\nabla^*}(\alpha \otimes s) = \sum_{i,j} -\nabla_{e_i}(\alpha(e_{ij})\epsilon^j \otimes s) + div(e_i)\alpha(e_{ij})\epsilon^j \otimes s$$

Not necessary $[J, d^{\nabla}] = 0$ (the Kähler property in the considered case).

Hodge theories - partial algebraic point of view

1) Forms on Riemannian and Kählerian manifolds - quite known

2) Forms on Symplectic: Symplectic Laplacian is zero $\Longrightarrow$ replace $K'_{harm,symp} := \mathrm{Ker}\, d \cap \mathrm{Ker}\, \delta_{symp}$, $\delta_{symp} = * \, d \, *$, $*$ symplectic star.

Mathieu: Symplectic manifold has hard Lefschetz property iff $K'_{harm,symp} \simeq H_{dRham}(M)$ (Brylinsky condition).

Suppl.: Definition of metaplectic structure

(M, ω) symplectic manifold
$\mathcal{Q} = \{A : V \to T_m M \,|\, \omega_0(u, v) = \omega_m(Au, Av), u, v \in V, m \in M\}$ is a principal G-bundle, bundle of symplectic frames, $\pi_Q : \mathcal{Q} \to M$
If $\pi_P : \mathcal{P} \to M$ is principal $\widetilde{G}$-bundle and $\Lambda : \mathcal{P} \to \mathcal{Q}$ is a fibre bundle map, $(\mathcal{P}, \Lambda)$ is called **metaplectic structure** on (M, ω) if the diagram

$$\begin{array}{ccccc} P \times \widetilde{G} & \longrightarrow & P & & \\ & & & \overset{\pi_P}{\searrow} & \\ \Big\downarrow{\scriptstyle \Lambda \times \lambda} & & \Big\downarrow{\scriptstyle \Lambda} & & M \\ & & & \underset{\pi_Q}{\nearrow} & \\ Q \times G & \longrightarrow & Q & & \end{array}$$

commutes.
Thm. (Forger, Hess): A metaplectic structure exists iff $c_1(TM^c)$ is even, i.e. an element of $H^2(M, 2\mathbb{Z})$ iff $w_1(TM) = 0$.

Suppl.: Ellipticity of the subcomplexes

Theorem: If ∇ is symplectic and Weyl flat, then

$$\left(\Gamma(\mathcal{E}^{i+k,k}), T^{i+k,k,+}\right)_{-1\leq k\leq \lfloor \frac{2n-i}{2}\rfloor}$$

$i = 0, \ldots, 2n-2$, and

$$\left(\Gamma(\mathcal{E}_{\lfloor \frac{i+1}{2}\rfloor+k,\lfloor \frac{i}{2}\rfloor-k}), T^{\lfloor \frac{i+1}{2}\rfloor+k,\lfloor \frac{i}{2}\rfloor-k,-}\right)_{0\leq k\leq \lfloor \frac{i}{2}\rfloor+1}$$

are elliptic for $i = 2, \ldots, 2n$.

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