

$$\lim_{x \rightarrow 16} \frac{\sqrt[4]{x} - 2}{\sqrt{x} - 4} = \lim_{x \rightarrow 16} \frac{\sqrt[4]{x} - 2}{(\sqrt[4]{x} - 2)(\sqrt[4]{x} + 2)} = \lim_{x \rightarrow 16} \frac{1}{\sqrt[4]{x} + 2} = \frac{1}{4}$$

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+1} - 1}{x} = \lim_{x \rightarrow 0} \frac{(\sqrt{x+1} - 1)(\sqrt{x+1} + 1)}{x(\sqrt{x+1} + 1)} = \lim_{x \rightarrow 0} \frac{x+1-1}{x(\sqrt{x+1} + 1)} =$$

$$= \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+1} + 1} = \frac{1}{2}$$

! pp. $1-x \geq 0$

$$\lim_{x \rightarrow 0} \frac{\sqrt{1-2x-x^2} - (1-x)}{x} = \lim_{x \rightarrow 0} \frac{\sqrt{1-2x-x^2} - \sqrt{(1-x)^2}}{x} \cdot \frac{\sqrt{1-2x-x^2} + \sqrt{(1-x)^2}}{\sqrt{1-2x-x^2} + \sqrt{(1-x)^2}} =$$

$$= \lim_{x \rightarrow 0} \frac{1-2x-x^2 - 1+2x-x^2}{x(\sqrt{1-2x-x^2} + \sqrt{(1-x)^2})} = \lim_{x \rightarrow 0} \frac{-2x}{\sqrt{1-2x-x^2} + \sqrt{(1-x)^2}} = -\frac{2 \cdot 0}{1+1} = 0$$

$$\lim_{x \rightarrow 0} \frac{(\sqrt[3]{27+x} - \sqrt[3]{27-x})((27+x)^{\frac{2}{3}} + (27+x)^{\frac{1}{3}}(27-x)^{\frac{1}{3}} + (27-x)^{\frac{2}{3}})}{x + 2\sqrt[3]{x^4}} = \lim_{x \rightarrow 0} \frac{27+x-27+x}{x + 2\sqrt[3]{x^4}}$$

atto

$$\frac{1}{atto} = \lim_{x \rightarrow 0} \frac{2x}{x + 2x^{4/3}} \cdot \frac{1}{atto} = \lim_{x \rightarrow 0} \frac{2}{1 + 2x^{1/3}} \cdot \frac{1}{3^2 + 3 \cdot 3 + 3^2}$$

podm. nové proc.?

$$= \frac{2}{1+2 \cdot 0} \cdot \frac{1}{3 \cdot 3^2} = \frac{2}{27}$$

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt[3]{1-x}}{\sqrt[2]{1+x} - \sqrt{1-x}} = \lim_{x \rightarrow 0} \frac{[(1+x)^3]^{\frac{1}{6}} - [(1-x)^2]^{\frac{1}{6}}}{[(1+x)^2]^{\frac{1}{6}} - [(1-x)^3]^{\frac{1}{6}}} \cdot \frac{[(1+x)^3]^{\frac{5}{6}} + [(1+x)^3]^{\frac{4}{6}}[(1-x)^3]^{\frac{1}{6}} + \dots + [(1-x)^2]^{\frac{5}{6}}}{atto}$$

$$\frac{atto'}{[(1+x)^3]^{\frac{5}{6}} + \dots + [(1-x)^2]^{\frac{5}{6}}} = \lim_{x \rightarrow 0} \frac{1+3x+3x^2+x^3 - (1-2x+x^2)}{1+2x+x^2 - (1-3x+3x^2-x^3)} \cdot \frac{1}{1} \frac{atto'}{1} =$$

$$= \lim_{x \rightarrow 0} \frac{x^3+2x^2+5x}{x^3-2x^2+5x} \cdot \frac{6}{6} = \lim_{x \rightarrow 0} \frac{x^2+2x+5}{x^2-2x+5} = 1$$

(2)

$$\lim_{x \rightarrow 0} \frac{(1+x)^{\frac{1}{m}} - (1+x)^{\frac{1}{n}}}{x} = \lim_{x \rightarrow 0} \frac{[(1+x)^n]^{\frac{1}{mn}} - [(1+x)^m]^{\frac{1}{mn}}}{x}$$

$$= \left| \begin{array}{l} \sqrt[k]{x'} - \sqrt[k]{y'} = x'^{\frac{1}{k}} - y'^{\frac{1}{k}} / \underbrace{\left(x'^{\frac{k-1}{k}} + x'^{\frac{k-2}{k}} y' + \dots + y'^{\frac{k-2}{k}} x' + y'^{\frac{k-1}{k}} \right)}_{A=} \\ x' - y' = (x'^{\frac{1}{k}} - y'^{\frac{1}{k}}) A \end{array} \right| =$$

$$= \lim_{x \rightarrow 0} \frac{(1+x)^n - (1+x)^m}{x \cdot A} = \lim_{x \rightarrow 0} \frac{1 + nx + \dots + x^n - (1 + mx + \dots + x^m)}{x \cdot A} =$$

$$= \lim_{x \rightarrow 0} \frac{nx + \dots + x^m - mx - \dots - x^m}{x \cdot A} = \lim_{x \rightarrow 0} \frac{(n-m + x \binom{n}{2} + \dots + x^n - x \binom{m}{2} - \dots - x^m)}{A}$$

$$A = \lim_{x \rightarrow 0} \left[(1+x)^{\frac{mn-1}{mn}} + (1+x)^{\frac{mn-2}{mn}} (1+x)^{\frac{1}{mn}} + \dots + (1+x)^{\frac{mn-1}{mn}} \right] = mn$$

$$= \lim_{x \rightarrow 0} \frac{(n-m + x \binom{n}{2} + \dots)}{mn} = \frac{n-m}{mn}.$$

$$\lim_{x \rightarrow 0} \frac{(1+x)(1+2x)\dots(1+nx) - 1}{x} = \lim_{x \rightarrow 0} \frac{1+x+2x+\dots+nx + \dots}{x} \quad (3)$$

musím vykrátit x,
než ujde VOAL

$$\frac{+x^2A+\dots+x^m n! - 1}{x} = \lim_{x \rightarrow 0} \frac{\frac{1}{2}n(n+1)x + x^2A + \dots + x^m n!}{x} =$$

$$= \lim_{x \rightarrow 0} \left(\frac{1}{2}n(n+1) + Ax + x^{m-1}n! \right) = \frac{1}{2}n(n+1).$$

$$\lim_{x \rightarrow 1} \frac{x^{100} - 2x + 1}{x^{50} - 2x + 1} = \lim_{x \rightarrow 1} \frac{(x-1)(x^{99} + \dots + x^1 - 1)}{(x-1)(x^{49} + \dots + x^1 - 1)} =$$

$$= \lim_{x \rightarrow 1} \frac{x^{99} + \dots + x^1 - 1}{x^{49} + \dots + x^1 - 1} = \frac{98}{48}.$$

$$\lim_{x \rightarrow 0} \frac{(1+mx)^n}{x^2} = \lim_{x \rightarrow 0} \frac{1 + nm x + \binom{n}{2} m^2 x^2 + \dots + m^n x^n}{x^2}$$

bikonn

$$\frac{(1 - nm \binom{n}{2} x^2 - \dots - m^n x^n)}{x^2} = \lim_{x \rightarrow 0} \frac{m^2 \binom{n}{2} x^2 - n^2 \binom{n}{2} x^2 + A_3 x^3 + \dots + A_m x^m}{x^2}$$

$$= \lim_{x \rightarrow 0} \left(m^2 \binom{n}{2} - n^2 \binom{n}{2} + A_3 x + \dots + A_m x^{m-1} \right) = m^2 \binom{n}{2} - n^2 \binom{n}{2}$$

$$= \frac{1}{2} m m^2 (n+1) - \frac{1}{2} n n^2 (n+1) = \frac{1}{2} mn (mn + m - nm - n) =$$

$$= \frac{1}{2} mn (m - n).$$

Pro $m \leq n$: $\lim_{x \rightarrow 0} \frac{m^2 \binom{n}{2} x^2 - n^2 \binom{n}{2} x^2 + B_3 x^3 + \dots + B_m x^m}{x^2} =$ dále stejně jako

$$\lim_{x \rightarrow 1} \frac{x^{n+1} - (n+1)x + n}{(x-1)^2} = \lim_{x \rightarrow 1} \frac{x^{n+1} - (n+1)x + n}{(x-1)^2} = \lim_{x \rightarrow 1} \frac{x^{n+1} - x + n(1-x)}{(x-1)^2}$$

(4)

$$= \lim_{x \rightarrow 1} \frac{x(x^{n-1}) - n(x-1)}{(x-1)^2} = \lim_{x \rightarrow 1} \frac{x(x^{n-1} + \dots + x + 1) - n}{(x-1)} =$$

$$= \lim_{x \rightarrow 1} \frac{(x^n + x^{n-1} + x^2 + x - n)}{(x-1)} = \lim_{x \rightarrow 1} \frac{x^n - 1 + x^{n-1} - 1 + \dots + x - 1}{(x-1)} =$$

$$= \lim_{x \rightarrow 1} (x^{n-1} + \dots + x + 1) + \lim_{x \rightarrow 1} (x^{n-2} + \dots + x + 1) + \dots + \lim_{x \rightarrow 1} 1 =$$

$$= n + (n-1) + \dots + 1 = \frac{1}{2}n(n+1).$$

bylo indukce

Některé limity jsme počítali pomocí vět o limite slož. fce:

V1: $\lim_{x \rightarrow a} g(x) = b \wedge \lim_{y \rightarrow b} f(y) = A \wedge \exists \delta > 0 \quad \forall x \in P_\delta(a) \quad g(x) \neq b$

$$\Rightarrow \lim_{x \rightarrow a} (f \circ g)(x) = A \quad (= \lim_{y \rightarrow b} f(y) = \lim_{x \rightarrow a} f(g(x))).$$

V2: $\lim_{x \rightarrow a} g(x) = b \wedge f \text{ spoj. v } b \Rightarrow \lim_{x \rightarrow a} (f \circ g) = f(b) (= A = \lim_{y \rightarrow b} f(y))$

Dle V2 stačí spočítat limitu vnitřní a dosadit do f , pokud f spojitá!

Př. Vzorově (s odvozením dle vět) spočítej: $\lim_{x \rightarrow 0} \frac{\frac{2}{x^2} + 1}{\sqrt{\frac{2}{x^4} - \frac{6}{x^2} + 5}} =$

lemma o rovnosti
a např. vektorově

VOAL (podíl)

$$= \lim_{x \rightarrow 0} \frac{2 + x^2}{\sqrt{3 - 6x^2 + 5x^4}}$$

$$\stackrel{VOAL}{=} \lim_{x \rightarrow 0} \frac{(2 + x^2)}{\sqrt{3 - 6x^2 + 5x^4}}$$

$$\stackrel{VOAL}{=} \frac{2}{\sqrt{\lim_{x \rightarrow 0} (3 - 6x^2 + 5x^4)}} = \frac{2}{\sqrt{3}}$$

VoLSF

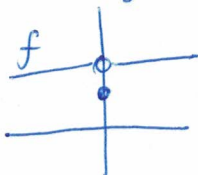
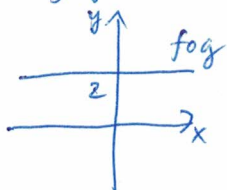
spojitá

v každém bod $\mathbb{R}^+$

Př.: $g(x) = 0, f(x) = \begin{cases} 3, & x \neq 0 \\ 2, & x = 0 \end{cases}$

$$\lim_{y \rightarrow 0} f(y) = 3$$

$$f(g(x)) = f(0) = 2$$



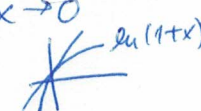
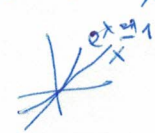
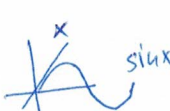
$$\text{Tj. } \lim_{x \rightarrow 0} (f \circ g)(x) = 2 \neq 3 = \lim_{y \rightarrow 0} f(y)$$

$$\lim_{x \rightarrow 0} g(x) = b$$

$$\begin{aligned}
 \text{Zpět: } \lim_{x \rightarrow 0} \frac{\sqrt{8+x^2} - \sqrt{8-x^2}}{x[(x+1)^2 - (x-1)^2]} &= \lim_{x \rightarrow 0} \frac{\sqrt{8+x^2} - \sqrt{8-x^2} \cdot [(\sqrt{8+x^2})^3 + \dots + (\sqrt{8-x^2})^3]}{x[4x]} \quad (5) \\
 &= \lim_{x \rightarrow 0} \frac{(8+x^2) - (8-x^2)}{4x^2} \cdot \frac{1}{[\text{atto}]} = \lim_{x \rightarrow 0} \frac{1}{2} \frac{1}{[\text{atto}]} = \\
 &= \lim_{x \rightarrow 0} \frac{1}{2} \frac{1}{4+2+4} = \frac{1}{2 \cdot 3 \cdot 4} = \frac{1}{24}.
 \end{aligned}$$

• Ted' začneme u některých příkladů se "zákl." limity. Dokážte je až později! (například se).

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1, \quad \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1, \quad \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1$$



Dále: $\lim_{x \rightarrow a} f(x) g(x) = e^{\lim_{x \rightarrow a} [\ln f(x)] g(x)}$ (Princip: spojitost $y = e^x$ (později) a "definice" $\beta^x := e^{x \ln \beta}$; odtud lim vlevo existuje $\Leftrightarrow$ limita napravo v x).
Tento vzorec je nebyřale užitečný!

! Př.:

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} &= \left| \text{Chci tam vidět } \frac{\sin x}{x} \right| = \\
 &= \lim_{x \rightarrow 0} \frac{(1 - \cos x)(1 + \cos x)}{x^2 (1 + \cos x)} = \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{x^2 (1 + \cos x)} = \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2} \\
 &= \frac{1}{1 + \cos x} = 1^2 \cdot \frac{1}{1 + 1} = \underline{\underline{\frac{1}{2}}} \quad (\text{do paměti}).
 \end{aligned}$$