POSLOUP.

Věty: ① Omezená mono. má limitu. — Opakování před. (kreslete si obrázky.) ①

Detailněji: Shora om. neklesající má limitu.
Zdola om. nerostoucí má limitu

② Z omezené lze vybrat konvergentní! (Bolzano–Weierstrass, půlení) „mající limitu."

! ③ $f \le g$ na $P r(a)$ $\wedge \exists \lim_{x\to a} f, \lim_{x\to a} g \Rightarrow \lim_{x\to a} f \le \lim_{x\to a} g$

④ Bolzano–Cauchy: $\forall \varepsilon > 0\ \exists n_0\ \forall n, m \ge n_0\ |a_n - a_m| < \varepsilon$.
Pak $\exists \lim_{n\to\infty} a_n$. (I $\Leftarrow$)
(Někdy se tomu říká věta o úplnosti $\mathbb{R}$. „Co se blíží, má limitu.")

⑤ $\lim_{x\to a^+} f(x) = \inf_{x\in(a,b)} f(x)$ $\quad \lim_{x\to b^-} f(x) = \sup_{x\in(a,b)} f(x)$, jif f neklés.

↑ „intuitivní"!!!

$\lim_{x\to a^+} f(x) = \sup_{x\in(a,b)} f(x)$ $\quad \lim_{x\to b^-} f(x) = \inf_{x\in(a,b)} f(x)$, jif f nerost.

Př.: $\lim_{x\to 0^+} \frac{1}{x} = \infty$, $\lim_{x\to\infty} = \inf\left(\frac{1}{x}\right) = 0$
f nerost, $f(x) = \frac{1}{x}, x \in \mathbb{R}^+$

⑥ Heine: $a \in \mathbb{R}^*, A \in \mathbb{R}^*$.
$\lim_{x\to a} f(x) = A \iff \forall (a_n)_n\ a_n \to a, n \to \infty,$
$a_n \ne a$ (pro $n \ge n_0$)
$\lim_{n\to\infty} f(a_n) = A$.

Př. Limita posloupnosti

1. $$\lim_{n\to\infty}\frac{\sqrt{n^3-2n^2+1}+\sqrt[3]{n^4+1}}{\sqrt{n^6-6n^5+2}+\sqrt[5]{n^7+n^3+1}} \overset{\text{„tvrz. Heine nebo def}}{=} \lim_{x\to\infty}\frac{\sqrt{x^3-2x^2+1}+\sqrt[3]{x^4+1}}{\sqrt{x^6-6x^5+1}+\sqrt[5]{x^7+x^3+1}}$$ [illegible]

moc. $\boxed{\frac{3}{2}}$ moc. $\boxed{\frac{4}{3}}$ $\boxed{3}$ $\boxed{7/5}$ $0\,?$

$$= \lim_{x\to 0^+}\frac{\sqrt[2]{\frac{1}{x^3}-\frac{2}{x^2}+1}+\sqrt[3]{\frac{1}{x^4}+1}}{\sqrt{\frac{1}{x^6}-\frac{6}{x^5}+1}+\sqrt[5]{\frac{1}{x^7}+\frac{1}{x^3}+1}}\cdot\frac{x^3}{x^3} =$$

$$\frac{n^{\frac{3}{2}}}{n^3} = \frac{1}{n^{3/2}}$$

$$= \lim_{x\to 0^+}\frac{\sqrt{x^3-2x^4+x^6}+\sqrt{x^5+x^9}}{\sqrt{1-6x+x^6}+\sqrt{x^8+x^{12}+x^{15}}} = \frac{0+0}{1+0} = 0$$

2. $\lim_{n\to\infty}\frac{a^n}{n!}$, $a\in\mathbb{R}^+$ $\quad \lim_{n\to\infty}\frac{a^n}{n!} = \lim_{n\to\infty}\left(\frac{a}{n}\,\frac{a}{n-1}\cdots\frac{a}{1}\right)$; Zvol $\varepsilon<1$.

$\forall n\geq n_0 \quad \frac{a}{n}<\varepsilon$

$b_n := \frac{a^n}{n!}$ Najdi $n_0\in\mathbb{N}$, že $\frac{a}{n_0}<\varepsilon$

$$b_n \le \frac{a^n}{n!} = \frac{a}{1}\cdots\frac{a}{n_0-1}\,\varepsilon^{n-n_0+1} \qquad \lim_{n\to\infty} b_n \le \frac{a}{1}\cdots\frac{a}{n_0-1}\lim_{n\to\infty}\varepsilon^{n-n_0+1} = \frac{a}{1}\cdots\frac{a}{n_0-1}\,0 = 0.$$

3. $$\lim_{n\to\infty}\sqrt[n]{n} = \lim_{n\to\infty} n^{\frac{1}{n}} \overset{\text{Heine}}{=} \lim_{x\to\infty} x^{\frac{1}{x}} = e^{\lim_{x\to\infty}(\ln x)\frac{1}{x}} = e^0 = 1$$

($\frac{\ln x}{x}\to 0, x\to\infty$ růstová ř.

nebo l'Hosp.)

4. $$\lim_{n\to\infty}\left[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}\right] = \lim_{n\to\infty}\left[\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\right] = \lim_{n\to\infty}\left(\frac{1}{1}-\frac{1}{n+1}\right) = 1$$

③

1. $\lim\limits_{n\to\infty} \sqrt[n]{3} = \lim\limits_{n\to\infty} 3^{\frac{1}{n}} = \lim\limits_{n\to\infty} e^{\frac{1}{n}\ln 3} = e^{\lim\limits_{n\to\infty}\frac{\ln 3}{n}} = e^0 = 1$

2. $\lim\limits_{n\to\infty} \sqrt[n]{3^n+5^n} \Rightarrow e^{\left[\frac{\ln(3^n+5^n)}{n}\right]}$; $\frac{\ln 5^n\left(1+\left(\frac{3}{5}\right)^n\right)}{n} = \frac{n\ln 5 + \overbrace{\ln\left(1+\left(\frac{3}{5}\right)^n\right)}^{\to 0}}{n}$

$= \ln 5 + \frac{\ln\left(1+\left(\frac{3}{5}\right)^n\right)}{n} \xrightarrow{n\to\infty} \ln 5 + \frac{0}{\infty} = \ln 5 \Rightarrow e^{\ln 5} = 5$

Nebo obdobně \ Strážníci $\lim\limits_{n\to\infty} \sqrt[n]{3^n+5^n} = 5$: $5 = \sqrt[n]{5^n} \le \sqrt[n]{3^n+5^n} \le \sqrt[n]{5^n+5^n} = \sqrt[n]{5}\,5$

$\downarrow$ 5 $\qquad$ $\downarrow$ $1\cdot 5$

3. $\lim\limits_{n\to\infty} \frac{3^n+5^n}{n!+6^n} = \lim\limits_{n\to\infty} \frac{\frac{3^n}{n!}+\frac{5^n}{n!}}{1+\frac{6^n}{n!}} = \frac{0+0}{1+0} = 0$

$\frac{a^x}{x^B}$ $a>1$

4. $\lim\limits_{n\to\infty} \frac{3^n+5^n}{7^n+6^n} = \lim\limits_{n\to\infty} \frac{\left(\frac{3}{7}\right)^n+\left(\frac{5}{7}\right)^n}{1+\left(\frac{6}{7}\right)^n} = \frac{0+0}{1+0} = 0$

Výhodné pro faktoriály $\underline{a_n > 0}$ $\quad \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = L \begin{cases} <1 & \lim a_n = 0 \\ =1 & \text{nevíme} \\ >1 & \lim a_n = +\infty \end{cases}$

Dk.: 1. $0 \le \frac{a_{n+1}}{a_n} < \overbrace{L+\varepsilon}^{<1} \Rightarrow a_{n+1} < (L+\varepsilon)a_n <$

$\dots < \underbrace{(L+\varepsilon)^n}_{<1} a_0 \to 0$

3. $1 < L-\varepsilon < \frac{a_{n+1}}{a_n}$ obdobně.

5. $\lim\limits_{n\to\infty} \frac{3^n}{n!}$ $\qquad \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} = \frac{3}{n+1} \to 0 \overset{<1}{} \quad \lim\limits_{n\to\infty} \frac{3^n}{n!} = 0$

6. $\lim \frac{(2n)!}{(n!)^2}$ $\qquad \frac{(2n+2)!\; n!\, n!}{(n+1)!\,(n+1)!\,(2n)!} = \frac{(2n+2)(2n+1)}{(n+1)^2} \to 4 \overset{>1}{\Rightarrow}$

$\Rightarrow \lim \frac{(2n)!}{(n!)^2} = +\infty$ $\qquad \frac{4n^2+8n+2}{n^2+2n+1} = \frac{4+\frac{8}{n}+\frac{2}{n^2}}{1+\frac{2}{n}+\frac{1}{n^2}}$

$= \lim_{n \to \infty} \left[\frac{1}{1} - \frac{1}{n+1} \right] = 1.$

Př.: Pro která $\cancel{x} \in \mathbb{R}$ $\exists \lim_{n \to +\infty} (\sin nx)$.

1. $\cancel{x} = 1$ (~~k~~ zapamatování) : $\lim_{n \to \infty} \sin \cancel{n} \; \nexists$.

Dk (Heine): $\alpha_{n} = n\pi \qquad \alpha_n \to \infty, n \to \infty$

$\beta_n = 2n\pi + \frac{\pi}{2} \qquad \beta_n \to \infty, n \to \infty$

$\sin \alpha_n = 0$

$\sin \beta_n = 1$

$\lim_{n \to \infty} \sin n\pi = 0 \qquad \Rightarrow \lim_{n \to \infty} \sin n \; \nexists$

$\lim_{n \to \infty} \sin\left(\frac{\pi}{2} + 2\pi n\right) = 1$

2. $x \in \mathbb{R}$. Nechť $\exists \lim_{n \to \infty} \sin(nx) = L$. Ziskame tim omezeni na x?

Ukazeme, ze nutne sin(x) = 0. Víme: sin[(n+1)x] = sin nx cos x + cos nx sin x a
sin[(n- 1)x] = sin nx cos x - cos n x sin x

Limita "posunutych" posloupnosti sin[(n +/- 1)x] pro x--> oo take existuji a jsou rovny L.
Pak sectenim vzorcu vyse a limitou x--> infty: 2L = 2L cosx. Odtud L= 0 nebo cos x =1.
Pokud cos x = 1, tak sin x = 0.

Pokud L = 0, pak souctove vztahy odecteme a uvazme x--> infty:
L - L = 2 lim_n sinx cos nx, tj. lim_n sin x cos nx = 0. Pokud je ale lim_n sin(nx) = 0, pak
lim_n |cos(nx)| = lim_n [1-sin^2(nx)]^(1/2) = 1, tj. limita lim_n sin x cos nx = sin(x) lim_n cos(nx) =
(+1/-1) sin(x), ale limita je 0, tj. opet sin x = 0.

V obou pripadech sin x = 0, tj. x = k pi.

...sme, ze lim sin(nx) existuje

Limes superior a inferior — nečíst: viz násl. strana

$\limsup\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(\sup\limits_{n\ge k} a_n\right)$, $\liminf\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(\inf\limits_{n\ge k} a_n\right)$

1. Najděte $\limsup a_n$, $\liminf a_n$

$$a_n = \frac{n-1}{n+1}\cos\left(\frac{2}{3}n\pi\right), \quad \frac{n-1}{n+1} = 1-\frac{2}{n+1} \to 1 \; !$$

$$\cos\left(\frac{2}{3}\pi n\right) \in \left\{\cos\frac{2}{3}\pi, \cos\frac{4}{3}\pi, \cos 2\pi\right\} = \left\{-\frac{1}{2}, 1\right\} \quad \forall n$$

$\{$

2. $a_n = \cos^n\left(\frac{2}{3}\pi n\right)$ $\begin{cases} 1 & n = 3k \to 1 \quad \limsup a_n = 1 \\ \left(-\frac{1}{2}\right)^n & \text{jinde} \to 0 \quad \liminf a_n = 0 \end{cases}$

Používám $\limsup\limits_{n\to\infty} a_n = \sup\left(\lim\limits_{k\to\infty} a_{n_k}\right)$ a $\liminf a_n = \inf\left(\lim\limits_{k\to\infty} a_{n_k}\right) = 0$

Limes superior a inferior

$$\limsup_{n\to\infty} a_n := \lim_{k\to\infty} (\underbrace{\sup_{n\geq k} a_n}_{b_k})$$ lim nerostoucí

$$\liminf_{n\to\infty} a_n := \lim_{k\to\infty} (\inf_{n\geq k} a_n)$$ lim neklesající

Věta: $\limsup_{n\to\infty} a_n = \sup_{(n_k)_k} (\lim_{k\to\infty} a_{n_k})$, pro $\lim_{k\to\infty} a_{n_k}$ $\exists$

$\liminf a_n = \inf_{(n_k)_k} (\lim a_{n_k})$ pro $\lim_{k\to\infty} a_{n_k}$ $\exists$

(Množinově: $\sup \{ \lim_{k\to\infty} a_{(n_k)} \mid (n_k)$ rostoucí & $\lim_{k\to\infty} (a_{n_k})$ existuje $\}$)

n_k rostoucí říkám a_{n_k} vybraná z a_n

Př.: Najdi $\limsup a_n$, $\liminf a_n$, $a_n = \frac{n-1}{n+1} \cos\left(\frac{2}{3}\pi n\right)$.

$n = 3k$, $\cos(2\pi k) = \cos(0) = 1$

$n = 3k+1$, $\cos(2\pi k + \frac{2}{3}\pi) = \cos \frac{2}{3}\pi = -\frac{1}{2}$

$n = 3k+2$, $\cos(\frac{4}{3}\pi) = -\frac{1}{2}$

$A \subseteq B$

$\inf A \geq \inf B$

$\sup A \leq \sup B$

$1 \leftarrow \frac{3k-1}{3k+1} \cdot 1$, $\frac{3k}{3k+2} \cdot \left(\frac{-1}{2}\right) \to -\frac{1}{2}$

$\limsup = 1$

$\liminf = -\frac{1}{2}$

Př.: Dtto $a_n = \cos^n\left(\frac{2}{3}\pi n\right)$.

$\limsup = 1$

$\liminf = 0$

$1^{3k} = 1 \to 1$

$\left(-\frac{1}{2}\right)^{3k+1} = -\frac{1}{2}\left(-\frac{1}{2}\right)^{3k} \to 0$

$\left(-\frac{1}{2}\right)^{3k+2} = \frac{1}{4}\left(-\frac{1}{2}\right)^{3k} \to 0$

$\boxed{\text{Př.:}}$ $a_n = \begin{cases} \frac{1}{n} & n = 2k \\ 0 & n = 2k+1 \end{cases}$ obě 0

Př. $a_n = \begin{cases} \frac{1}{2n} & 2k \\ \frac{1}{n} & 2k+1 \end{cases}$ obě 0

Př. $a_n = \begin{cases} \frac{1}{n} & n \text{ sudé} \\ \frac{(-1)^n}{n} & n \text{ liché} \end{cases}$ obě 0

Taylorův polynom

Nechť $f \in C^k(I)$, $a \in I$, pak $T_k^a(f) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(k)}(a)}{k!}(x-a)^k$ sluje Taylorův polynom f se středem a stupně k, $k \in \mathbb{N}_0$.

Př.: Tayl. $f(x) = \sin x$, $x = 0$, $k = 3$.

$\sin(0) = 0$, $\sin'(0) = \cos(0) = 1$, $\sin''(0) = \cos'(0) = -\sin(0) = 0$, $\sin'''(0) = -\sin'(0) = -\cos(0) = -1$

$$T_3^0 \sin(x) = 1 \cdot x - \frac{1}{3!}x^3 = x - \frac{x^3}{6}.$$

Př.: Urči $T_4^{0=a}(\cos)$.

sami

$\cos(0) = 1$, $\cos'(0) = -\sin(0) = 0$, $\cos''(0) = -\cos'(0) = -1$

$\cos'''(0) = 0$, $\cos^{(IV)}(0) = 1$

$$(T_4^0 \cos)(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

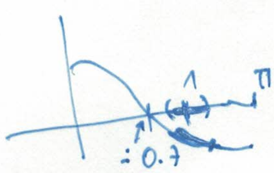
Věta o Tayl. pol. $\cos x = 1 - \frac{x^2}{2} + o(x^2) \to o(x^3), x \to 0$

$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^4) \to o(x^5), x \to 0$

Př.: $T_2^1 \cos$?

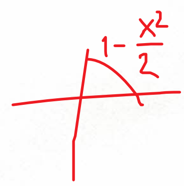
sami

$$(T_2^{1=a} \cos)(x) = \cos(1) + \frac{\overset{=\cos'(1)}{-\sin(1)}}{1!}(x-1)^1 - \frac{\overset{=\cos''(1)}{\cos(1)}}{2!}(x-1)^2.$$



Další Taylory: mocnina, logaritmus.

$$T_k^0 e^x = \sum_{n=0}^{k} \frac{x^n}{n!}$$



Sčítání: $f(x) = T_f^a + o(x^n)$

$g(x) = T_g^a + o(x^m)$

$f(x) + g(x) = T_f^a + T_g^a + o(x^{\min\{m,n\}})$

Násobení: $o(x^n)\, o(x^m) = o(x^{n+m})$, avšak

$x\, o(x^n) = o(x^{n+1})$: $\lim_{x\to 0} \frac{x f}{x^{n+1}} = \lim_{x\to 0} \frac{f}{x^n} = 0$

$\lim_{x\to 0} \frac{fg}{x^n x^m} = \lim \frac{f}{x^n} \lim \frac{g}{x^m} = 0 \cdot 0 = 0$

Př.: Tayl. pol. spočítejte limity

$1 + y + \frac{y^2}{2} +$

2. $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^4} = \lim_{x\to 0} \frac{1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^4) - \left(1 - \frac{x^2}{2} + \frac{x^4}{4}\frac{1}{2} + o(x^4)\right)}{x^4}$

$4! = 6 \cdot 4 = 24$

$\frac{1}{24} - \frac{1}{8} = -\frac{1}{12}$

$= \lim_{x\to 0} \frac{-\frac{1}{12}x^4 + o(x^4)}{x^4} = -\frac{1}{12} + \lim_{x\to 0} \frac{o(x^4)}{x^4} = -\frac{1}{12}$

1. $\lim_{x\to 0} \frac{e^x \sin x - x(x+1)}{x^3} = \lim_{x\to 0} \frac{1}{x^3} \left[\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)\right)\left(x - \frac{x^3}{3!} + o(x^3)\right) - x(x+1)\right] = \lim_{x\to 0} \frac{1}{x^3}\left[x - \frac{x^3}{\underset{6=3!}{}} + x^2 - \boxed{\frac{x^4}{6}} + o(x^3) + \frac{x^3}{2} + o(x^3) - x^2 - x\right] =$

$= \lim_{x\to 0} \frac{x^3}{x^3}\left(\frac{1}{2} - \frac{1}{3!}\right) + \lim \frac{o(x^3)}{x^3} = \frac{1}{3}.$

3. $\lim_{x\to 0} \frac{a^x + a^{-x} - 2}{x^2}$, $a > 0$ $\quad (a^x)' = \ln a\, a^x \quad (a^x)^{(k)} = (\ln a)^k a^x$

$\lim_{x\to 0} \frac{1}{x^2}\left[1 + \ln a\, a^x x + \ln^2 a\, a^x \frac{x^2}{2} + o(x^2) + 1 - \ln a\, a^x x + \ln^2 a\, a^x \frac{x^2}{2}\right.$

$+ o(x^2) - 2\Big] = \lim_{x\to 0} (\ln a)^2 a^x + \lim_{x\to 0}\frac{o(x^2)}{x^2} =$

$= (\ln a)^2 a^x.$ Zde obdobně l'Hospitalem.

Př: Spočtěte přibližně $\sqrt[5]{250}$.

Znám: $(1+x)^\alpha = \sum_{k=1}^{n}\binom{\alpha}{k}x^k + o(x^n)$ $\boxed{(\Delta)}$, $x\in(-1,1)$

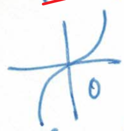
$\sqrt[5]{250} = \sqrt[5]{\frac{250}{3^5}\cdot 3^5} = 3\sqrt[5]{\frac{250}{243}} = 3\sqrt[5]{1+\frac{7}{243}}$, $\frac{7}{243}\in(-1,1)$

$f(x) = x^{\frac{1}{5}}$. Dále stačí dosadit do (Δ).

Průběhy funkcí

1. Základní věty: 1) $\exists f'(a)$ a f má lok. extrém v a $\Rightarrow f'(a)=0$.

Ne naopak: $(x^3)'(0) = 3\cdot 0^2 = 0$



nemá v 0 extr.

2) $f\in C(a,b)\wedge \exists f'$: $f'>0 \Rightarrow f$ roste, $f'\geq 0 \Rightarrow f$ neklesá

$f'<0 \Rightarrow f$ klesá, $f'\leq 0 \Rightarrow f$ neroste

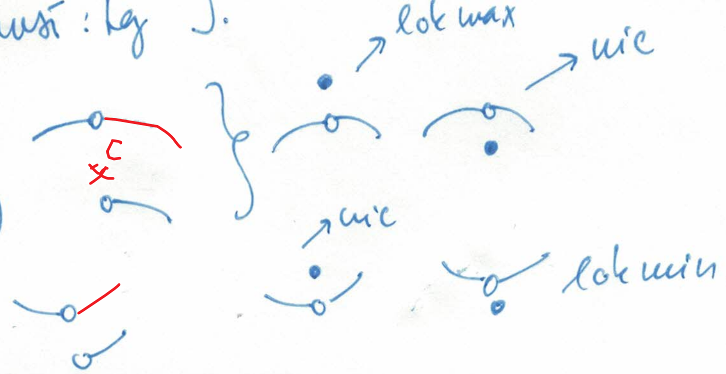
3) f spoj. na uzavř. $[a,b]$ má (globální) maximum i minimum

[Na otevřeném nemusí: tg].

4) a) f neklesá (a,c)

f neroste (c,b)

b) naopak

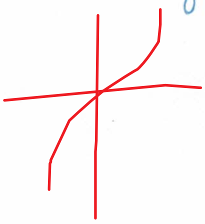


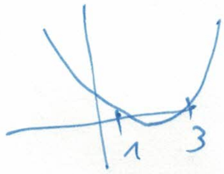
4) Používáme k vyšetřování ~~glob.~~ lok. extr.

Př.: Nalezněte lok. extrémy $f(x)=x^3-6x^2+9x-4,\ x\in\mathbb{R}$.

$f'(x)=3x^2-12x+9=3(x-1)(x-3)$

$f'=0 \iff x=1 \ \vee\ x=3$





	$(-\infty,1)$	$(1,3)$	$(3,\infty)$
$x-1$	$-$	$+$	$+$
$x-3$	$-$	$-$	$+$
f'	$+$	$-$	$+$

f

$x=1$ lok. max (dokonce ostré)

$x=3$ lok. min (dokonce ostré)

[$\lim\limits_{x\to\pm\infty} f(x)=\pm\infty$ (aritm. lim.) $\Rightarrow$ nejde o glob. extrémy]

Př.: Lok. extr. $f(x)=e^x\sin x,\ x\in\mathbb{R}$

$f'(x)=e^x\cos x+e^x\sin x=e^x(\sin x+\cos x)=0 \iff$

$\sin x=-\cos x$

$x_1=\frac{3}{4}\pi+2k\pi$, $x_2=\frac{7}{4}\pi+2k\pi$ $\Big\}$ $(x_0=\frac{3}{4}\pi+k\pi)$

a) $\cos x\neq 0$

$\operatorname{tg} x=-1$

b) $\cos x=0$ $\quad \sin x=0$, pak ovšem $-\cos x=-1$

	$(-\frac{\pi}{4},\frac{3}{4}\pi)$	$(\frac{3}{4}\pi,\frac{7}{4}\pi)$	$(\frac{7}{4}\pi,\dots)$
f'	$+$	$-$	

$\frac{3}{4}\pi+2k\pi$ lok max

$\frac{7}{4}\pi+2k\pi$ lok min

[$\lim\limits_{x\to\frac{\pi}{2}+k\pi} f(x)=\lim\limits_{k\to\infty} e^{\frac{\pi}{2}+k\pi}\cdot 1=+\infty$; analog. $x\to-\frac{\pi}{2}+k\pi$ $\to-\infty$

glob. extr. $\nexists$]