Záverečný test ZS 2022/23, Varianta C

Vzorové riešenie

1)
$$\lim_{x\to 3} \frac{\ln(x^2+x-11)}{x^2-7x+12} \overset{\text{"}\frac{0}{0}\text{"}\ (1b)}{\underset{L.H.}{=}} \lim_{x\to 3} \frac{\frac{1}{x^2+x-11}\cdot(2x+1)}{2x-7} \quad (1b)$$

$$= \lim_{x\to 3} \frac{2x+1}{(x^2+x-11)(2x-7)} \quad (1b)$$

môžeme dosadiť

$$= \frac{2\cdot 3+1}{(3^2+3-11)(2\cdot 3-7)} = \frac{7}{1\cdot(-1)} = \underline{\underline{-7}} \quad (1b)$$

Bodovanie:
- Zistenie, že je nutné použiť L.H pravidlo, overenie predpokladu vety (1b)
- Správne zderivovaný čitateľ (0,5b)
- Správne zderivovaný menovateľ (0,5b)
- Správne upravený výraz pred dosadením (1b)
- Dosadenie a správny výsledok (1b)

2)
$$f(x) = \frac{(x+2)^2}{\sqrt{x+1}}$$

podmienka: $x+1 \geq 0 \wedge x+1 \neq 0$

$\Rightarrow x+1 > 0$

$x > -1$

$$\boxed{D_f = (-1,\infty)} \quad (0{,}5b)$$

$$f'(x) = \frac{((x+2)^2)'\sqrt{x+1} - (x+2)^2(\sqrt{x+1})'}{(\sqrt{x+1})^2} \quad (0{,}5b)$$

$$= \frac{2(x+2)\sqrt{x+1} - \frac{(x+2)^2}{2\sqrt{x+1}}}{x+1} \quad (0{,}5b) = \frac{4(x+2)(x+1) - (x+2)^2}{2\sqrt{x+1}\,(x+1)}$$

$$= \frac{4x^2+4x+8x+8-(x^2+4x+4)}{2\sqrt{x+1}\,(x+1)} = \frac{3x^2+8x+4}{2\sqrt{x+1}\,(x+1)} \quad (2b)$$

alternatíve

$$= \frac{(x+2)(3x+2)}{2(x+1)^{\frac{3}{2}}}$$

podmienka: $\sqrt{x+1} > 0$

$$\boxed{D_{f'} = (-1,\infty) = D_f} \quad (0{,}5b)$$

3 $f(x) = \frac{x+1}{2x-6}$ $\qquad k = -2$

Dotyčnica: $t: y = kx + q$ $\qquad k = f'(x_0) = \frac{1\cdot(2x_0-6) - (x_0+1)\cdot 2}{(2x_0-6)^2}$

$$= \frac{2x_0 - 6 - 2x_0 - 2}{(2x_0-6)^2}$$

$$= -\frac{8}{(2x_0-6)^2} \quad 1b$$

$$-2 = -\frac{8}{(2x_0-6)^2} \quad 0{,}5b$$

$$(2x_0-6)^2 = 4$$

$$4x_0^2 - 24x_0 + 36 = 4$$

$$x_0^2 - 6x_0 + 8 = 0$$

$$(x_0-4)(x_0-2) = 0$$

$$x_0 = +4 \lor x_0 = +2$$

Body, kde je smernica $k = -2$ 1b

Body dotyku:

$f(+4) = \frac{+4+1}{2\cdot(+4)-6} = \frac{5}{2}$ $\Rightarrow$ $\boxed{T_1 = [4, \frac{5}{2}]}$

$f(+2) = \frac{2+1}{2\cdot 2-6} = -\frac{3}{2}$ $\Rightarrow$ $\boxed{T_2 = [2, -\frac{3}{2}]}$ 0,5b

Rovnica dotyčnice:

Bod $T_1 \in t$: $\frac{5}{2} = 4\cdot(-2) + q$

$q = \frac{5}{2} + 8 = \frac{5+16}{2} = \frac{21}{2}$ $\Rightarrow$ $\boxed{t_1: y = -2x + \frac{21}{2}}$ 1b

Bod $T_2 \in t$: $-\frac{3}{2} = 2\cdot(-2) + q$

$q = 4 - \frac{3}{2} = \frac{8-3}{2} = \frac{5}{2}$ $\Rightarrow$ $\boxed{t_2: y = -2x + \frac{5}{2}}$ 1b

Priesečníky s osami pre dotyčnice

t_1: $y = 0 \Rightarrow 0 = -2x + \frac{21}{2}$

$x = \frac{21}{4}$ $\Rightarrow$ $\boxed{P_{t_1x} = [\frac{21}{4}, 0]}$ 0,25b

$x = 0$ $\quad y = -2\cdot 0 + \frac{21}{2}$

$y = \frac{21}{2}$ $\Rightarrow$ $\boxed{P_{t_1y} = [0, \frac{21}{2}]}$ 0,25b

t_2: $y = 0 \Rightarrow 0 = -2x + \frac{5}{2}$

$x = \frac{5}{4}$ $\Rightarrow$ $\boxed{P_{t_2x} = [\frac{5}{4}, 0]}$ 0,25b

$x = 0$

$y = -2\cdot 0 + \frac{5}{2}$

$y = \frac{5}{2}$ $\quad \boxed{P_{t_2y} = [0, \frac{5}{2}]}$ 0,25b

Hyperbola: $f(x) = \frac{x+1}{2x-6}$

$2x-6 \neq 0 \quad x \neq -3$

$D_f = \mathbb{R} \setminus \{3\}$ (0,5b)

$\rightarrow$ z vlastností lineárnych lomených fcií je teda $\boxed{x = 3}$ vertikálna asymptota (0,5b)

(prípadne vypočítať jednostranné limity $\lim\limits_{x \to 3^\pm} f(x)$)

Stred: Vzorec alebo:

$$(x+1) : (2x-6) = \frac{1}{2} + \frac{4}{2(x-3)} \Rightarrow \boxed{S = [3, \tfrac{1}{2}]}$$ (1b)

$$\underline{-x+3}$$
$$4$$

$\Rightarrow \boxed{y = \frac{1}{2}}$ je horizontálna asymptota (0,5b)

(Prípadne cez výpočet $\lim\limits_{x \to \pm\infty} f(x)$)

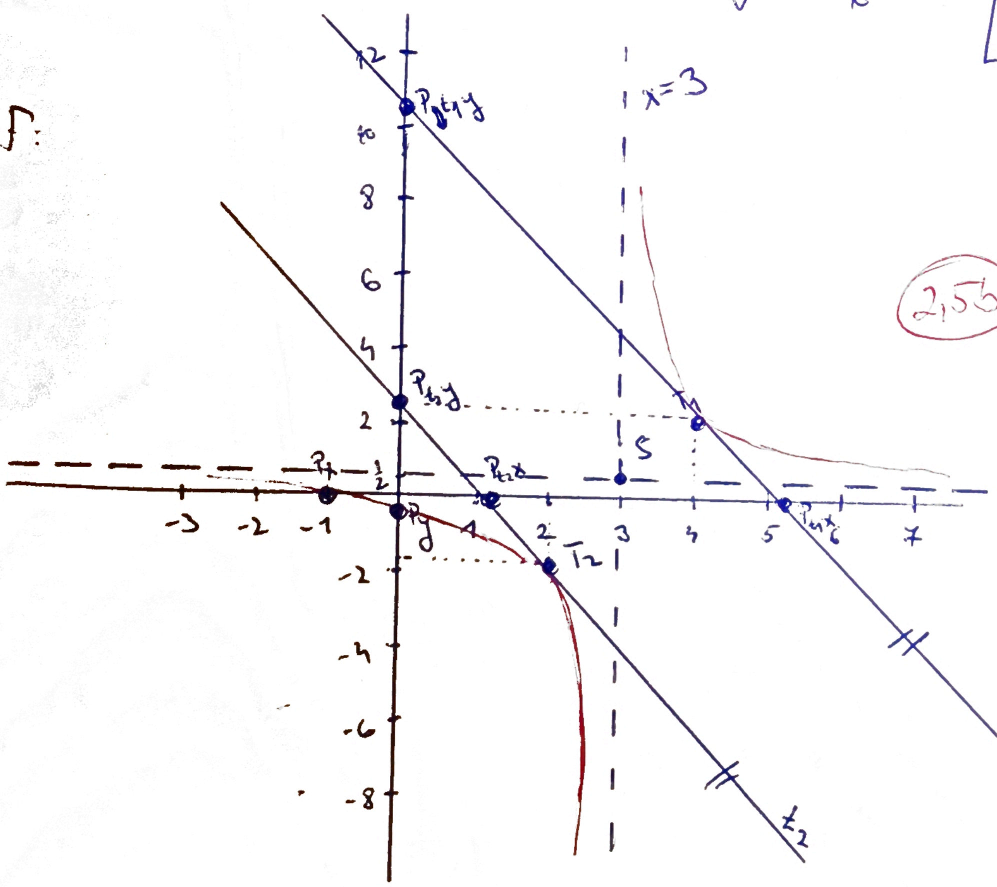
Priesečníky s osami pre hyperbolu:

P_x: $y = 0 \Rightarrow 0 = \frac{x+1}{2x-6} \Rightarrow x = -1 \Rightarrow$ $P_x = [-1, 0]$

P_y: $x = 0 \Rightarrow y = \frac{0+1}{2 \cdot 0 - 6} \Rightarrow y = -\frac{1}{6} \Rightarrow$ $P_y = [0, -\frac{1}{6}]$

(1b)

Graf:



(2,5b)

$\boxed{4)}$ $f(x) = \dfrac{x^2+2x-15}{x-4}$

1. $D_f = \mathbb{R} \setminus \{4\}$ (1b)

2. D_f nie je symetrický $\Rightarrow$ fcia není ani sudá ani lichá (1b)

3. P_x: $y = 0 \Leftrightarrow 0 = \dfrac{x^2+2x-15}{x-4}$

$$0 = x^2+2x-15$$
$$0 = (x+5)(x-3)$$
$$x = +3, \quad x = -5 \Rightarrow \boxed{P_{x_1} = [-5,0], \; P_{x_2} = [3,0]}$$ (1b)

P_y: $x = 0 \Rightarrow y = \dfrac{0^2+2\cdot 0-15}{0-4} = \dfrac{15}{4} \Rightarrow \boxed{P_y = [0, \tfrac{15}{4}]}$ (0,5b)

4. dosádzame do f(x)

$\ominus$ | -5 | $\oplus$ | 3 | $\ominus$ | 4 | $\oplus$ (1b)

Fcia je kladná na $(-5,3) \cup (4,\infty)$

Fcia je záporná na $(-\infty,-5) \cup (3,4)$

5. $$\lim_{x\to-\infty} \frac{x^2+2x-15}{x-4} = \lim_{x\to-\infty} \frac{x^2(1+\underset{\to 0}{\frac{2}{x}}-\underset{\to 0}{\frac{15}{x^2}})}{x(1-\underset{\to 0}{\frac{4}{x}})} = \lim_{x\to-\infty} x \cdot \lim_{x\to-\infty} \frac{1+\frac{2}{x}-\frac{15}{x^2}}{1-\frac{4}{x}} =$$
$$= -\infty \cdot 1 = -\infty$$ (0,5b)

$$\lim_{x\to+\infty} \frac{x^2+2x-15}{x-4} = \text{...analogicky...} = +\infty \cdot 1 = +\infty$$ (0,5b) → nie sú globálne extrémy

$$\lim_{x\to 4^-} \frac{x^2+2x-15}{x-4} = \frac{16+8-15}{4^- - 4} = \frac{9}{0^-} = -\infty$$ (0,5b)

$$\lim_{x\to 4^+} \frac{x^2+2x-15}{x-4} = \frac{9}{0^+} = +\infty$$ (0,5b) → zvislá asymptota $x = 4$

6. $$f'(x) = \frac{(2x+2)(x-4)-(x^2+2x-15)\cdot 1}{(x-4)^2} = \frac{2x^2-8x+\cancel{2x}-8-x^2-\cancel{2x}+15}{(x-4)^2}$$
$$= \frac{x^2-8x+7}{(x-4)^2}$$ (1,5b) $D_{f'} = \mathbb{R} \setminus \{4\}$

Stac. body: $x^2-8x+7 = 0$

$(x-1)(x-7) = 0$

$x_1 = 1$ (0,5b) $f(1) = 4$

$x_2 = 7$ $f(7) = 16$

7.

$\oplus \quad \ominus \quad \ominus \quad \oplus$

$\nearrow \; 1 \; \searrow \; 4 \; \searrow \; 7 \; \nearrow$

Fcia je rastúca na $(-\infty, 1) \cup (7, \infty)$

Fcia je klesá na $(1,4) \cup (4,7)$

Bod $[1,4]$ je lokálne maximum

Bod $[7,16]$ je lokálne minimum

2b

8. Asymptoty: Vertikálna asymptota $x = 4$ (2 limit)

~~...~~ Horizontálna neexistuje, lebo sme vypočítali limity.

0,5b

Šikmé: $y = kx + q$

$$k = \lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \frac{x^2 + 2x - 15}{x(x-4)} = \lim_{x \to \pm\infty} \frac{x^2\left(1 + \frac{2}{x} - \frac{15}{x^2}\right)}{x^2\left(1 - \frac{4}{x}\right)} = 1$$

$$q = \lim_{x \to \pm\infty} (f(x) - kx) = \lim_{x \to \pm\infty} \left(\frac{x^2 + 2x - 15}{x - 4} - x \right) = \lim_{x \to \pm\infty} \frac{x^2 + 2x - 15 - x^2 + 4x}{x - 4}$$

$$= \lim_{x \to \pm\infty} \frac{6x - 15}{x - 4} = \lim_{x \to \pm\infty} \frac{x\left(6 - \frac{15}{x}\right)}{x\left(1 - \frac{4}{x}\right)} = 6$$

Šikmá asymptota $\boxed{y = x + 6}$ v $\pm\infty$.

1b

9. $$f''(x) = \frac{(2x-8)(x-4)^2 - (x^2 - 8x + 7)2(x-4)}{(x-4)^4}$$

$$= \frac{(2x-8)(x-4) - 2(x^2 - 8x + 7)}{(x-4)^3} = \frac{2x^2 - 8x - 8x + 32 - 2x^2 + 16x - 14}{(x-4)^3}$$

$$= \frac{18}{(x-4)^3}$$

$D_{f''} = \mathbb{R} \setminus \{4\}$ 2b

$f''(x) \neq 0$ pre žiadne $x \in D_f$ → nemáme inflexný bod

$\ominus \quad \oplus$

$\cap \; 4 \; \cup$

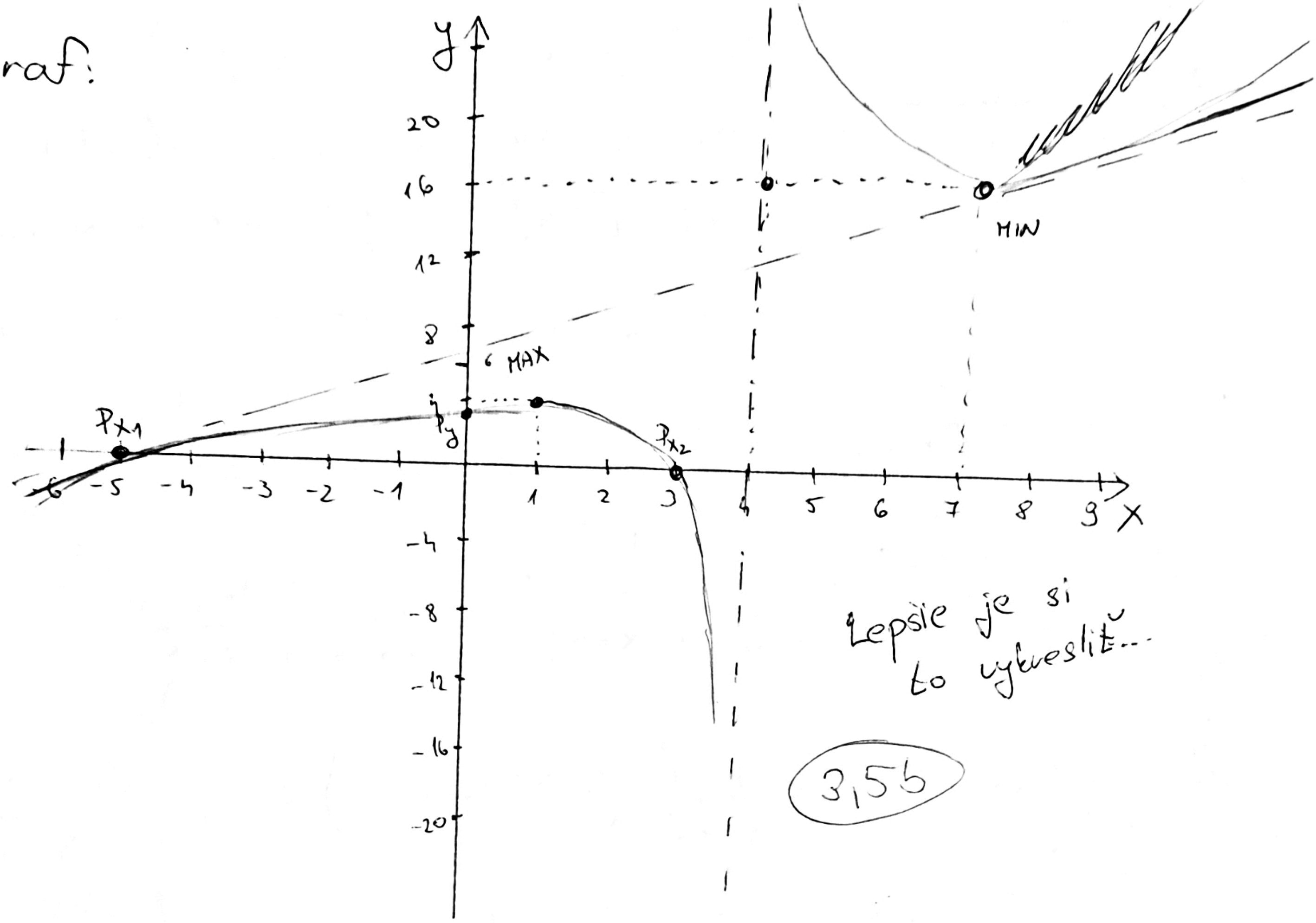
Fcia je konvexná pre $x \in (4, \infty)$

Fcia je konkávna pre $x \in (-\infty, 4)$

1,5b

$H_f = \mathbb{R} \setminus (4,16) = (-\infty, 4\rangle \cup \langle 16, \infty)$ 1b

Graf:



Lepšie je si to vykresliť...

3,5b

5. $f(x,y) = x^3 - 3x^2 + x + xy - y$

$M = \{[x,y] \in \mathbb{R}^2 : 2x - 4 \leq y \leq -x^2 + 3x + 2\}$

Náčrt množiny M: $M = M^o \cup \partial M$

$y \geq 2x - 4 \rightarrow$ polrovina oddelená priamkou danou bodmi $P_y = [0,-4]$ a $P_x = [2,0]$

$y \leq -x^2 + 3x + 2 \rightarrow$ polrovina oddelená parabolou (konkávna)

vrchol: $-x^2 + 3x + 2 = -(x^2 - 3x - 2) = -((x - \frac{3}{2})^2 + \frac{17}{4})$

$V = [\frac{3}{2}, \frac{17}{4}]$

Priesečníky dvoch kriviek:

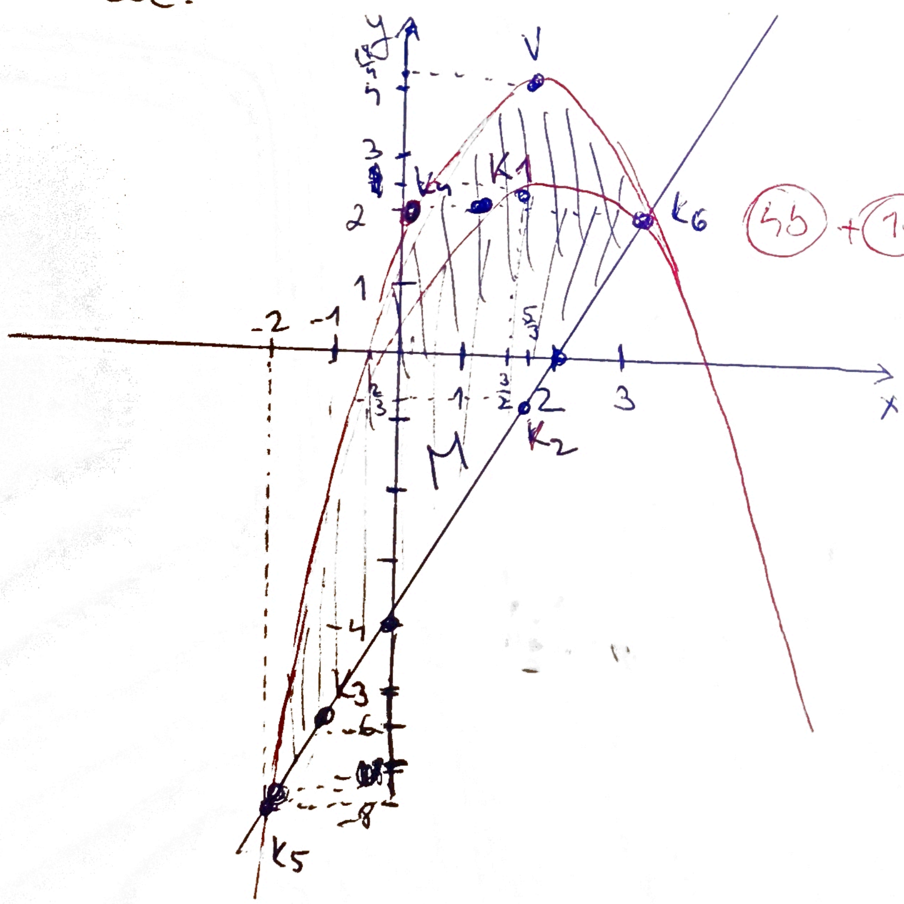
$2x - 4 = -x^2 + 3x + 2$

$x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

$x = -2$	$f(-2) = -8$
$x = 3$	$f(3) = 2$

Obrázok:



4b + 1b za body (kandidátov)

a) Na M°:

$$\frac{\partial f}{\partial x} = 3x^2 - 6x + 1 + y = 0$$

$$\frac{\partial f}{\partial y} = x - 1 = 0 \quad \Rightarrow x = 1$$

$$3 - 6 + 1 + y = 0$$

$$y = 2$$

3b

Kandidát $K_1 = [1, 2] \in M$

b) Na ∂M:

priamka: $g(x) = f(x, 2x-4) = x^3 - 3x^2 + x + x(2x-4) - (2x-4)$

$$= x^3 - 3x^2 + x + 2x^2 - 4x - 2x + 4$$

$$g(x) = x^3 - x^2 - 5x + 4$$

$$g'(x) = 3x^2 - 2x - 5 = 0$$

$$D = 4 + 4 \cdot 3 \cdot 5 = 64$$

$$x_{1,2} = \frac{2 \pm 8}{6} = \begin{cases} \frac{5}{3} \\ -1 \end{cases}$$

$$f\left(\tfrac{5}{3}\right) = 2 \cdot \tfrac{5}{3} - 4 = -\tfrac{2}{3}$$

$$f(-1) = 2 \cdot (-1) - 4 = -6$$

Kandidáti na priamke $K_2 = \left[\frac{5}{3}, -\frac{2}{3}\right]$

$K_3 = [-1, -6]$

3b

Na parabole → Lagrangeova funkcia: väzba $0 = -x^2+3x+2-y$

$$L(x,y,\lambda) = x^3-3x^2+x+xy-y+\lambda(-x^2+3x+2-y)$$

$$\frac{\partial L}{\partial x} = 3x^2-6x+1+y-2\lambda x+3\lambda = 0$$

$$\frac{\partial L}{\partial y} = x-1-\lambda = 0 \Rightarrow x = \lambda+1$$

$$\frac{\partial L}{\partial \lambda} = -x^2+3x+2-y = 0$$

$$-(\lambda+1)^2+3(\lambda+1)+2-y=0$$

$$-\lambda^2-2\lambda-1+3\lambda+3+2-y=0$$

$$y = -\lambda^2+\lambda+4$$ → dosadenie do 1) za x, y a vyjde $\boxed{x=0,\ y=2,\ \lambda=-1}$

kandidát $K_4 = [0,2]$ 4b

Napokon ešte kandidáti ako priesečník: $K_5 = [-2,-8]$ a $K_6 = [3,2]$

Vyhodnotenie:

$f(1,2) = -1$

$f\left(\frac{5}{3},-\frac{2}{3}\right) = -\frac{67}{27}$ → MIN

$f(-1,-6) = 7$ ~~MAX~~ → MAX

$f(0,2) = -2$

$f(-2,-8) = 2$

$f(3,2) = 7$ → MAX

dve maximá

3b