

Scott 67 - p. 96

(1.)

Set-up: (Ω, \mathcal{A}, P)

non-empty set
of samples

σ -algebra
of subsets
of Ω

σ

σ -additive
probability measure
 $P(\Omega) = 1$

$[P=0] :=$ the ideal in \mathcal{A} of sets $X, P(X) = 0$

$\mathcal{B} := \mathcal{A}/[P=0]$: Boolean algebra

Lemma (p. 96 in Scott's paper)

\mathcal{B} is complete: $\forall U \subseteq \mathcal{B} \exists \text{sup } U$.

Proof:

Claim 1: \mathcal{B} is σ -algebra and $\mu: \mathcal{B} \rightarrow [0, 1]$

defined by

$$\mu(A/[P=0]) := P(A)$$

is a σ -additive measure on \mathcal{B} that
is strictly positive (i.e. $\forall b \in \mathcal{B} \setminus \{\emptyset\}, \mu(b) > 0$).

(2.)

Prf-claim¹: Use that $[P=0]$ is closed

under countable unions (as P is σ -additive)

 \square_1

Claim 2: B has ccc ($=$ countable chain condition)

 \square_2

Every antichain $I \subseteq B$ is countable.

 \square_3

$$\forall b_1, b_2 \in I, b_1 \neq b_2 \rightarrow b_1 \wedge b_2 = \emptyset_B$$

Prf-claim²: By the strict positivity of μ

There can be $\leq k$ elements of I with μ -measure in the interval $[\frac{1}{k+1}, \frac{1}{k}]$,

for all $k = 1, 2, \dots$.

 \square_2

Claim 3: For $U \subseteq B$, $U \neq \emptyset$, denote \tilde{U} the ideal generated by U ($b \in \tilde{U}$ iff $b \leq a_1 \vee \dots \vee a_n$, some $a_i \in U$).

Then the upper bounds of U and \tilde{U} are the same: use B : $b \geq U \Leftrightarrow b \geq \tilde{U}$.

 \square_3

(3.)

Claim 4 : Any ideal \tilde{U} has the same upper bounds as any maximal antichain $I \subseteq \tilde{U}$.
 (Proved by Zorn's lemma.)

Prf-claim 4: Clear $b \geq \tilde{U} \Rightarrow b \geq I$.

If for some b : $b \geq I$ but $b \notin \tilde{U}$, say
 $b \geq a$ for $a \in \tilde{U}$, then $a - b (= a \cap \bar{b}) \neq \emptyset$
 and $I \cup \{a - b\}$ is also antichain, contradicting
 the maximality of I .

□₄

Claim 5 : $\forall U \subseteq B \exists V \subseteq \mathbb{C}_l$, V countable
 s.t. (U and V have the same upper bounds)?

Prf-claim 5 : U has the same app. bounds as \tilde{U}
 by Claim 3, which has the same upper bounds
 as any max. antichain $I \subseteq \tilde{U}$ by Claim 4,
 and I is countable by Claim 2. Define:

V : some countable $\subseteq U$ s.t. $I \subseteq \tilde{V}$ (The ideal generated by)

exists, because each element of \tilde{U}
 is majorized by the union of
 finitely many elmts of U . □₅

(3)

To conclude the proof of the lemma
note that V produced by Claim 4 has to
sup V by Claim 1 (i.e. by the δ -additivity)

(joined now to V produced by Claim 4)

(which is not yet true)

(join this to V produced by Claim 4)

q.e.d.

Lemma 3.2

V has anticipated to V' (i.e.

$V' \leq V$)

PROOF

By definition of V it is clear that V is not greater than V' .
So we have to prove that V is not less than V' .

Assume that $V < V'$. Then there exist some δ such that $V + \delta > V'$.
This contradicts the fact that V is the supremum of V .
So we have $V = V'$.

Corollary 3.3

If V is a sum of large numbers of V (i.e. if V is a sum of large numbers of V produced by Claim 4) then V is the supremum of V .

PROOF

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So V is the supremum of V .