Propositional logic in Lean

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Outline

Natural deduction

Lean's syntax (see code)

Some examples (see code)

Propositional logic in natural deduction

Lean's prover is built upon the calculus of natural deduction.
The system consists of several inference rules that fall into two classes:

1. introduction rules
2. elimination rules

There are no axioms and every proof of a proposition P is a proof from some hypotheses P_1, P_2 ... P_n. Derivations (proofs) are built from smaller ones.
Hypotheses can be *canceled*, e.g. in the introduction rule for implication a hypothesis is moved into the antecedent of the conclusion.

Rules for $\to$ and $\wedge$

$$\cfrac{\cfrac{}{A}\,1 \\ \vdots \\ B}{A \to B}\;1\ {\to}\mathrm{I}$$

$$\frac{A \to B \qquad A}{B}\;{\to}\mathrm{E}$$

$$\frac{A \qquad B}{A \wedge B}\;\wedge\mathrm{I}$$

$$\frac{A \wedge B}{A}\;\wedge\mathrm{E}_{\mathrm{l}}$$

$$\frac{A \wedge B}{B}\;\wedge\mathrm{E}_{\mathrm{r}}$$

Rules for ∨ and ¬

$$\frac{A}{A \vee B}\ \vee\mathrm{I_l} \qquad \frac{B}{A \vee B}\ \vee\mathrm{I_r} \qquad \frac{A \vee B \quad \begin{array}{c}\overline{A}^{\ 1}\\ \vdots \\ C\end{array} \quad \begin{array}{c}\overline{B}^{\ 1}\\ \vdots \\ C\end{array}}{C}\ 1\ \ \vee\mathrm{E}$$

$$\frac{\begin{array}{c}\overline{A}^{\ 1}\\ \vdots \\ \bot\end{array}}{\neg A}\ 1\ \ \neg\mathrm{I} \qquad \frac{\neg A \qquad A}{\bot}\ \neg\mathrm{E}$$

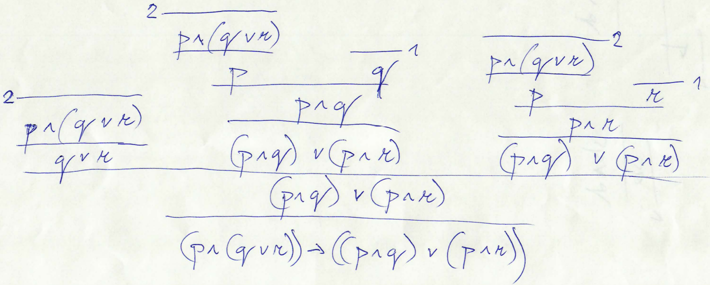
Constants and proof by contradiction

$$\frac{\bot}{A}\ \bot\text{E} \qquad \frac{}{\top}\ \top\text{I}$$

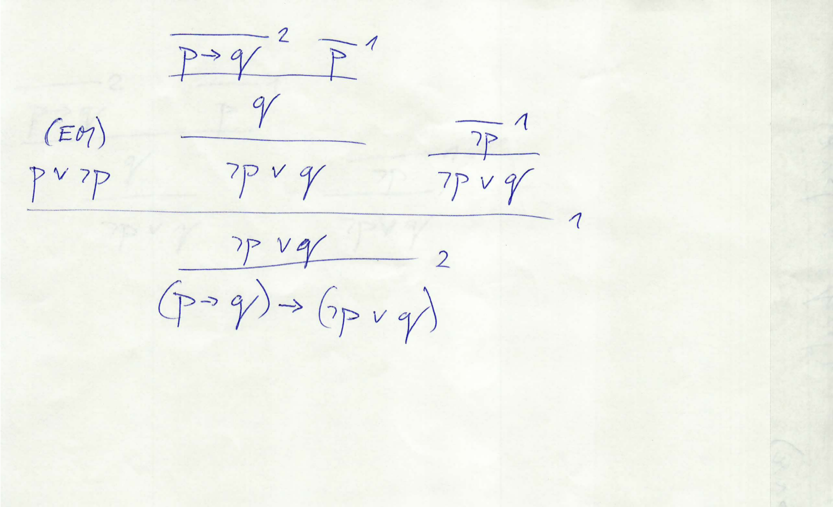
These rules constitute a system for intuitionistic logic, if we add *reductio ad absurdum*, we obtain classical logic.

$$\begin{array}{c} \overline{\neg A}^{\,1} \\ \vdots \\ \dfrac{\bot}{A}\ 1\ \text{RAA} \end{array}$$

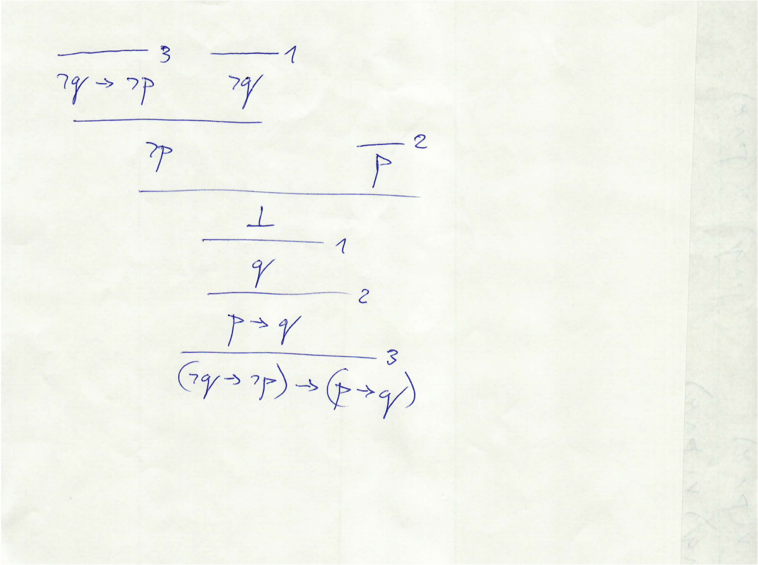
Example derivation: $\vee$-elim



Example derivation: EM



Example derivation - RAA



References

- Avigad, *Logic and Proof* (pictures of natural deduction inference rules are taken from Chapter 3)
- Avigad, *Theorem proving in Lean*