Our eventual goal (after solving the following problem and Problem 6) is to show that both versions of the WPHP:

$$\mathrm{WPHP}_x^{2x} \quad \text{and} \quad \mathrm{WPHP}_x^{x^2} \tag{1}$$

are also provable in $I\Delta_0$, although we will need to include this time axiom Ω_1 (it is not known whether it is indeed needed). Recall that Ω_1 says that $\forall x \exists y\ y = x^{|x|}$, cf. Section 5.1 in my 1995 book.

In solving this problem you will need to code in $I\Delta_0$ sequences of length $\sim |x|$ of numbers $< x$: each such number has length up to $|x|$ and hence the total bit length of such a sequence is $|x|^2$, and the code y may be of size up to $2^{|x|^2} = x^{|x|}$. Axiom Ω_1 is exactly what you need to prove that such y exists.

Problem 5: *Show that the two principles (1) accepted for all Δ_0 maps are in theory $I\Delta_0 + \Omega_1$ equivalent.*

Clearly the first principle implies the second so the task is to prove the opposite implication. In other words, if $f : 2x \to x$ violates WPHP_x^{2x} we want to define from f using bounded quantifiers (i.e. $\Delta_0(f)$-define) a map $g : x^2 \to x$ violating $\mathrm{WPHP}_x^{x^2}$.

Hint: define injective maps into x with bigger and bigger domains $4x$, $8x$, all the way up to $2^{|x|}x = x^2$.