# Building models by games pt. 2

## Forcing with games

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March 17, 2021

# A quick recap

- Canonical model theorem

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- $\forall p \in N$ : have a model.
- We introduced games.

# Notion of forcing

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- Notice that property (13) of notions of consistency here is not needed at all. Unions of (short enough) chains here are trivially in $N$, since "short enough" here means finite chains with finite differences of successors.

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- $A^+(\overline{p})$ is called the **structured compiled by** $\overline{p}$.

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- A game $G_N(P; X)$ is called **standard** iff $X$ is both infinite and coinfinite subset of $\omega \setminus \{0\}$. In other words the players alternate whose turn it is countably many times and $p_0$ is picked by the $\forall$-player.

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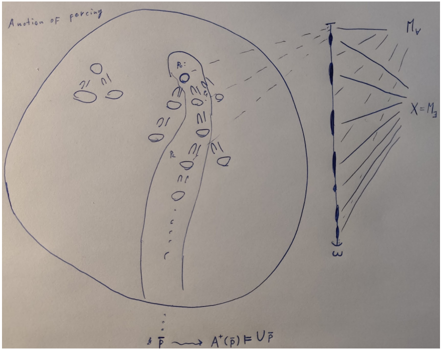
# A picture of $G_N(P; X)$



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*A player has a winning strategy for $G_N(P; X)$ iff the same player has a winning strategy for $G_N(P; odds)$.*

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**Proof**

*Let $p_i, \ldots, p_{i+k}$ be consecutive moves of one player. This player loses nothing if they instead set $p_i := p_k$ and let the other player play sooner. On the other had a single move can be prolonged into a constant sequence of moves.* □

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**Lemma**

*Let $N$ be a notion of forcing. Then $P :=$ "The compiled structure $A^+(\overline{p})$ is a model of $\bigcup \overline{p}$ and each element of $A^+(\overline{p})$ is of the form $c^{A^+(\overline{p})}$ for infinitely many witnesses $c$." is $N$-enforceable.*

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**Proof**

*Recall the proof of the theorem "$p \in N$ has a model". We again organize the moves of $\exists$-player indexed by $X$ into countable families of tasks as in this theorem and add the following countably many tasks:*
*"(For a closed $L(W)$-term $t$ and $n < \omega$) put $t = c_i$ into $\bigcup \overline{p}$ for some witness $c_i$ with $i \geq n$."*
*These tasks can be carried out thanks to the additional properties of notions of forcing.* □

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- Now follows an equivalent condition for $q$ to force $P$.

# The forcing relation $\Vdash$ cont.

**Lemma**

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- *In $G_N(P; odds)$, if the $\forall$-player chooses $p_0 \supseteq q$, then he puts the $\exists$-player in the winning position.*

## Proof

(1) $\Rightarrow$ (2) *trivially.* (2) $\Rightarrow$ (1)*: Let* $(p_0, \ldots, p_k)$ *be a position and* $q \subseteq p_k$*. Assume that* $p_{k+1}$ *is to be chosen by the* $\exists$*-player, otherwise let her wait until it is her turn. She can pretend that the choices of* $(p_0, \ldots, p_{k-1})$ *were simply a warming-up, and that the game actually begins at* $p_k$*.*

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## Proof (cont.)

*The* $\exists$*-player imagines that she plays a new game* $G_N(P; Y)$*, where* $Y = \{n - k; n \in X, n \geq k\}$ *and the* $\forall$*-player had chosen* $p_0 \supseteq q$ *and therefore put the* $\exists$*-player into winning position.*
*She can proceed using this strategy and win* $G_n(P; X)$*.* □

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4. *If for every* $p \supseteq q$ *exists an* $r \supseteq p$ *such that* $r \Vdash P$*, then* $q \Vdash P$.
5. *(Conjugation lemma) Let* $P$ *be a conjunction of the properties* $(P_i)_{i<\omega}$*. Then* $q \Vdash P$ *iff* $\forall i < \omega : q \Vdash P_i$.

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## Proof

*(1)-(3) follow trivially from the definitions.*

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## Proof

*(4) In a play of $G_N(P; odds)$ suppose that the $\forall$-player picks $p_0 \supseteq q$, then the $\exists$ player can choose $p_1 := r$ such that $r \Vdash P$ this puts her into winning position. Therefore $(p_0)$ was already a winning position for her.*

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*This corresponds with the previous lemma about equivalent condition for the forcing relation.*

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## Proof

*(5) Left to right is trivial.*
*Right to left: Partition $\omega$ into $(X_i)_{i<\omega}$ a countable family of countable sets. Let the $\forall$-player choose $p_0 \supseteq q$. Then the $\exists$-player has a winning strategy $\sigma_i$ for each the games $G_N(P_i; X_i)$. She can play the game $G_N(P; odds)$ by picking $p_j$ using $\sigma_i$ whenever $j \in X_i$.*

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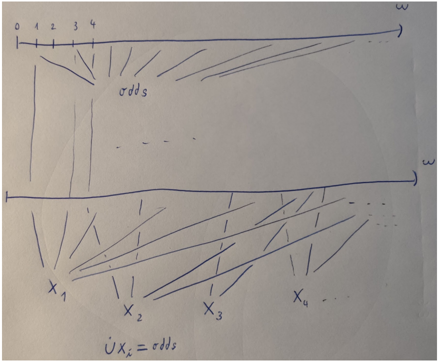
*Let $\overline{p}$ be the resulting play, then for each $i < \omega$, $\overline{p}$ is also a play of $G_N(P_i; X_i)$ winning for the $\exists$-player. Which means that each property $P_i$ holds.* □

# A picture fo the proof



# Formulas as properties

- Let $\phi$ be an $L(W)$-sentence. Then we say $\phi$ is $N$-enforceable iff the property $P :=$ "$A^+(\overline{p}) \models \phi$" is $N$-enforcable. Simmilarly $q \Vdash \phi$ iff $q \Vdash P$.

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- $\phi$ **does not** have to be a first-order sentence!
- If $\phi$ is an $L(W)_{\omega_1,\omega}$ sentence (Sentence in the language of infinitary logic with countable disjunctions and conjunctions but finitely many quantifiers.), then we can characterize those conditions which force $\phi$.

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# Forcing of sentences

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**Proof**

*The statements (1) and (2) follow trivially from the definitions. The statement (4) is just a special case of the conjugation lemma from earlier.*

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## Proof

*(5): Left to right is a special case of (2) and therefore is trivial.*
*Right to left: Let $\psi(\overline{c})$ be N-enforceable for every $\overline{c}$.*

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*By the conjugation lemma we have that $\forall \overline{x} : \psi(\overline{x})$ is N-enforacble.*

# Forcing of sentences cont.

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*Let $N$ be a notion of forcing and $q \in N$.*

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*"$\Rightarrow$": We will prove the converse. If the $\forall$-player chooses $r \supseteq p$ containing $\phi$ as $p_0 := r$ in the game $G_N(\neg\phi, odds)$, then he puts the $\exists$-player in the losing position.*

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*"⇐": Suppose that no condition $\supseteq p$ contains $\phi$. Then let the $\exists$-player play $G_N(\neg\phi; odds)$ so that $\bigcup \overline{p}$ is $=$-closed.*
*Then we have $A^+(\overline{p}) \models \phi$ iff $\phi \in \bigcup \overline{p}$. If the $\forall$-player began with $p_0 \supseteq$, then by ours assumption the $\exists$-player wins.* □

# Forcing of sentences cont.

## Lemma (*)

*If* $\phi$ *is an atomic sentence and* $p$ *is an N-condition, then* $p \Vdash \neg\phi$ *iff no condition* $\supseteq p$ *contains* $\phi$*.*

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## Proof (of (6))

***Claim:*** *Either some $p \supseteq q$ forces $\phi$ or some $p \supseteq q$ forces $\neg\phi$.*

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***Prf. of the claim:*** *By induction on the complexity of $\phi$.*

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*(i) If $\phi$ is atomic suppose no $p \supseteq q$ forces $\neg\phi$. Then by the $(*)$-lemma there is an $r \supseteq p$ which contains $\phi$. By (3) $p$ already force $\phi$.*

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*(i) If $\phi$ is atomic suppose no $p \supseteq q$ forces $\neg\phi$. Then by the $(*)$-lemma there is an $r \supseteq p$ which contains $\phi$. By (3) $p$ already force $\phi$.*
*(ii) If the claim holds for some sentence $\phi$, then it already holds for $\neg\phi$.*

# Forcing of sentences cont.

## Theorem

*Let $N$ be a notion of forcing and $q \in N$.*

3. *Let $\phi$ be an atomic $L(W)$-sentence. Then $q \Vdash \phi$ iff for every $N$-condition $p \supseteq q$, there is an condition $r \supseteq p$ with $\phi \in r$.*
5. *Let $\psi(x_1, \ldots, x_n)$ be a formula. Then $q \Vdash \forall \bar{x} : \psi(\bar{x})$ iff for every $n$-tuple $\bar{c}$ of witnesses $q \Vdash \psi(\bar{c})$.*
6. *Let $\phi$ be an $L(W)_{\omega_1,\omega}$-sentence. Then $q \Vdash \neg\phi$ iff there is no $N$-condition $p \supseteq q$ which forces $\phi$.*

***Lemma(*):*** *If $\phi$ is an atomic sentence and $p$ is an $N$-condition, then $p \Vdash \neg\phi$ iff no condition $\supseteq p$ contains $\phi$.*

## Proof (of (6))

***Claim:*** *Either some $p \supseteq q$ forces $\phi$ or some $p \supseteq q$ forces $\neg\phi$.*
***Prf. of the claim:*** *By induction on the complexity of $\phi$.*
*(i) If $\phi$ is atomic suppose no $p \supseteq q$ forces $\neg\phi$. Then by the $(*)$-lemma there is an $r \supseteq p$ which contains $\phi$. By (3) $p$ already force $\phi$.*
*(ii) If the claim holds for some sentence $\phi$, then it already holds for $\neg\phi$.*
*(iii) Let $\phi := \bigwedge_{i<\omega} \phi_i$, and suppose that no $p \supseteq q$ forces $\neg\phi$. This means that for every $i < \omega$ no $p \supseteq q$ forces $\neg\phi_i$. By induction hypothesis if $p \supseteq q$ then there is $r_i \supseteq p$ such that $r_i \Vdash \phi_i$. This means that $q$ forces all $\phi_i$-s and by (4) it forces $\phi$.*

# Forcing of sentences cont.

## Theorem

*Let $N$ be a notion of forcing and $q \in N$.*

3. *Let $\phi$ be an atomic $L(W)$-sentence. Then $q \Vdash \phi$ iff for every $N$-condition $p \supseteq q$, there is an condition $r \supseteq p$ with $\phi \in r$.*
5. *Let $\psi(x_1, \ldots, x_n)$ be a formula. Then $q \Vdash \forall \overline{x} : \psi(\overline{x})$ iff for every $n$-tuple $\overline{c}$ of witnesses $q \Vdash \psi(\overline{c})$.*
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*(iv) If $\phi := \forall \overline{x} : \psi$ then then $(5)$ reduces this to $(iii)$.* $\square$

# Forcing of sentences cont.

**Theorem**

*Let $N$ be a notion of forcing and $q \in N$.*

# Forcing of sentences cont.

**Theorem**

*Let $N$ be a notion of forcing and $q \in N$.*

6. *Let $\phi$ be an $L(W)_{\omega_1,\omega}$-sentence. Then $q \Vdash \neg\phi$ iff there is no $N$-condition $p \supseteq q$ which forces $\phi$.*

# Forcing of sentences cont.

## Theorem

*Let $N$ be a notion of forcing and $q \in N$.*

6. *Let $\phi$ be an $L(W)_{\omega_1,\omega}$-sentence. Then $q \Vdash \neg\phi$ iff there is no $N$-condition $p \supseteq q$ which forces $\phi$.*

## Proof

***Claim:*** *Either some $p \supseteq q$ forces $\phi$ or some $p \supseteq q$ forces $\neg\phi$.* □

# Forcing of sentences cont.

## Theorem

*Let $N$ be a notion of forcing and $q \in N$.*

6. *Let $\phi$ be an $L(W)_{\omega_1,\omega}$-sentence. Then $q \Vdash \neg\phi$ iff there is no $N$-condition $p \supseteq q$ which forces $\phi$.*

## Proof

***Claim:*** *Either some $p \supseteq q$ forces $\phi$ or some $p \supseteq q$ forces $\neg\phi$.* □

# Forcing of sentences cont.

## Theorem

*Let $N$ be a notion of forcing and $q \in N$.*

6. *Let $\phi$ be an $L(W)_{\omega_1,\omega}$-sentence. Then $q \Vdash \neg\phi$ iff there is no $N$-condition $p \supseteq q$ which forces $\phi$.*

## Proof

***Claim:*** *Either some $p \supseteq q$ forces $\phi$ or some $p \supseteq q$ forces $\neg\phi$.* □

*(6)"$\Rightarrow$": If there is some $p \supseteq q$ such that $p \Vdash \phi$, then by picking $p_0 := p$ can the $\forall$-player get himself into winning position for $G_N(\neg\phi; odds)$, which means that $q$ does not force $\neg\phi$.*

# Forcing of sentences cont.

## Theorem

*Let $N$ be a notion of forcing and $q \in N$.*

6. *Let $\phi$ be an $L(W)_{\omega_1,\omega}$-sentence. Then $q \Vdash \neg\phi$ iff there is no $N$-condition $p \supseteq q$ which forces $\phi$.*

## Proof

***Claim:*** *Either some $p \supseteq q$ forces $\phi$ or some $p \supseteq q$ forces $\neg\phi$.* □
*(6)"$\Rightarrow$": If there is some $p \supseteq q$ such that $p \Vdash \phi$, then by picking $p_0 := p$ can the $\forall$-player get himself into winning position for $G_N(\neg\phi; odds)$, which means that $q$ does not force $\neg\phi$.*
*(6)"$\Leftarrow$": If no condition $p \supseteq q$ forces $\phi$, then no condition $r \supseteq p$ forces $\phi$, then by the **Claim** some $r \supseteq p$ forces $\neg\phi$. Therefore $q$ forces $\neg\psi$.* □