Building models by games

Ondřej Ježil

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A way of building models

The canonical model

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Let L contain at least one constant and let T be an =-closed L-theory which contains only atomic sentences, then there exists a structure $\mathcal{A}$ *such that:*

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- ***Existence***: *Choose* $\{t; t \text{ a closed } L\text{-term}\}/\sim$ *as the underlying set of* $\mathcal{A}$ *where* $t \sim s : \iff (s = t) \in L$*, define the interpretation of functional and relational symbols as:*

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- *This is a sound definition because of the $=$-closedness of T.*

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- ***Uniqueness***: *Notice, that for every $\mathcal{B} \models T$, there is a unique homomorphism $f : \mathcal{A} \to \mathcal{B}$ sending $t_\sim \mapsto t^{\mathcal{B}}$, this follows just from the two requirements we have one $\mathcal{A}$.*

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- *So if $\mathcal{A}'$ were satisfy the statement of the theorem we would have $g : \mathcal{A} \to \mathcal{A}'$ and $h : \mathcal{A}' \to \mathcal{A}$, so $h \circ g : \mathcal{A} \to \mathcal{A}$ but only such homomorphism is the identity.* □

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- We can do so without proof calculus using the following notion.
- You can think of it like this: What are the conditions for some theory such that the theory “could be true in some structure”?

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For all variables x and L-formulas $\phi(x)$:

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For all variables x *and L-formulas* $\phi(x)$*:*

9. $\forall x : \phi(x) \in p \Rightarrow p \cup \{\phi(t)\} \in N$ *for every closed L-term* t
10. $\neg\forall x : \phi(x) \in p \Rightarrow p \cup \{\neg\phi(t)\} \in N$ *for some closed L-term* t

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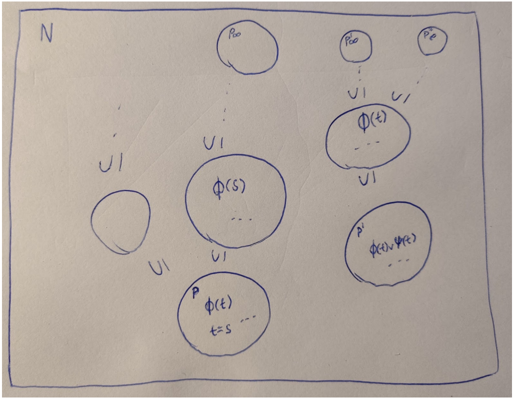
Finally:

13. *if $\alpha < \lambda$, $(p_i)_{i<\lambda}$ is an increasing chain in N, $|\bigcup_{i<\alpha} p_i \setminus p_0| < \lambda$, then $(\bigcup_{i<\alpha} p_i) \in N$.*

A picture!



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- Elements $p \in N$ are called **conditions**.

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Theorem

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How to construct the chain? We will use tasks that correspond to the instances of properties (1),(2),(4)-(12) of N. For example task corresponding to (7) and formulas ϕ, ψ is:

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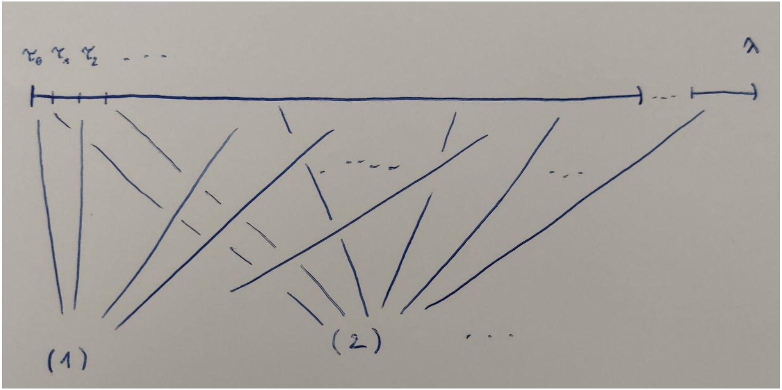
Organize the tasks into a sequence $(\tau_i)_{i<\lambda}$ such that for each property we have λ many instances and the task as a whole is completed by the time λ.

A picture for the task organization



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Since $p \subseteq \bigcup \overline{p}$, then $\mathcal{A} \models p$. □

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Let $W := \{c_i; i < \lambda\}$ be a set of new constants. And let N be a set of sets of $L(W)$-sentences such that

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- Every model theorist τ^* can think of himself as playing a game against all other model theorists: he wins the game is the task τ has been completed by the time the chain $(p_i)_{i<\lambda}$ is finished.
- **Then each model theorist has a winning strategy of his own game, provided that he can pick λ of the sets p_{i+1}.**

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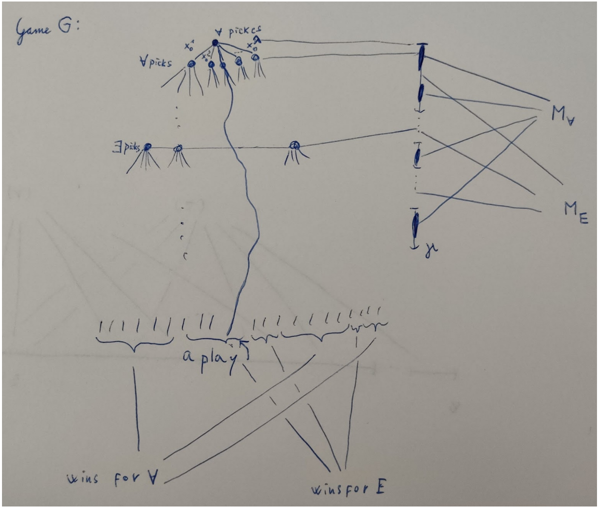

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- *If $\overline{y}$ is not winning for a player, then the other player can guarantee that the former player is still not in the winning position after the next move.*

This implies that games of finite length are determined!

Forcing with games (a teaser)

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- By noticing, that notions of forcing look like trees, we can play games on them.
- Studying these games, we will be able to understand properties of models which we will get from notions of forcing.

A picture of the situation.