Determinacy of games

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Ehrenfeucht Fraïssé games

Set-up: finite language L with no function symbols, $\mathbb{A}, \mathbb{B}$ - L-structures

Players: $\exists$-player Eloise (moves first), $\forall$-player Abelard (moves after Eloise)

Game:

- finite game $G_n(\mathbb{A}, \mathbb{B})$: finite number of moves $= n$. Each step $i \leq n$ starts with Eloise picking an element $a_i \in A$ or $b_i \in B$. Abelard then picks an element $b_i' \in B$ or $a_i' \in A$ from the opposite structure.

  After the n-th step players investigate whether the resultant set of pairs $(a_0, b_0), \dots, (a_n, b_n)$ is a partial isomorphism from $\mathbb{A}$ into $\mathbb{B}$.

  If so, Abelard wins. Otherwise Eloise wins

- infinite game $G_\omega(\mathbb{A}, \mathbb{B})$: same as above, but the number of moves is infinite (countable) and so in the end we are left with an infinite number of pairs $(a_0, b_0), \dots, (a_n, b_n), \dots$

Ehrenfeucht Fraïssé (cont.)

A strategy for Eloise is a function σ defined on sequences of the form $(a_0, b_0), \ldots, (a_{i-1}, b_{i-1})$. It tells Eloise what to do at the beginning of each step. Similarly for Abelard.

A strategy is winning iff for any moves of the opposite player the resultant sequence is winning for the one, who has used the given strategy.

Thm. for countable $\mathbb{A}$ and $\mathbb{B}$ it holds that Abelard has a winning strategy for $G_\omega(\mathbb{A}, \mathbb{B})$ iff $\mathbb{A} \cong \mathbb{B}$.

Thm. For countable $\mathbb{A}$ and $\mathbb{B}$ it holds that Abelard has a winning strategy for $G_k(\mathbb{A}, \mathbb{B})$ for all $k \geq 1$ iff $\mathbb{A} \equiv \mathbb{B}$.

Pr: see model theory course or D. Marker, Model Theory: An Introduction

2.

Ehrenfeucht
Fraïssé
(end.)

Does there always exist a winning strategy for at least one player?
Note that they cannot both have winning strategies for an obvious reason.

<u>Finite case</u>. 1-step game: Eloise has a winning strategy iff $\exists$ a move m so that $\forall_{\text{move}}$ n of Abelard it holds that moves m and n produce a winning pair of elements for Eloise.

Formally: $\exists m \forall n$ $\underbrace{\text{"pair produced by } m \text{ and } n \text{ is not a partial isomorphism"}}_{\text{can be written in a finite way, because } L \text{ is finite.}}$

Abelard has a winning strategy iff $\forall$ move m of Eloise $\exists$ a move n so that these moves produce a winning pair for Abelard

Formally: $\forall m \exists n$ " ———— // ———— "

Existence of a winning strategy (finite case)

Note: De-Morgan laws imply $\neg \exists m \forall n$ "pair produced by m and n is winning for Eloise"

$$\Updownarrow$$

$\forall m \exists n$ "pair produced by m and n is <u>not</u> winning for Eloise"

$$\Updownarrow$$

$\forall m \exists n$ "pair produced by m and n is winning for Abelard".

The same works for any finite-step game $G_k(\mathbb{A}, \mathbb{B})$.

This proves that there is a winning strategy for $G_k(\mathbb{A}, \mathbb{B})$ or that such games are <u>determined</u>.

4.

Existence of a winning strategy (infinite case)

What about $G_\omega(\mathbb{A}, \mathbb{B})$?

Eloise has a winning strategy iff : $\exists m_0 \forall n_0 \exists m_1 \forall n_1 \ldots\ldots$ "resultant sequence is winning for Eloise"

Abelard has a winning strategy iff: $\forall m_0 \exists n_0 \forall m_1 \exists n_1, \ldots$ "resultant sequence is winning for Abelard"

We can make sense of these two sentences using infinitary ($L_{\omega_1\omega_1}$)-logic and it would hold that: $\neg$"resultant sequence is winning for Eloise"

$\Updownarrow$

"resultant sequence is winning for Abelard".

However, <u>De-Morgan rules do not work here</u> :

$$\neg \exists m_0 \forall n_0 \exists m_1 \forall n_1 \ldots \sim \; \cancel{\Leftrightarrow} \; \forall m_0 \exists n_0 \forall m_1 \exists n_1 \ldots \neg \sim$$

Topological games

We will consider much simpler games.

$\mathbb{R} = \omega^\omega$ (sequences of natural numbers).

Every set $A \subset \mathbb{R}$ induces a <u>topological game</u> G_A:

Two players I and $\overline{\mathrm{II}}$ subsequently pick natural numbers a_i (resp. b_i).

In the end they arrive at a real number $(a_0, b_0, a_1, b_1, \ldots) =: r$.

I wins iff $r \in A$. Otherwise $\overline{\mathrm{II}}$ wins.

Strategy and winning strategy are defined completely analogous to Ehrenfeucht-Fraïssé games.

We say that a set $A \subset \mathbb{R}$ is <u>determined</u> iff G_A is determined (I or $\overline{\mathrm{II}}$ have a winning strategy).

<u>Def</u>: <u>Axiom of Determinacy</u> (AD) = every set $A \subset \mathbb{R}$ is determined.

Undetermined sets

<u>Thm</u>: AD + ZFC is inconsistent, i.e. assuming axiom of choice there is an undetermined set $U \subset \mathbb{R}$.

<u>Pr</u>: Let us introduce a little bit of notation. If $(a_0, b_0, a_1, b_1, \dots)$ is the resultant play, then we denote sequences $(a_0, a_1, \dots)$ and $(b_0, b_1, \dots)$ by a and b. Here, a corresponds to moves made by I and similarly b corresponds to moves made by II. If σ is the strategy of I, then obviously the resultant play r is fully dependent on σ and b. We denote this by $r = \sigma * b$. Similarly, if τ is the strategy of II, then we denote $r = a * \tau$ the resultant play given by this strategy and the sequence of moves a made by I.

Let us count how many strategies are there. Each such strategy is a function, whose domain is the set of all finite sequences of natural numbers of even length (for I) or of odd length (for II). The cardinality of such domain is $\aleph_0$. The range of a strategy is the set of natural numbers. So, there are $\aleph_0^{\aleph_0} = 2^{\aleph_0} (= |\mathbb{R}|)$ possible strategies for I and similarly for II.

Undetermined sets
(cont.)

Let us now enumerate the sets of all possible strategies of $\overline{I}$ and $\overline{II}$ as $\{\sigma_\alpha \mid \alpha < 2^{\aleph_0}\}$ and $\{\tau_\alpha \mid \alpha < 2^{\aleph_0}\}$. We are now going to create a set $X \subset \mathbb{R}$ so that for any strategy σ_α of $\overline{I}$ there will always be a sequence of moves b_α made by $\overline{II}$ so that $\sigma_\alpha * b_\alpha \bar{\in} X$, meaning $\overline{I}$ cannot win in such a game and similarly, for any strategy τ_α of $\overline{II}$ there will always be a sequence of moves a_α made by $\overline{I}$ so that $a_\alpha * \tau_\alpha \in X$, meaning $\overline{II}$ cannot win in such a game, too. This will conclude the proof. Such a set X is created inductively (more precisely, by transfinite recursion) using a similar diagonal argument. We will also create a set $Y \subset \mathbb{R}$ that will help us.

So assume we have $X = \{x_\xi \in \mathbb{R} \mid \xi < \alpha < 2^{\aleph_0}\}$ and $Y = \{y_\xi \in \mathbb{R} \mid \xi < \alpha < 2^{\aleph_0}\}$. Consider the strategy σ_α of $\overline{I}$. We can then find a sequence of moves of $\overline{II}$ denoted by b_α so that $\sigma_\alpha * b_\alpha \bar{\in} X$. We then expand Y by adding $\sigma_\alpha * b_\alpha$ to it. We can then similarly find a sequence of moves of $\overline{I}$ denoted by a_α, so that $a_\alpha * \tau_\alpha \bar{\in} Y$ (here we take an expanded Y) and then add $a_\alpha * \tau_\alpha$ to X.

In the end, the resultant set X is precisely what we want. ■

Determined sets

So there exist undetermined sets. But do determined sets exist in the first place?

__Observation__: Finite and co-finite sets are determined.

The class of determined sets is even broader. We need a little detour to topology.

__Def__: for a finite sequence of natural numbers $(a_0, \dots, a_n) =: a$ we define a __base set__ $B_a := \{ b \in \mathbb{R} (= \omega^\omega) \mid b \text{ starts with } a \}$.

__Observation__: base sets are determined.

Determined sets are obviously closed under finite unions, intersections and complements and so finite unions, intersections and complements of base sets are determined.

__Observation__: base sets can be extended to a topology.

Standard topology

<u>Def</u>: Topology on ω^ω for which base sets form a basis of open sets is called <u>standard topology</u>

<u>Thm</u>: open and closed sets from standard topology are determined. ⊛

<u>Thm</u> (Kechris 1995): determinacy of closed sets of A^ω for arbitrary A is equivalent to the axiom of choice over ZF.

We can go even further.

<u>Def</u>: The smallest class of sets closed under countable unions, intersections and complements and containing open and closed sets is called <u>the class of Borel sets</u>.

<u>Thm</u> (Martin 1975): Borel sets are determined.

<u>Thm</u> (Friedman 1971): There is a model of ZFC without the axiom schema of replacement where Borel determinacy fails.

Projective determinacy

Borel sets are very important in descriptive set theory. It holds that this class can be constructed using transfinite recursion up to ω_1 starting from open (or closed) set. It also holds that Borel sets are measurable by the Lebesgue measure.

But there is even broader class of sets that appear in descriptive set theory.

<u>Def</u>: A set is called <u>analytic</u> iff it is a continuous image of a Borel set.

Borel sets are obviously analytic but there are analytic sets which are not Borel. Analytic sets are not closed under complements. However, analytic sets are Lebesgue measurable. Measurable sets are closed under complements which shows that the class of measurable sets is broader than the class of analytic sets.

Projective determinacy (cont.)

A natural question is whether analytic sets are determined.

<u>Thm</u>: ZFC cannot prove nor disprove that analytic sets are determined.

A positive answer to the question above is called <u>the axiom of projective determinacy</u>.

Such axiom is closely related to certain large cardinal axioms

<u>Def</u>: A cardinal number κ is called <u>measurable</u>, iff there exists a κ-additive, non-trivial, 0-1-valued measure on the $P(\kappa)$.
Here κ-additive means that, for any sequence A_α of length $\lambda < \kappa$, A_α being pairwise disjoint sets of ordinals less than κ, the measure of the $\bigcup_\alpha A_\alpha$ equals the sum of the measures of the individual A_α.

Projective determinacy (end.)

Measurable cardinals are <u>strongly inaccessible</u> ($\kappa > 2^\lambda$ for any $\lambda < \kappa$) and so their existence cannot be proved in ZFC.

<u>Thm</u>: Existence of a measurable cardinal implies axiom of projective determinacy.

<u>Proof of (*) (if we have enough time)</u>: Let us show the proof for an open $A \subset \omega^\omega$. Assume I does not have a winning strategy. Let us show that there is a winning strategy for II. The strategy is as follows: if I plays a_0, then there is some b_0 so that position (a_0, b_0) is not yet lost for II. Similar argument holds for any finite move and so we arrive at a play $r = (a_0, b_0, a_1, b_1, \ldots)$. We now want to show that $r \in \bar{A}$ (that II wins). Assume this fails and so $r \in A$. Since A is open there is a base set $B_a \subset A$, where $a = (a_0, b_0, \ldots, a_n, b_n)$ which contradicts the fact that I does not have a winning strategy. ■

Axiom of Determinacy

So we know that ZFC + AD is inconsistent. It turns out that ZF + AD is consistent (assuming ZF is consistent) and so we can ask how ZF + AD is different from ZFC.

Thm: AD implies a weak form of AC, i.e. every countable family of nonempty sets of reals has a choice function.

Pr: Assume $X = \{X_n \mid n < \omega\}$ is a family as above. We want to construct $f : X \to \mathbb{R}$ so that $f(X_n) \in X_n$ for every n. Consider the game so that a resultant play $(a_0, b_0, a_1, b_1, \ldots)$ is winning for II iff $(b_0, b_1, \ldots) \in X_{a_0}$. This is a well-defined topological game. Note that I cannot win as for any a_0 he picks, the II can find a element $b \in X_{a_0}$ (X_{a_0} is not empty) and stick to it. AD then implies that there is a winning strategy τ for II. We can then define a choice function $f(X_n) := (n, 0, 0, \ldots) * \tau$ composed with the projection onto odd coordinates.

Measure

What about other consequences? One of the most important and interesting is that AD implies all sets are Lebesgue measurable. We will prove this and also construct a non-measurable set in ZFC. That will also provide yet another (indirect) proof that ZFC + AD is inconsistent.

But first we need a quick review of measure theory.

<u>Def</u>: <u>A (real-valued) measure</u> is a function λ defined on <u>certain</u> subsets of $\mathbb{R}$ and satisfying the following properties:

- $\lambda(E) \geq 0$
- $\lambda(\emptyset) = 0$
- $\lambda\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \lambda(E_k)$, assuming λ is defined on $E, \emptyset, \bigcup_{k=1}^{\infty} E_k, E_k$ and E_k are pair-wise disjoint.

A family of subsets of $\mathbb{R}$ for which λ is defined is called <u>λ-measurable family</u> (or just measurable). We always assume that $\emptyset, \mathbb{R}$ are measurable and that measurable sets are closed under countable unions, intersections and complements.

Lebesgue measure

The most important measure for us is <u>the Lebesgue measure</u>. The class of measurable sets is then very large. It contains all Borel sets, analytic sets and much more. Lebesgue measure has many nice properties, i.e. it measures intervals as you would expect: $\lambda([a,b]) = \lambda((a,b]) = \lambda([a,b)) = \lambda((a,b)) = b-a$, even for $a,b = \pm\infty$.

All countable sets have measure 0 and λ is <u>monotonne</u>, i.e. if $A \subseteq B$ and both are measurable, then $\lambda(A) \leq \lambda(B)$. It is also <u>translation invariant</u>, i.e. if A is measurable then the set $A + a := \{r+a \mid r \in A\}$ is also measurable and it holds that $\lambda(A+a) = \lambda(A)$.

Lebesgue measure is also <u>complete</u>, i.e. if A is a <u>null-set</u> (is measurable and has measure 0), then every $B \subseteq A$ is also measurable (and has measure 0 by the monotonicity of λ).

There is also a concept of <u>Lebesgue outer measure</u> λ^* which is defined on <u>all</u> subsets of $\mathbb{R}$ and coincides with λ on measurable sets.

However, λ^* is not additive in general, i.e. there exist disjoint $A, B \subseteq \mathbb{R}$, so that $\lambda^*(A \cup B) \neq \lambda^*(A) + \lambda^*(B)$. But it holds that $\lambda^*(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1} \lambda^*(A_i)$

Non-measurable set

Even though λ is very powerful, it cannot measure everything.

<u>Thm</u> (Vitali 1905): There exist a non-measurable set V.

<u>Pr</u>: Consider an interval $[0,1]$. We introduce an equivalence $\sim$ s.t. $a \sim b \Leftrightarrow |a-b|$ is rational. Such relation is obviously reflexive, symmetric and transitive. $[0,1]$ is then partitioned into the equivalence classes. Note that each class is countable and so there are $2^{\aleph_0}$ classes. We can now form a set V by picking an element from each class. This is obviously possible with the help of AC. We claim that V is not measurable. So assume it is and note that then $0 \le \lambda(V) \le 1$.

For every $q \in \mathbb{Q} \cap [0,1]$ we can define $V_{+q} := V+q$ and $V_{-q} := V-q$.

Translation invariance of λ implies all V_{+q}, V_{-q} are measurable and $\lambda(V) = \lambda(V_{+q}) = \lambda(V_{-q})$. Note that all these sets are pair-wise disjoint and so

$$\lambda\Big(\underbrace{\dot{\bigcup_{q \in \mathbb{Q}\cap[0,1]}} V_{+q} \;\dot{\cup}\; \dot{\bigcup_{q \in \mathbb{Q}\cap[0,1]}} V_{-q}}_{=: W}\Big) = \sum_{q \in \mathbb{Q}\cap[0,1]} \lambda(V_{+q}) + \sum_{q \in \mathbb{Q}\cap[0,1]} \lambda(V_{-q}) = 2 \sum_{q \in \mathbb{Q}\cap[0,1]} \lambda(V).$$

Note that $[0,1] \subseteq W \subseteq [-1,2]$. But then $\lambda(V) = 0$ implies $\lambda(W) = 0$ and $\lambda(V) > 0$ implies $\lambda(W) = +\infty$ which is a contradiction. $\blacksquare$

Measurability

We are now going to prove the main theorem.

<u>Thm</u>: $ZF + AD$ implies all sets are Lebesgue measurable.

<u>Pr</u>(sketch):

The proof uses the following lemma:

<u>Lemma</u>: $ZF + AD$ implies that if S is set such that every measurable subset $Z \subseteq S$ is null, the S is also null.

Let us first show how this lemma implies the theorem.
Take any $X \subseteq \mathbb{R}$. It can be shown in ZF that there is a measurable $A \supseteq X$ so that every measurable $Z \subseteq A \setminus X$ is null.
But then $A \setminus X$ is null and so $X = A \setminus (A \setminus X)$ is measurable $\blacksquare$

<u>Pr (of lemma)</u>: Assume S has the property as above. We can further assume that $S \subseteq [0,1]$. To show S is null it suffices to show that $\lambda^*(S) = 0$, because ZF proves that if set has an outer measure 0, then it is Lebesgue measurable.

We will show that $\forall \varepsilon > 0$ it holds that $\lambda^*(S) \leq \varepsilon$.

Measurability
(cont.)

Fix a positive real ε. We now introduce <u>the covering game</u>. If $(a_0, a_1, \dots)$ is a sequence of 0's and 1's, then we define $a := \sum_{n=0}^{\infty} \frac{a_n}{2^{n+1}}$. Note that $0 \leq a \leq 1$. For $n \in \mathbb{N}$ we define K_n as a set of sets G so that G is a union of finitely many intervals with rational endpoints and $\lambda(G) \leq \frac{\varepsilon}{2^{2(n+1)}}$.

Note that each K_n is countable and so we can enumerate them. Denote G_k^n as the k-th set from K_n.

The game is that I tries to play a number $a \in S$ and II tries to cover this a by the union $\bigcup_{n=0}^{\infty} H_n$ so that $H_n \in K_n$ for any n. More precisely, a play $(a_0, b_0, a_1, b_1, \dots)$ is won by the player I iff:

(i) $a_n = 0$ or 1

(ii) $a \in S$

(iii) $a \overline{\in} \bigcup_{n=0}^{\infty} G_{b_n}^n$.

Measurability (cont.)

We claim that I cannot win. To show this assume σ is a winning strategy of I. We can then define a function F which to each $b:=(b_0, b_1, \ldots) \in \mathbb{R}$ assigns $a:=(a_0, a_1, \ldots) \in \mathbb{R}$, so that $(a_0, b_0, a_1, b_1, \ldots) = \sigma * b$. Note that this function is continuous. This shows that the set $Z := F(\mathbb{R})$ is analytic and hence measurable. Moreover $Z \subseteq S$ and hence is null. Going to back to the definition of K_n's we see that a null set Z can be covered by $\bigcup_{n=1}^{\infty} H_n$ so that $H_n \in K_n$. So if II plays $(b_0, b_1, \ldots)$ so that $G^n_{b_n} = H_n$ and I plays by σ, then II wins. This shows that I cannot have a winning strategy.

AD then implies that II has a winning strategy τ. For each finite sequence $s = (a_0, \ldots, a_n)$ of 0's and 1's let $G_s \in K_n$ be the set $G^n_{b_n}$, where $(b_0, \ldots, b_n)$ are the moves of II that are played by τ in response to $a_0, \ldots, a_n$.

Measurability (end.)

Since τ is a winning strategy, every $a \in S$ is in the set $\bigcup \{ G_s \mid s \text{ finite prefix of } a \}$

and hence $S \subseteq \bigcup \{ G_s \mid s \text{ finite sequence of 0's and 1'} \} = \bigcup_{n=1}^{\infty} \bigcup_{s \in \{0,1\}^n} G_s$.

For every $n \geq 1$, if $s \in \{0,1\}^n$, then $\lambda(G_s) \leq \frac{\varepsilon}{2^{2n}}$.

This implies $\lambda\left(\bigcup_{s \in \{0,1\}^n}\right) \leq \frac{\varepsilon}{2^{2n}} \cdot 2^n = \frac{\varepsilon}{2^n}$.

Finally, $\lambda\left(\bigcup_{n=1}^{\infty} \bigcup_{s \in \{0,1\}^n} G_s\right) \leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$ and thus $\lambda^*(S) \leq \varepsilon$.

This completes the proof. $\blacksquare$

Other corollaries

There are other interesting topological properties of $\mathbb{R}$ which can be proved by AD but fail assuming AC.

Thm: ZF+AD implies that every set of reals has the property of Baire.

It means that for any set A there is an open set U so that $A \triangle U$ is meager (= countable union of nowhere dense sets (= closure has empty interior)).

It essentially means that all sets are almost open.

Thm: ZF+AD implies that every uncountable set of reals contains a perfect subset (= closed and has no isolated points)

But are there any disasters with AD?

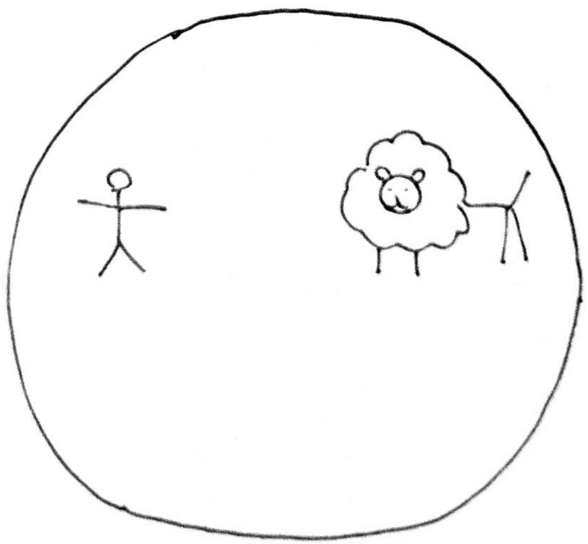
Thm (Sierpinski): ZF+AD proves that there is no linear order on $\mathbb{R}/\mathbb{Q}$. Furthermore, it is consistent with ZF+AD that $\mathbb{R}/\mathbb{Q}$ has strictly larger cardinality than $\mathbb{R}$.

Lion and Man

We are now going to consider a differend kind of games, namely <u>continuous games</u>. They posses a radher curious property: there are games in which both player have winning stradegies even though there is no draw!

To define continuous games led us consider a famous example given by Richard Rado. This is known as "Lion and Man" problem.

A lion and a man in a closed disc have equal maximum speeds; can the lion catch the man?



Lion and Man
(cont.)

We can generalize this problem to an arbitrary metric space (X, d). A run of the game takes place in time $[0,1]$.

Moves are just points in X, so the final object a player produces is a function $f: [0,1] \to X$ (obviously, f has to be continuous).

We further specify initial points $x_e \neq x_m$ and $t = 0$.

The fact that they both have equal maximum speed is imposed by stating that final f is <u>Lipschitz with constant 1</u>, i.e.

$d(f(a), f(b)) \leq |a-b|$ for any $a, b \in [0,1]$ (this, by the way, implies continuity of f).

Denote lion's play by f_e and man's as f_m. For any $t \in [0,1]$, to choose $f_e(t)$ lion has access to $f_m \restriction [0,t)$ and similarly for the man.

The lion wins if it catches the man, i.e. there $\exists\, t \in [0,1]$ so that $f_e(t) = f_m(t)$

Lion and Man
(end.)

A strategy for the lion is a function Φ s.t. to each path m of the man it assigns a path $\Phi(m)$ of the lion s.t. if m_1 and m_2 coincide on $[0,t)$, then $\Phi(m_1)$ and $\Phi(m_2)$ also agree on $[0,t)$.
Strategy Φ is winning, if for any path m there is $t \in [0,1]$ s.t.

$\Phi(m)(t) = m(t)$.

The same can be defined similarly for the man.

Let us now go back to the original problem (so (X,d) is just a unit disc with the Euclidean metric).

Thm: running directly towards the man is not a winning strategy for the lion.

Pr: google circular escape curve. The question is said to be solved only in 1921 and it turns out that lion can get arbitrarily close to the man. ■

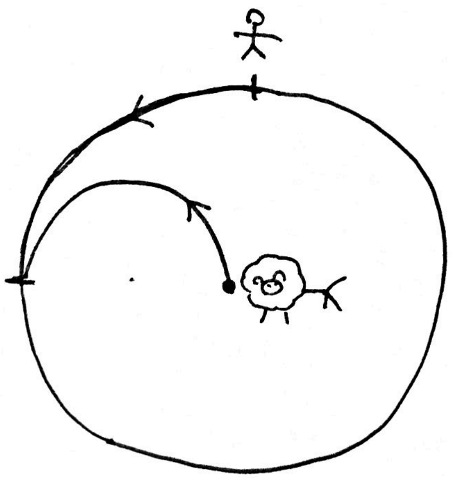
Besicovitch strategy

Turns out, under certain assumptions lion can win.

<u>Thm</u>: Assuming man starts and stays on the border, lion ~~have~~ has a winning strategy.

<u>Pr</u>: The idea is to always stay on the same radius as the man and move at a full speed towards him.

Indeed, in the time it takes the man to run, say, a quarter-circle at full speed the lion (say starting at the center) performs a semicircle of half the radius of the disc, thus catching the man.



Besicovitch strategy (cont.)

It was presumed then that there is no difference for man staying on the border or moving away and so the lion can always win. However, in 1952 Besicovitch proved this wrong & actually showing that man can escape the lion forever.

<u>Thm</u> (Besicovitch 1952): The man has a winning strategy.

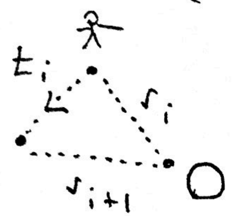
<u>Pr</u>: We assume the man stands not on the border, but somewhere inside a disc (if not, he can spend a small amount of time moving inside).

We then split time into a sequence of intervals $t_1, t_2, t_3, \dots$.

At the i-th step the man runs for time t_i in a straight line that is perpendicular to his radius vector at the start of the step. He chooses to run into the half plane that does not contain the lion (if the lion is on the radius vector, choose half plane arbitrarily). So if the man has the room to move, he is not caught by the end of the i-th step.

Besicovitch strategy
(end.)

Now, note that if r_i is the distance between the man and the center of the disc at the beginning of the i-th step, then $r_{i+1}^2 = r_i^2 + t_i^2$.



Hence as long as $\sum_i t_i$ is infinite, the man is never caught.

But if $\sum_i t_i^2$ is finite, then r_i are bounded and so we can always remain inside the arena. So taking, for example, $t_i = \frac{1}{i}$ we have a winning strategy for the man. $\blacksquare$

So the ~~lion~~ man can always win. But what about the ~~man~~ lion, does he have a winning strategy?

Such question is "obviously nonsense". Assume both the man and the lion have winning strategies M and L, respectively. Now let them both playing by their strategies, who then wins? Such "proof" is, however, wrong!

Winning strategies

What does it mean that they both followed their strategies? If the lion uses strategy L and the man M, then we would need paths l for lion and m for man such that $l = L(m)$ and $m = M(l)$, but why do such common fixed points should even exist?

It turns out, that in the case of unit disc the lion does not have a winning strategy. But there is a metric space where they both can win.

__Def__: let X and Y be metric spaced. The the __ℓ_∞-sum of X and Y__ is defined as a metric space on $X \times Y$, where $d((x,y),(x',y')) = \max(d_X(x,x'), d_Y(y,y'))$.

__Thm__ (Bollobás-Leader-Walters, 2009)

Consider X to be the ℓ_∞ sum of the closed unit disc D and $[0,1]$. Then, if the lion stards at $(\overline{0},1)$ and the man stards at $(\overline{0},0)$, then they both have winning strategies.

Winning strategies (cont.)

<u>Pr</u>: the lion has an obvious winning strategy: Keep the disc coordinates the same as the man and run towards him in the interval coordinate. The nature ℓ_∞ shows that the lion catches the man in time ≤ 1.

For a winning strategy for the main we first aim to change our disc coordinates so that they will differ from the lion's.

For the time say $\frac{1}{2}$ the man acts as follows: we ask if there exists a positive time t such that the lion's disc coordinates were exactly $(s,0)$ for all $0 \leq s \leq t$. If so, the man runs straight to the point $((-\frac{1}{2},0),0)$ and if not, he runs straight to $((\frac{1}{2},0),0)$. Note that this satisfies the "no-lookahead rule", i.e. for any given time $t \leq \frac{1}{2}$ the man's position at t depends only on the lion's position at arbitrarily early time. So at time $\frac{1}{2}$ the man has different disc coordinate to the lion. So he can then play the Besicovitch strategy inside the disc, while not changing interval coordinate $\blacksquare$

Winning strategies (cnd.)

For the previous result it was important that the man and the lion had the same disc coordinates at the beginning. Otherwise, lion cannot win.

Turns out there are space for which the above assumption is not necessary.

<u>Thm</u>: if X is the closed unit ball in ℓ_∞^2 (the ℓ_∞ sum of $[-1,1]$ with itself), then both players have winning strategies for any distinct starting positions.

There are other interesting properties of continuous games, e.g. for an n-dimensional unit disc the man can simultaneously escape n-1-lions, though n lions can catch him.

See "Lion and Man- Can Both Win" by B. Bollobás, I. Leader and M. Walters