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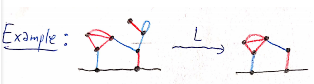

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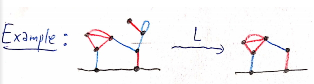

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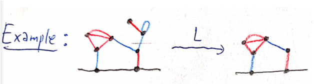

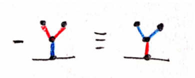

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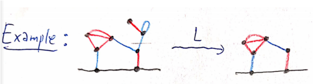
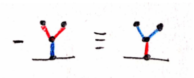
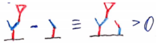


- Cutting blue edge always decreases the game (e.g. , because ), similarly cutting red edge increases the game.

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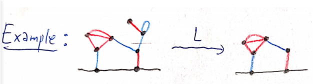
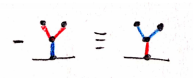
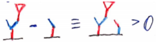


- Cutting blue edge always decreases the game (e.g. , because ), similarly cutting red edge increases the game.
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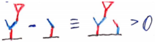
- Cutting blue edge always decreases the game (e.g. , because ), similarly cutting red edge increases the game.
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- These games have finite birthday, so they are actually dyadic fractions.

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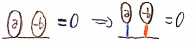
- If $a = b$, then $1{:}a = 1{:}b$ ()

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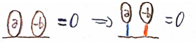
- If $a = b$, then $1{:}a = 1{:}b$ ()
- We just need to be able to compute $1{:}x$ for given dyadic fraction x.

Hackenbush

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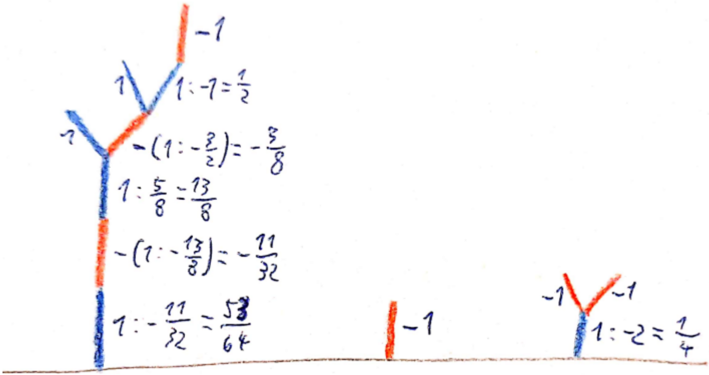

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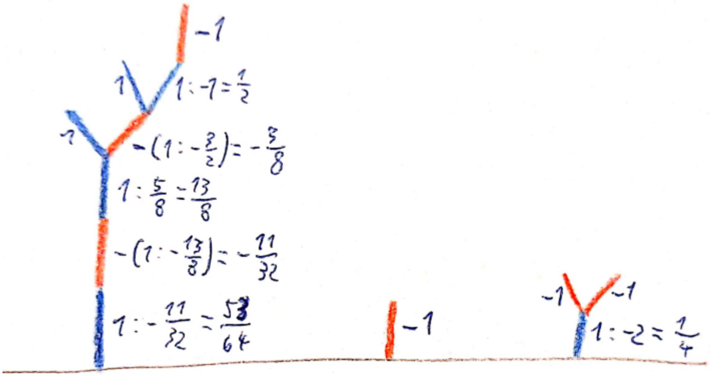

$\frac{53}{64} + (-1) + \frac{1}{4} = \frac{5}{64} > 0$, so this is a winning position for Left.

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We could define multiplication in the same way for general games, but here it turns out to be not nice.

Remarkable subfields

Definition

Surreal number x is a real number, if $-n < x < n$ for some natural number n and $x = \left\{x - 1, x - \frac{1}{2}, x - \frac{1}{3}, \ldots \mid \ldots, x + \frac{1}{3}, x + \frac{1}{2}, x + 1\right\}$.

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Surreal number α is an ordinal number, if α can be expressed as $\{L \mid \}$, where L is a set of numbers.

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Ordinals are closed under $+$ and $\times$. However, these operations do not correspond to standard ordinal addition and multiplication. This is easy to see, because in surreal numbers $1 + \omega = \omega + 1$.

Numbers and general games

Numbers and general games

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Let g be a game and α be its birthday. Then $-\alpha \leq g \leq \alpha$.

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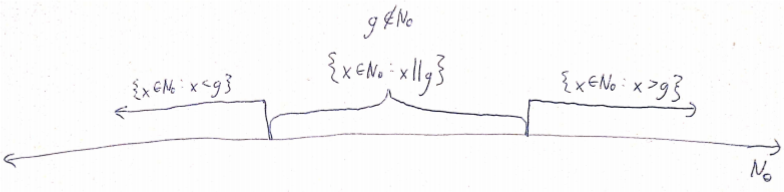
Let g be a game, which is not a number. Then for any number x either $x < g$, $x \parallel g$, or $x > g$. In this way, g divides **No** into 3 disjoint convex sections. Since $-\alpha \leq g \leq \alpha$, the middle section is bounded.



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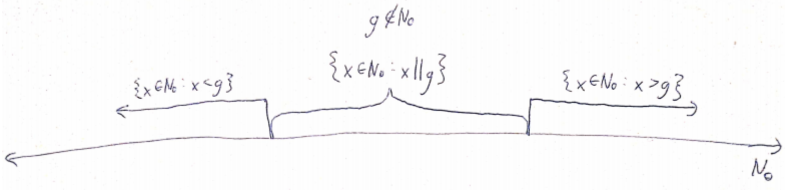

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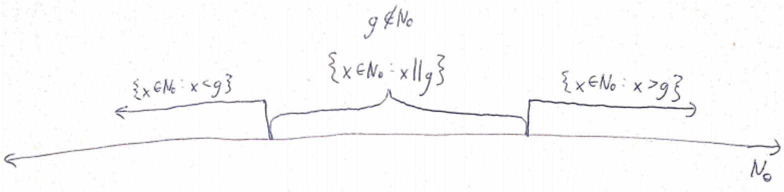

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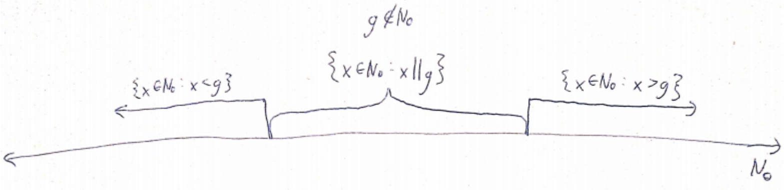

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Example - Schrinking rectangles

We have a number of rectangles of integer sides. Left can decrease the breadth of any rectangle, Right the height. A rectangle whose breadth or height is decreased to zero disappears. Who can win?

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- From previous point it follows, that if $a, b > 0$, then the value of (a, b) depends only on $a - b$. Let $(a, b) = g(a - b)$.
- For $n \geq 0$ we have $g(n) = (n + 1, 1)$ and $g(-n) = (1, n + 1) = -g(n)$.

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It turns out $\uparrow^n$ are quite easy to compute with. It can be shown that:

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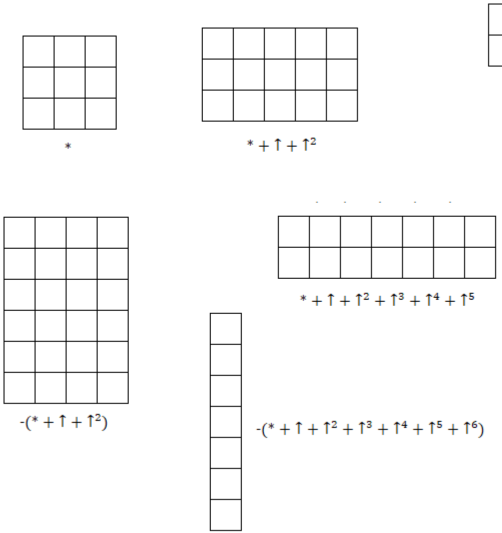
So $\uparrow, \uparrow^2, \uparrow^3, \cdots$ is a sequence of positive games, in which every game is infinitely smaller then the previous one.

Schrinking rectangles - even case



whole game

$= *$

$+ (* + \uparrow + \uparrow^2)$

$+ (* + \uparrow + \uparrow^2)$

$+ (* + \uparrow + \uparrow^2 + \uparrow^3 + \uparrow^4 + \uparrow^5)$

$- (* + \uparrow + \uparrow^2)$

$- (* + \uparrow + \uparrow^2 + \uparrow^3 + \uparrow^4 + \uparrow^5 + \uparrow^6)$

$= (* + * + * + * + * + *)$

$+ (\uparrow + \uparrow + \uparrow - \uparrow - \uparrow)$

$+ (\uparrow^2 + \uparrow^2 + \uparrow^2 - \uparrow^2 - \uparrow^2)$

$+ (\uparrow^3 - \uparrow^3)$

$+ (\uparrow^4 - \uparrow^4)$

$+ (\uparrow^5 - \uparrow^5)$

$- \uparrow^6$

$= 0 + \uparrow + \uparrow^2 - \uparrow^6 > \uparrow > 0$, so Left wins

Schrinking rectangles - odd case

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- $* \,||\, \uparrow + \uparrow^2 + \cdots + \uparrow^n$ (because $g(n) = (n+1, 1) \,||\, 0$)

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- $* < \uparrow + \uparrow^2 + \cdots + 2\uparrow^n$ (because $2(n+1, 1) + (1, n) > 0$)

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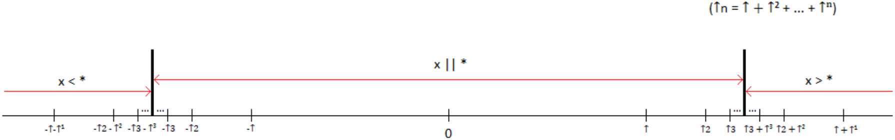

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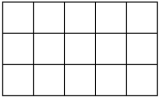
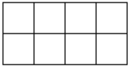


$* + \uparrow + \uparrow^2$



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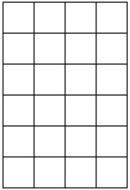
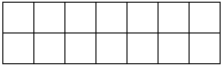


$-(* + \uparrow + \uparrow^2)$



$* + \uparrow + \uparrow^2 + \uparrow^3 + \uparrow^4 + \uparrow^5$

$-(* + \uparrow + \uparrow^2 + \uparrow^3 + \uparrow^4 + \uparrow^5 + \uparrow^6)$

whole game
$= (* + \uparrow + \uparrow^2)$
$+ (* + \uparrow + \uparrow^2)$
$+ (* + \uparrow + \uparrow^2 + \uparrow^3 + \uparrow^4 + \uparrow^5)$
$- (* + \uparrow + \uparrow^2)$
$- (* + \uparrow + \uparrow^2 + \uparrow^3 + \uparrow^4 + \uparrow^5 + \uparrow^6)$
$= * + \uparrow + \uparrow^2 - \uparrow^6$

We have $0 < \uparrow + \uparrow^2 - \uparrow^6 < \uparrow + \uparrow^2$, so $\uparrow + \uparrow^2 - \uparrow^6 \,||\, *$. By adding * on both sides we get $* + \uparrow + \uparrow^2 - \uparrow^6 \,||\, 0$, so the first player can win.

Thank you for your attention