Surreal numbers

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- Our order will be different from the order in the book
- We start by defining *games* and learn to make arithmetics on them
- Then we define *surreal numbers* as a special kind of games
- Field of surreal numbers turns out to be similar in structure to reals, but much richer

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- If $a = \{X \mid Y\}$ is a game, we write a_L for X and a_R for Y

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- $|M_2| = 256$
- The class of all games is a proper class.
- If x is a game and $y \in x_L \cup x_R$, then birthday$(y) <$ birthday(x).

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There is no infinite sequence of games $\{g_i\}$ such that $g_{i+1} \in (g_i)_L \cup (g_i)_R$ for each $i \in \mathbb{N}$. If there was, their birthdays would form an infinite decreasing sequence of ordinals.

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Lemma (Induction 2)

Let ϕ be a property such that for each x, y:

- $(\forall x' \in x_L \cup x_R : \phi(x', y)) \wedge (\forall y' \in y_L \cup y_R : \phi(x, y')) \to \phi(x, y)$

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Similar lemma holds for $\phi(x, y, z)$, etc.

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Consequences:

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- For given game and given starting player, exactly one player has a winning strategy.

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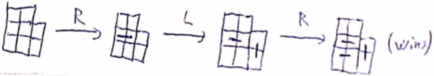
DOMINOES

We have a board consisting of squares (shape can be arbitrary, even disconnected). Players are alternately placing dominoes. Each domino covers two adjacent squares, no two dominoes can overlap. Left must place his dominoes vertically, Right horizontally. A player with no valid move loses.

Example:  We see that  $\neq 0$

 $\equiv \{ \mid \} \equiv 0$

 $\equiv \{0 \mid \} \equiv 1$

 $\equiv \{ \mid 0\} \equiv -1$

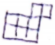
 $\equiv \{0 \mid 0\} \equiv *$

 $\equiv \{1 \mid -1\}$

 $\equiv \{-1, 0 \mid 1\} \equiv \frac{1}{2}$

$-x \equiv$ (x rotated by 90° degrees)

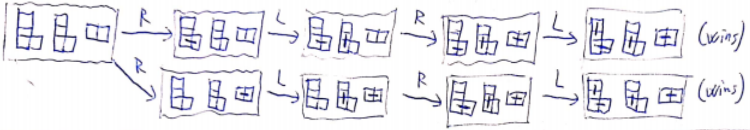
$a + b \equiv$ (a composed with b)

$-$  $\equiv$ 

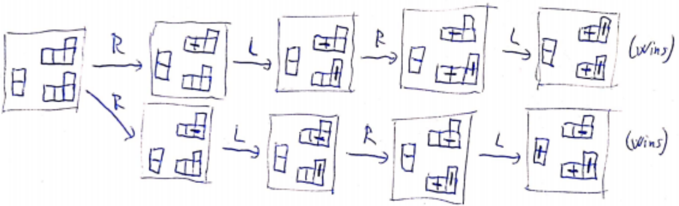
 $+$  $\equiv$ 

Let's prove, that $\frac{1}{2} + \frac{1}{2} = 1$. That is, … = …. We need to check two inequalities:

(1.) … ≥ … : We need … ≥ 0, so Right starts, can Left win?



(2.) … ≥ … : We need … ≥ 0



Arithmetics on games

Lemma

Let a, b, c be games. Then:

1. $a + 0 \equiv a$

Arithmetics on games

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Let a, b, c be games. Then:

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Lemma

Let a, b, c be games. Then:

1. $a + 0 \equiv a$
2. $a + b \equiv b + a$
3. $(a + b) + c \equiv a + (b + c)$

Arithmetics on games

Lemma

Let a, b, c be games. Then:

1. $a + 0 \equiv a$
2. $a + b \equiv b + a$
3. $(a + b) + c \equiv a + (b + c)$
4. $-(-a) \equiv a$

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Let a, b, c be games. Then:

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Arithmetics on games

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Let a, b, c be games. Then:

1. $a + 0 \equiv a$
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3. $(a + b) + c \equiv a + (b + c)$
4. $-(-a) \equiv a$
5. $-(a + b) \equiv -a - b$

Before, we defined $a \geq b$ as $a - b \geq 0$. Now we know, that $a - 0 \equiv a + 0 \equiv a$, so the definition of $\geq$ is consistent.

Arithmetics on games

Proof

We prove all of them by induction.

1. $a + 0$

Arithmetics on games

Proof

We prove all of them by induction.

1. $a + 0 \equiv \{a_L \mid a_R\} + \{ \mid \}$

Arithmetics on games

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We prove all of them by induction.

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Arithmetics on games

Proof

We prove all of them by induction.

1. $a + 0 \equiv \{a_L \mid a_R\} + \{\ \mid\ \} \equiv \{a_L + 0 \mid a_R + 0\} \equiv \{a_L \mid a_R\}$

Arithmetics on games

Proof

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Arithmetics on games

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We prove all of them by induction.

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2. $a + b$

Arithmetics on games

Proof

We prove all of them by induction.

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2. $a + b$
   $\equiv \{(a_L + b) \cup (a + b_L) \mid (a_R + b) \cup (a + b_R)\}$

Arithmetics on games

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We prove all of them by induction.

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$$\equiv \{(a_L + b) \cup (a + b_L) \mid (a_R + b) \cup (a + b_R)\}$$
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Proof

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   $\equiv \{(b + a_L) \cup (b_L + a) \mid (b + a_R) \cup (b_R + a)\} \equiv b + a.$
3. Both sides evaluate to $(a_L + b + c) \cup (a + b_L + c) \cup (a + b + c_L)$ on the left and analogously on the right.

Arithmetics on games

Proof

We prove all of them by induction.

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   $\equiv \{(b + a_L) \cup (b_L + a) \mid (b + a_R) \cup (b_R + a)\} \equiv b + a$.
3. Both sides evaluate to $(a_L + b + c) \cup (a + b_L + c) \cup (a + b + c_L)$ on the left and analogously on the right.
4. $-(-a)$

Proof

We prove all of them by induction.

1. $a + 0 \equiv \{a_L \mid a_R\} + \{\ \mid\ \} \equiv \{a_L + 0 \mid a_R + 0\} \equiv \{a_L \mid a_R\} \equiv a.$
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   $\equiv \{(a_L + b) \cup (a + b_L) \mid (a_R + b) \cup (a + b_R)\}$
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Arithmetics on games

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Arithmetics on games

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Arithmetics on games

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   $\equiv -a - b$

Arithmetics on games

Lemma

Let a, b be games. Then:

1. $a \leq b$ iff $-a \geq -b$.

Arithmetics on games

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Let a, b be games. Then:

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Proof

1. Follows from the definition and $b - a \equiv -a + b \equiv -a - (-b)$.

Arithmetics on games

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Let a, b be games. Then:

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1. Follows from the definition and $b - a \equiv -a + b \equiv -a - (-b)$.
2. $a = b \leftrightarrow (a \leq b \wedge b \leq a) \leftrightarrow (-a \geq -b \wedge -b \geq -a) \leftrightarrow -a = -b$
3. $a \leq 0$ is equivalent to $-a \geq 0$. Thus in $-a$, given Right starts, Left can win. But $-a$ is just a with roles switched.

Arithmetics on games

Consequence

Let a be a game. Then:

- $a > 0$ iff Left has a winning strategy.

Arithmetics on games

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Arithmetics on games

Consequence

Let a be a game. Then:

- $a > 0$ iff Left has a winning strategy.
- $a < 0$ iff Right has a winning strategy.
- $a = 0$ iff second player has a winning strategy.
- $a \mid\mid 0$ iff first player has a winning strategy.

Consequence

$-1 < 0 < 1$; $* \mid\mid 0$; $\{* \mid *\} = 0$

Arithmetics on games

Lemma

Let a, b be games. Then:

1. If $a \geq 0$ and $b \geq 0$, then $a + b \geq 0$.

Arithmetics on games

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1. We are playing $a + b$, Right starts. Left can just combine winning strategies for a and b, always playing in the same game as Right.

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Proof

1. We are playing $a + b$, Right starts. Left can just combine winning strategies for a and b, always playing in the same game as Right.
2. In b, second player can win. Whoever can win in a, can also win in $a + b$ - just use your winning strategy for a and respond to opponent's moves in b with second player's winning strategy.

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From point 3 it follows that $a = a$.

Arithmetics on games

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Proof

- $a \geq b$ is equivalent to $a - b \geq 0$.

Arithmetics on games

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Let a, b, c be games. Then:

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Proof

- $a \geq b$ is equivalent to $a - b \geq 0$. $a + c \geq b + c$ is equivalent to $(a + c) - (b + c) \geq 0$, thus $(a - b) + (c - c) \geq 0$.

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- $a \geq b$ is equivalent to $a - b \geq 0$. $a + c \geq b + c$ is equivalent to $(a + c) - (b + c) \geq 0$, thus $(a - b) + (c - c) \geq 0$. But $c - c = 0$, so these are equivalent.
- We know, that $a - b \geq 0$ and $b - c \geq 0$. Thus $(a - b) + (b - c) \equiv (a - c) + (b - b) \geq 0$, thus $a - c \geq 0$.

Thus $a = b \leftrightarrow a + c = b + c$ and $(a = b \wedge b = c) \rightarrow a = c$.

Arithmetics on games

Consequence

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$=$ is an equivalence relation. $+$, $-$ and $\leq$ are well-defined operations on equivalence classes defined by $=$. The class of these equivalence classes together with $+$, $-$, 0 and $\leq$ forms a partially ordered abelian Group.

Surreal numbers

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Definition

Game a is a surreal number, if a_L and a_R are sets of surreal numbers and for each $x \in a_L$, $y \in a_R$ we have $x < y$.

Examples:

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Class **No** of all number is a proper class.

Surreal numbers

Theorem

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Let a, b be numbers. Then $a \geq b$ or $b \geq a$.

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Theorem

Let a, b be numbers. Then $a \geq b$ or $b \geq a$.

Theorem

If a, b are numbers, then so are $a + b$ and $-a$.

Surreal numbers - finite birthday

- In step 0 we have the only number $\{\ |\ \} = 0$.

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- In step 3 we get -3, $-\frac{3}{2}$, $-\frac{3}{4}$, $-\frac{1}{4}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{3}{2}$, 3.

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- ...
- In first ω step we get all fractions of the form $\frac{a}{2^n}$ (dyadic fractions)

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- In step 3 we get -3, $-\frac{3}{2}$, $-\frac{3}{4}$, $-\frac{1}{4}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{3}{2}$, 3.
- ...
- In first ω step we get all fractions of the form $\frac{a}{2^n}$ (dyadic fractions)
- We define $n+1$ as $\{n \mid \}$, $-n-1$ as $\{ \mid -n\}$ and $\frac{2a+1}{2^{n+1}}$ as $\left\{\frac{a}{2^n} \mid \frac{a+1}{2^n}\right\}$

Surreal numbers - finite birthday

- In step 0 we have the only number $\{\ |\ \} = 0$.
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Theorem

These are all the numbers with finite birthday. $+$, $-$ and $\leq$ work as you would expect.

Surreal numbers - birthday ω

- This is where we start getting infinite sets.

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- $\{0, 1, 2, 3, \ldots \mid \} = \omega$

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Surreal numbers - birthday $\omega + 1$

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- Analogously $-\omega - 1$ and $-\omega + 1$

Surreal numbers - higher birthdays

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- $\sqrt[3]{\omega + 1} - \frac{\pi}{\omega}$, $\omega^{\frac{1}{\omega}}$, ...

Surreal numbers

Simplicity theorem

Let x be a game and z be a number such that all of $x_L \ngeq z \ngeq x_R$ hold and no element of $z_L \cup z_R$ satisfies the same condition. Then $x = z$.

Examples:

Surreal numbers

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Let $\{L \mid R\}$ be a number of unknown value. Then this number is equal to simplest such x, that all of $L < x < R$ hold. Here 'simplest' means 'with the smallest birthday'.

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