Σ_1-definability of RE sets

Let $\mathcal{M}$ be any T. machine (Lect. 7). We shall identify:

(i) its set of states Q with $\{0, \ldots, |Q|-1\}$ and the initial state q_0 with 0,

(ii) its working alphabet Σ with $\{0, \ldots, |\Sigma|-1\}$, the blank symbol with 0, and the input alphabet $\Gamma \subseteq \Sigma \setminus \{\text{blank}\}$ with $\{1, \ldots, |\Gamma|\}$.

(iii) Let $\delta : \subseteq Q \times \Sigma \to Q \times \Sigma \times \{L, R\}$ be its transition (partial) function.

An input ~~...~~ $u_0 \cdots u_{n-1} \in \Gamma^n$ we'll be coded by $u \in \mathbb{N}$ s.t. $\forall i < n : (u)_i = u_i$.

We want to express – by a Σ_1-sentence – that $\mathcal{M}$ halts on u after t steps (w.l.o.g. $t \geq n$).

$\mathcal{M}$ can visit at most t squares of the tape. If ~~...~~ we want the initial sq. by t

Then all sq's M visits are in $[0, 2t]$. Then a content of the tape is given by a word in Σ^{2t+1} (all other sq's are blank). ~~For~~ Word $a \in \Sigma^{2t+1}$ corresponds to the initial situation iff

$$(1) \quad [\forall j < n,\ (u)_j = (a)_{t+j}] \wedge$$
$$[\forall j \leq 2t,\ (j < t \vee t+n \leq j) \to (a)_j = 0]$$

The entire computation corresponds to $t+1$ such strings $A = (a^0, \ldots, a^t)$, with $a^i \in \Sigma^{2t+1}$ describing the tape after step i. Hence $(A)_i \in \Sigma^{2t+1}$ and it is more readable to use the matrix notation:

$$A_{ij} := ((A)_i)_j = \text{"the symbol in sq. } j \text{ after } i \text{ steps"}$$

Let q code sequence of machine states after each step $i \leq t$. I.e.:

(2a) $\quad len(q) = t+1$

(2b) $\quad \forall i \leq t,\ (q)_i \leq |Q|-1$

(2c) $\quad (q)_0 = 0$ (the initial state).

We also need a record of head's positions: h codes positions $h_0, \ldots, h_t \le 2t$. I.e.:

(3a) $\quad len(h) = t+1$

(3b) $\quad \forall i \le t,\ (h)_i \le 2t$

(3c) $\quad (h)_0 = t$ (the initial position).

Now we want to write a formula expressing that "A, q, h are records of a valid computation on input u and it halts after step t."

Take subsequently all pairs $(u,v) \in Q \times \Sigma$ s.t. δ is defined on (u,v) and the instruction is, say: $(u,v) \longrightarrow (u', v', L)$.

Then add condition.

$$(4_{(u,v)}) \quad \forall i < t\ \forall j \le 2t$$

$$[(h)_i = j \wedge (q)_i = u \wedge (A)_{i,j} = v] \Rightarrow$$

$$[(h)_{i+1} = j-1 \wedge (q)_{i+1} = u' \wedge A_{i+1,j} = v'].$$

condition that $\mathcal{P}$ holds is arranged by the disjunction:

$$(5) \qquad \bigvee_{(u,v)\notin \operatorname{dom}(\delta)} \left(\exists j \le 2t,\ (h)_t = j \wedge (q)_t = u \wedge A_{tj} = v\right)$$

It say that after time t there is no instruction to apply.

Now we can express that $\mathcal{P}$ holds after t steps by Σ_1-formula:

$$\exists A, h, q \in \alpha(A, h, q, t, u)$$

where $\alpha \in \Delta_0$ is the conjunction of all conditions (1), (2), (3), $(4)_{uv}$ (all $(u,v) \in \operatorname{dom}\delta$) and (5).

Put: $\varphi(x) \rightleftharpoons \exists t \exists A, h, q \ \alpha(A, h, q, t, x)$.

<u>Theorem</u>: $\varphi(x)$ is a Σ_1-formula defining the set of $u \in \mathbb{N}$ on which $\mathcal{P}$ holds.