Ehrenfeucht-Fraïssé games and their pebbling versions

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Back-and-forth equivalence

$\cong$ isomorphism — more intrinsic property - defined directly in terms of structural properties

$\equiv$ elementary equivalence

involves a language.

- The existence of isomorphism can depend on some subtle questions about the surrounding universe of sets
- $\equiv$ only on A and B

- Family of equivalence relations - somewhere between $\cong$ and $\equiv$
  - purely structural, no languages involved
  - independent of the surrounding universe of sets
  - look at isomorphism, but only between a finite number of elements at a time

Back-and-forth games

- L signature, A and B L-structures
- $\forall$ (Abelard), $\exists$ (Eloise)
- $\forall$ wants to prove that A is <u>different</u> from B

  $\exists$ wants to show that A is <u>the same</u> as B
- $\forall$ wins if he manages to find a difference between A and B before the game finishes;

  Otherwise player $\exists$ wins

<u>The game</u> ... length γ (ordinal) $\rightarrow$ <u>γ steps</u>

<u>i-th step</u>

- $\forall$ takes one of the structures A, B and chooses an element of this structure
- $\exists$ chooses an element of <u>the other</u> structure

$\Rightarrow$ they choose a_i of A and b_i of B ... $\boxed{\text{At the end: } \bar{a} = (a_i : i < \gamma), \bar{b} = (b_i : i < \gamma) \quad \text{the pair } (\bar{a}, \bar{b}) = \text{the play}}$

- can choose an element which was choosen at an earlier step
- each player - see and remember all previous moves in the play (<u>game of perfect information</u>)

- win for player $\exists$ if there is an <u>isomorphism $f : \langle \vec{a} \rangle_A \to \langle \vec{b} \rangle_B$</u> such that $\boxed{f\bar{a} = \bar{b}}$
- A play which is not a win for player $\exists$ counts as a win for player $\forall$

<u>Definition</u>

L a signature, A, B L-structures

- <u>homomorphism</u> f from A to B, $f : A \to B$ is a function from dom(A) to dom(B) that:
    - for each constant c of L, $\boxed{f(c^A) = c^B}$
    - for each $n > 0$ and each n-ary relation symbol R of L and n-tuple $\bar{a}$ from A, if $\boxed{\bar{a} \in R^A \text{ then } f\bar{a} \in R^B}$
    - for each $n > 0$ and each n-ary function symbol F of L and n-tuple $\bar{a}$ from A, $\boxed{f(F^A(\bar{a})) = F^B(f\bar{a})}$
- <u>embedding</u> of A into B ... homomorphism $f : A \to B$, injective, and
    - for each $n > 0$, each <u>n-ary</u> relation symbol R of L and each n-tuple $\bar{a}$ from A, $\boxed{\bar{a} \in R^A \iff f\bar{a} \in R^B}$
- <u>isomorphism</u> is a surjective embedding

Example: Rationals vs integers

- $\eta \geq 2$
- A ... the additive group $\mathbb{Q}$
- B ... the additive group $\mathbb{Z}$

Player $\forall$ can win:

- he chooses $a_0 \in \mathbb{Q}$, $a_0 \neq 0$
- $\exists$ must choose b_0 to be a non-zero integer (otherwise she loses)

Now, there is some integer n, which doesn't divide b_0 in $\mathbb{Z}$

- $\forall$ chooses $a_1 \in \mathbb{Q}$, $na_1 = a_0$

There is no way that player $\exists$ can choose an element b_1 of $\mathbb{Z}$ so that $nb_1 = b_0$.

$\Rightarrow$ For $\eta \geq 2$, $\forall$ can always arrange to win the game on $\mathbb{Q}$ and $\mathbb{Z}$.

$\boxed{A \equiv_0 B}$ mean that For every atomic sentence ϕ of L

$$A \models \phi \iff B \models \phi$$

Then

$\boxed{\text{player } \exists \text{ wins the play } (\bar{a},\bar{b}) \iff (A,\bar{a}) \equiv_0 (B,\bar{b})}$

[which is equivalent ~~to~~ our definition of a win for $\exists$ by a lemma that for A, B L-structures, f a map from dom(A) to dom(B), f is an embedding iff for every atomic formula $\phi(\bar{x})$ of L and tuple $\bar{a}$ from A, $A \models \phi[\bar{a}] \iff B \models \phi[f\bar{a}]$]

The game is called the <u>Ehrenfeucht-Fraïssé game of length η on A and B</u>, $EF_\eta(A,B)$

- If $\exists$ knows an isomorphism $i: A \to B$, she can be sure of winning every time.
- Follow the rule:
    - $\forall$ has choosen a of A $\Rightarrow$ choose $i(a)$
    - $\forall$ has choosen b of B $\Rightarrow$ choose $i^{-1}(b)$
- <u>strategy</u> for a player = set of rules, how to move, depending on what has happened earlier
- player <u>uses</u> the strategy σ = obeys the rules of σ
- <u>winning strategy</u> ... the player wins every play in which he/she uses σ
- $\boxed{A \sim_{\gamma} B}$ player $\exists$ has a winning strategy in the game $EF_{\gamma}(A,B)$

Lemma

Let L be a signature and let A, B be L-structures

(a) If $A \cong B$, then $A \sim_\eta B$ for all ordinals η.

(b) If $\beta < \eta$ and $A \sim_\eta B$ then $A \sim_\beta B$

(c) If $A \sim_\eta B$ and $B \sim_\eta C$ then $A \sim_\eta C$. ($\sim_\eta$ is an equivalence relation on the class of L-structures)

Proof of (c) [$\sim_\eta$ is reflexive and symmetric]

Suppose $A \sim_\eta B$, $B \sim_\eta C$ ... $\exists$ has winning strategies ρ and τ for $EF_\eta(A,B)$, $EF_\eta(B,C)$.

Match $EF_\eta(A,C)$. We will find a winning strategy for $\exists$.

2 private games: $EF_\eta(A,B)$, $EF_\eta(B,C)$

- $\forall$ chooses a_i of A, she imagines $\forall$ made this move in $EF_\eta(A,B) \xrightarrow{\rho}$ "picks" b_i of B then she imagines b_i was the choice of $\forall$ in $EF_\eta(B,C) \xrightarrow{\tau}$ picks $\boxed{c_i \text{ in } C}$

  c_i will be the answer in public game

- $\forall$ chooses c_i of A ... respond in the other direction

End: $\bar{a}$ of A, $\bar{b}$ of B, $\bar{c}$ of C. $(\bar{a}, \bar{b})$ is win for $\exists$, $(\bar{b}, \bar{c})$ is win for $\exists$

$(A, \bar{a}) \equiv_0 (B, \bar{b}) \equiv_0 (C, \bar{c}) \Rightarrow (A, \bar{a}) \equiv_0 (C, \bar{c})$ ... $(\bar{a}, \bar{c})$ is win for $\exists$ in $EF_\eta(A, C)$
$\Rightarrow A \sim_\eta C$ $\square$

$\boxed{\text{Back-and-forth systems}}$

- two L-structures A and B are <u>back-and-forth equivalent</u> if $\boxed{A \sim_\omega B}$
  ($\exists$ has a winning strategy for the game $EF_\omega(A,B)$)
- A <u>back-and forth system</u> from A to B is a set I of pairs $(\bar{a},\bar{b})$
  of tuples with $\bar{a}$ from A and $\bar{b}$ from B, such that

  (2.4) if $(\bar{a},\bar{b})$ is in I then $\bar{a}$ and $\bar{b}$ have the same length and $(A,\bar{a}) \equiv_0 (B,\bar{b})$

  (2.5) I is not empty

  (2.6) for every pair $(\bar{a},\bar{b})$ in I and every element c of A
  there is an element d of B such that the pair $(\bar{a}c,\bar{b}d)$ is in I

  (2.7) For every pair $(\bar{a},\bar{b})$ in I and every element d of B
  there is an element c of A such that the pair $(\bar{a}c,\bar{b}d)$ is in I
- (2.4) and the lemma about embeddings give us that
  - if $(\bar{a},\bar{b})$ is in I then there is an isomorphism $f: \langle\bar{a}\rangle_A \to \langle\bar{b}\rangle_B$ such that $f\bar{a} = \bar{b}$
  - f is unique since $\bar{a}$ generates $\langle\bar{a}\rangle_A$

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$I^{\circledast}$ ... the set of all such functions f corresponding to pairs of tuples in I

Similar conditions for $J = I^{\circledast}$:

(2.4') each $f \in I^{\circledast}$ is an isomorphism from a finitely generated substructure of A to a finitely generated substructure of B

(2.5') $I^{\circledast}$ is not empty

(2.6') for every $f \in I^{\circledast}$ and c in A, there is $g \supseteq f$ such that $g \in I^{\circledast}$ and $c \in \operatorname{dom} g$

(2.7') For every $f \in I^{\circledast}$ and d in B, there is $g \supseteq f$ such that $g \in I^{\circledast}$ and $d \in \operatorname{im} g$

- If J is any set obeying the conditions (2.4') - (2.7') then there is a back-and forth system I such that $J = I^{\circledast}$.

namely: I - the set of all pairs of tuples $(\bar{a}, \bar{b})$, $\bar{a}$ from A, $\bar{b}$ from B and J contains a map $f: \langle\bar{a}\rangle_A \to \langle\bar{b}\rangle_B$ such that $f\bar{a} = \bar{b}$

Lemma

Let L be a signature and let A, B be L-structures. Then A and B are $\boxed{\text{back-and-forth equivalent}}$ if and only if there is a $\boxed{\text{back-and-forth system}}$ from A to B.

Proof

$\boxed{\Rightarrow}$ $\exists$ has a winning strategy σ for $EF_\omega(A,B)$. Define I to consist of the pairs of tuples which are of the form $(\bar{c}|n, \bar{d}|n)$ for some $n<\omega$ and some play $(\bar{c},\bar{d})$ in which player $\exists$ uses σ.

The set I is a back-and forth system from A to B. $\Rightarrow$ (2.5)

- putting $n=0$ in the definition of I ... I contains the pair of 0-tuples $(\langle\rangle, \langle\rangle)$
- (2.6) and (2.7) express that σ tells player $\exists$ what to do at each step
- (2.4) holds because the strategy is winning

$\boxed{\Leftarrow}$ Suppose there exists a back-and-forth system I from A to B.

- define the set $I^{\circledR}$ of maps as before
- choose an arbitrary well-ordering of I^{R}
- strategy σ for $\exists$ in the game $EF_\omega(A,B)$:
  - at each step, if the play so far is $(\bar{a},\bar{b})$ and $\exists$ has choosen c from A,
    - find the <u>first map f in I^*</u> such that <u>$\bar{a}$ and c are in the domain of f</u> and <u>$f\bar{a} = \bar{b}$</u>
    - then choose d to be fc
  - likewise in the other direction
  - if there is no such map f, choose some arbitrarily assigned element of the appropriate structure
- by (2.5')-(2.7') if $\exists$ follows σ, there always will be a map f in I^* as required
- resulting play $(\bar{a},\bar{b})$ ... then by (2.4') and the embeddings lemma we have $(A,\bar{a}) \equiv_0 (B,\bar{b})$ and so player $\exists$ wins

$\square$

Example: Algebraically closed fields

- A, B algebraically closed fields of the same characteristic and finite transcendence degree
- We shall show that A is back-and-forth equivalent to B.
  - Let I be the set of all isomorphisms $e: A' \to B'$ where A', B' are finitely generated subfields of A, B (smallest subfield of A containing some given finite set of elements of A)
  - I satisfies (2.4')
  - I is not empty since the prime subfields of A and B are isomorphic $\Rightarrow$ (2.5') ✓
  - Suppose $f: A' \to B'$ is in I and c is an element of A. We want a matching element d in B. 2 cases:
    - c is algebraic over A' $\Rightarrow$ c is determined up to isomorphism over A' by its polynomial $p(x)$ over A'. f carries $p(x)$ to a polynomial $fp(x)$ over B' and B contains a root d of $fp(x)$ since it is algebraically closed. $\Rightarrow$ f extends to an isomorphism $g: A'(c) \to B'(d)$

- c is transcendental over A'

  B' is finitely generated and B has infinite transcendence degree,

  $\Rightarrow$ there is an element d of B which is

  transcendental over B'

  again f extends to an isomorphism $g : A'(c) \to B'(d)$

  .... (2.6') is satisfied

- by symmetry (2.7') ✓
- so J defines a back-and-forth system from A to B

$\square$

Consequences of back-and-forth equivalence

- two structures A and B back-and-forth equivalent $\Rightarrow$ hard to tell apart
- a position of length n in a play $EF_\gamma(A,B)$ is a pair $(\bar{c},\bar{d})$ of n-tuples where $\bar{c}$ (resp. $\bar{d}$) lists in order the elements of A (resp. B) chosen in the first n moves
- the position is said to be winning for a player, if he/she has a strategy which enables him/her to win in $EF_\gamma(A,B)$ whenever the first n moves are $(\bar{c},\bar{d})$
- $(\bar{c},\bar{d})$ is a winning position for a player $\Leftrightarrow$ the player has a winning strategy for the game $EF_\gamma((A,\bar{c}),(B,\bar{d}))$
- the starting position (of length 0) is winning for $\exists$ $\Leftrightarrow$ A and B are back-and-forth equivalent

Theorem

Let L be any signature (not necessarily countable) and let A and B be L-structures.

(a) If $\boxed{A \cong B}$ then $\boxed{A \text{ is back-and-forth equivalent to } B}$

(b) Suppose A, B are at most countable. If $\boxed{A \text{ is back-and-forth equivalent to } B}$ then $\boxed{A \cong B}$.

In fact, if $\bar{c}, \bar{d}$ are tuples from A, B such that $(\bar{c}, \bar{d})$ is a winning position for $\exists$ in $EF_\omega(A, B)$, then there is an isomorphism from A to B which takes $\bar{c}$ to $\bar{d}$.

Proof

(a) special case of (first) lemma ($A \cong B \Rightarrow A \sim_\gamma B$ for all ordinals γ)

(b) $EF_\omega(A, B)$ has infinite length, A, B at most countable $\Rightarrow \forall$ can list all the elements of A and B among his choices.

Let $\exists$ play to win, and let $(\bar{a}, \bar{b})$ be the resulting play.

Since she wins, the diagram lemma gives an isomorphism

$f : A = \langle \bar{a} \rangle_A \to \langle \bar{b} \rangle_B = B$

The last sentence - proved the same way, starting the play at $(\bar{c}, \bar{d})$. $\square$

Theorem

Let A, B be L-structures and $\bar{a}, \bar{b}$ n-tuples from A, B.
If $(\bar{a}, \bar{b})$ is a winning position for player $\exists$ in $EF_\omega(A,B)$, then $(A, \bar{a}) \equiv_{\infty\omega} (B, \bar{b})$. In particular if A and B are back-and-forth equivalent, then they are $L_{\infty\omega}$-equivalent.

Proof

We prove that if $\phi(x)$ is any formula of $L_{\infty\omega}$ and $(\bar{a}, \bar{b})$ is a winning position for player $\exists$, then $A \models \phi(\bar{a}) \Leftrightarrow B \models \phi(\bar{b})$. By induction on the construction of ϕ.

ϕ atomic ... the result from the diagram lemma & the definition of winning

ϕ of the form $\neg\psi$, $\bigwedge\Phi$, $\bigvee\Phi$ ... straightforwardly from the induction hypothesis

ϕ ... $\exists y\, \psi(\bar{x}, y)$ Suppose $A \models \phi(\bar{a})$, then there is an element c in A such that $A \models \psi(\bar{a}, c)$. Since position $(\bar{a}, \bar{b})$ is winning for $\exists$, she has a winning strategy onward - choose d of B if $\forall$ chooses c. So $(\bar{a}c, \bar{b}d)$ must still be winning for $\exists$. Then the induction hypothesis: $B \models \psi(\bar{b}, d)$ so $B \models \exists y\, \psi(\bar{b}, y)$ as required. Likewise in the other direction from B to A.

ϕ ... $\forall y\, \psi$ We reduce this to the previous cases by writing $\neg\exists y \neg$ for $\forall y$ $\square$

Pebbling versions

- We explicitly deal with relational vocabularies only.
- Unless otherwise mentioned all vocabularies τ are finite and consist of relation symbols only.
- We denote finite maps p from $\mathrm{def}(p) \subseteq A$ to $\mathrm{im}(p) \subseteq B$ as

$$\boxed{p = (\bar{a} \mapsto \bar{b})}$$

if $\bar{a} = (a_1, \dots, a_n) \in A^n$ is such that $\mathrm{def}(p) = \{a_1, \dots, a_n\}$ and $\bar{b} = (b_1, \dots, b_n)$ where $b_i = p(a_i)$.

- $\boxed{p \subseteq p'}$ means that p' extends p in the sense that $\mathrm{def}(p) \subseteq \mathrm{def}(p')$ and $p'(a) = p(a)$ for all $a \in \mathrm{def}(p)$
- a map $p: (\bar{a} \mapsto \bar{b})$ is a partial isomorphism between τ-structures $\mathbb{A}$ and $\mathbb{B}$ if $p: \mathbb{A} \restriction \mathrm{def}(p) \simeq \mathbb{B} \restriction \mathrm{im}(p)$ is an isomorphism of induced substructures (admit $p = \emptyset$ as a special case)
- for $\mathbb{A}, \mathbb{B} \in \mathrm{STR}(\tau)$ let $\boxed{\mathrm{Part}(\mathbb{A}, \mathbb{B})}$ be the set of all finite partial isomorphisms between $\mathbb{A}$ and $\mathbb{B}$.

Example

- linear orderings $\mathbb{A}$ and $\mathbb{B}$ $\Rightarrow$ $\mathrm{Part}(\mathbb{A}, \mathbb{B})$ consists of all order-preserving maps $p = (\bar{a} \mapsto \bar{b})$ these are representable by $\bar{a}$ and $\bar{b}$ such that $\bar{a}$ is strictly increasing $<^{\mathbb{A}}$ and b is strictly increasing $<^{\mathbb{B}}$.

strictly increasing $\mathcal{L}^{P}$.

The basic EF game ; FO pebble game

Basic idea : two players $\forall$ (challenger, spoiler, male) and $\exists$ (duplicator, female)

play over two structures $\mathbb{A}, \mathbb{B} \in STR(\tau)$

$\forall$ tries to demonstrate differences, $\exists$ similarity between $\mathbb{A}$ and $\mathbb{B}$

Game positions

configurations $(\mathbb{A}, \bar{a} ; \mathbb{B}, \bar{b})$ where $\bar{a} = (a_1, \dots, a_n)$, $\bar{b} = (b_1, \dots, b_n)$, $n \in \mathbb{N}$

In pebble game terms: two sets of pebbles numbered $i = 1, \dots, n$,

placed on elements a_i and b_i of $\mathbb{A}$ and $\mathbb{B}$ respectively

Single round

$\forall$ places next pebble on some element of either $\mathbb{A}$ or $\mathbb{B}$

$\exists$ responds by placing the opposite pebble in the opposite structure

Leads the play from some position $(\mathbb{A}, \bar{a} ; \mathbb{B}, \bar{b})$ to a new position $(\mathbb{A}, \bar{a}a ; \mathbb{B}\bar{b}b)$.

The newly placed pebbles extends the correspondence $\bar{a} \mapsto \bar{b}$ to $\bar{a}a \mapsto \bar{b}b$.

Winning conditions

$\forall$ loses (and $\exists$ wins) as soon as the mapping $\bar{a} \mapsto b$ induced by the current position is not a partial isomorphism.

Otherwise we speak of isomorphic pebble configurations if $(\bar{a} \to \bar{b}) \in Part(\mathbb{A}, \mathbb{B})$ and play may continue

The m-round game

Definition

The m-round game $G_m(A, \bar{a}; B, \bar{b})$ continues for m rounds starting from position $(A, \bar{a}, B, \bar{b})$. $\exists$ wins any play in which she maintains isomorphic pebble configurations through all m rounds, and loses otherwise.

- we say $\exists$ wins the game if she has a winning strategy
- in any m-round game over finite structures precisely one of the players has a winning strategy
  (here this even follows directly from the finiteness of the game tree of all possible plays, which can also be analysed by exhaustive search to determine who can force a win)

Winning strategies and back-and-forth systems

Winning strategies and back-and-forth systems

Definition

(i) Let $I \subseteq \mathrm{Part}(\mathbb{A}, \mathbb{B})$, $p = (\bar{a} \mapsto \bar{b}) \in \mathrm{Part}(\mathbb{A}, \mathbb{B})$.

p has back-and-forth extensions in I if

forth $\quad \forall a \in A \ \exists b \in B: (\bar{a}a \mapsto \bar{b}b) \in I$

back $\quad \forall b \in B \ \exists a \in A: (\bar{a}a \mapsto \bar{b}b) \in I$

(ii) Let $I_i \subseteq \mathrm{Part}(\mathbb{A}, \mathbb{B})$ for $0 \leq i \leq m$. Then $(I_i)_{0 \leq i \leq m}$ is a back-and-forth system for $G_m(\mathbb{A}, \bar{a}; \mathbb{B}, \bar{b})$ if

- $(\bar{a} \mapsto \bar{b}) \in I_m$
- for $1 \leq k \leq m$, every $p \in I_k$ has back-and-forth extensions in I_{k-1}

(iii) If $(I_i)_{0 \leq i \leq m}$ is a back-and-forth system for $G_m(\mathbb{A}, \bar{a}; \mathbb{B}, \bar{b})$, we write

$$(I_i)_{0 \leq i \leq m} : \boxed{\mathbb{A}, \bar{a} \simeq_m \mathbb{B}, \bar{b}}$$

and say that $\mathbb{A}, \bar{a}$ and $\mathbb{B}, \bar{b}$ are m-isomorphic, $\mathbb{A}, \bar{a} \simeq_m \mathbb{B}, \bar{b}$.

Observation

$\boxed{\exists \text{ wins } G_m(\mathbb{A}, \bar{a}; \mathbb{B}, \bar{b})}$ (ie. she has a winning strategy for this game)

iff $\boxed{\mathbb{A}, \bar{a} \simeq_m \mathbb{B}, \bar{b}}$ (ie. if there is a back-and-forth system for $G_m(\mathbb{A}, \bar{a}; \mathbb{B}, \bar{b})$.

Proof (sketch)

$\boxed{\Leftarrow}$

- winning strategy from back-and-forth conditions with k more rounds to play
- $\exists$ can maintain positions in I_k

$\boxed{\Rightarrow}$

- the system $I_k = \{(\bar{a} \mapsto \bar{b}) : \exists \text{ wins } G_k(\mathbb{A}, \bar{a}; \mathbb{B}, \bar{b})\}$ satisfies the back-and-forth conditions

$\square$

Definition

For any formula ϕ of the first-order language L, we define the <u>quantifier rank of ϕ</u>, $qr(\phi)$, by induction on the construction of ϕ:

- If ϕ is atomic then $qr(\phi)=0$
- $qr(\neg\psi)=qr(\psi)$
- $qr(\bigwedge\Phi)=qr(\bigvee\Phi)=\max\{qr(\psi):\psi\in\Phi\}$
- $qr(\forall x\psi)=qr(\exists x\psi)=qr(\psi)+1$

Thus $qr(\phi)$ measures the nesting of quantifiers in ϕ.

$\boxed{\equiv_m}$ stands for <u>elementary equivalence up to qr-rank m</u>

$\boxed{A,\bar{a}\equiv_m B,\bar{b}}$ iff <u>for all $\varphi(\bar{x})\in FO$ with $qr(\varphi)\leq q$</u> we have $\boxed{A\models\varphi[\bar{a}]\Leftrightarrow B\models\varphi[\bar{b}]}$

Theorem (Ehrenfeucht-Fraïssé Theorem)

The following are equivalent for all $\mathbb{A}, \bar{a}; \mathbb{B}, \bar{b}$ and m:

(i) $\mathbb{A}, \bar{a} \simeq_m \mathbb{B}, \bar{b}$

(ii) $\exists$ wins $G_m(\mathbb{A}, \bar{a}; \mathbb{B}, \bar{b})$

(iii) $\mathbb{A}, \bar{a} \equiv_m \mathbb{B}, \bar{b}$

Proof

(ii) $\Rightarrow$ (iii) can be shown by induction on m

one outermost quantifier corresponds to the first round in the game

(iii) $\Rightarrow$ (i)

- show that the system $I_k := \{ (\bar{a} \mapsto \vec{b}) : \mathbb{A}, \bar{a} \equiv_k \mathbb{B}, \bar{b} \}$ satisfies the back-and-forth condition
- alternatively use the following lemma

Lemma

For $\mathbb{A}, \bar{a}$ and m, there is a formula $\boxed{\chi(\bar{x}) = \chi^m_{\mathbb{A},\bar{a}}(\bar{x})}$ of qFr-rank m, that characterises the $\simeq_m$-class of $\mathbb{A}, \bar{a}$ in the sense that for all $\mathbb{B}, \bar{b}$:

$$\boxed{\mathbb{B} \models \chi[\bar{b}] \quad \text{iff} \quad \mathbb{B}, \bar{b} \simeq_m \mathbb{A}, \bar{a}}$$

The $\chi^m_{\mathbb{A},\bar{a}}(\bar{x})$ are constructed by induction on m, for all $\mathbb{A}, \bar{a}$ simultaneously:

$\chi^0_{\mathbb{A},\bar{a}}$ consists just of conjunctions over all atomic and negated formulae true of $\bar{a}$ in $\mathbb{A}$

Inductively, χ^{m+1} expresses the back-and-forth conditions relative to the given χ^m, in the following typical format:

$$\boxed{\chi^{m+1}_{\mathbb{A},\bar{a}}(\bar{x}) := \underbrace{\bigwedge \{ \exists y\, \chi^m_{\mathbb{A},\bar{a}a}(\bar{x},y) : a \in A \}}_{\text{forth: responses for challenges in } \mathbb{A}} \wedge \underbrace{\forall y \bigvee \{ \chi^m_{\mathbb{A},\bar{a}a}(\bar{x},y) : a \in A \}}_{\text{back: responses for challenges in } \mathbb{B}}}$$

Corollary

A query (global relation) Q on $FIN(\tau)$ is FO-definable at q fr-rank m iff Q is closed under $\simeq_m$ in the sense that for $\mathbb{A}, \bar{a} \simeq_m \mathbb{B}, \bar{b}$ we have $\bar{a} \in Q^{\mathbb{A}} \Leftrightarrow \bar{b} \in Q^{\mathbb{B}}$.

It follows that Q is FO-definable iff Q is closed under $\simeq_m$ for some $m \in \mathbb{N}$.

Proof

(i)$\Rightarrow$(ii) Let $\varphi(\bar{x}) \in FO$ define Q, $qr(\varphi) = m$ and let $\mathbb{A}, \bar{a} \simeq_m \mathbb{B}, \bar{b}$.

By the theorem, $\mathbb{A}, \bar{a} \equiv \mathbb{B}, \bar{b}$, so $\mathbb{B} \models \varphi[\bar{b}] \Leftrightarrow \mathbb{A} \models \varphi[\bar{a}]$, and thus $\bar{a} \in Q^{\mathbb{A}} \Leftrightarrow \bar{b} \in Q^{\mathbb{B}}$.

(ii)$\Rightarrow$(i) Let Q be closed under $\simeq_m$, hence under $\equiv_m$.

The claim follows with the Proposition: Let $\Delta(\bar{x})$ be a class of formulae that is closed under boolean conectiveness and finite up to logical equivalence over $FIN(\tau)$, Q a query over $FIN(\tau)$. Then the following are equivalent:

(i) Q is definable by a formula from Δ

(ii) Q is closed under $\equiv_\Delta$ ..... iff $\underbrace{\mathbb{A}, \bar{a} \equiv_\Delta \mathbb{B}, \bar{b}}_{\text{for all } \varphi(\bar{x}) \in \Delta : \mathbb{A} \models \varphi[\bar{a}] \Leftrightarrow \mathbb{B} \models \varphi[\bar{b}]}$ implies that $a \in Q^{\mathbb{A}}$ iff $b \in Q^{\mathbb{B}}$

A defining formula for Q is

$$\varphi(\bar{x}) := \bigvee \{ \chi^m_{\mathbb{A}, \bar{a}}(\bar{x}) : \bar{a} \in Q^{\mathbb{A}} \}$$ ... this disjunction is essentially finite $\square$

this distinction is essentially finite $\square$

Inexpressibility via games

Example

$EVEN \subseteq FIN(\emptyset)$ is not FO-definable. Trivially any two naked sets of sizes $\geq m$ are m-isomorphic. Taking sets of sizes m and $m+1$ we see that EVEN is not closed under $\simeq_m$.

Example

The class Q of even length finite linear orderings is not FO-definable.

Proof

Let $\mathbb{A}_n$ be the standard ordering of $\mathbb{N}$ in restriction to $[n] := \{1, \dots, n\}$.

On $(\mathbb{N}, <)$ consider the usual distance $d(i,j) = |j-i|$. We use truncated distance d_t (for $t \in \mathbb{N}$) with values in $\{0, \dots, 2^t - 1\} \cup \{\infty\}$ defined as

$$d_t(i,j) := \begin{cases} d(i,j) & \text{if } d(i,j) < 2^t \\ \infty & \text{else} \end{cases}$$

Consider strictly increasing $\bar{a} = (a_1, \dots, a_s)$ in $[n]$ and $\bar{b} = (b_1, \dots, b_s)$ in $[n']$.

Put $(\bar{a} \mapsto \bar{b})$ into $I_t \subseteq Part(\mathbb{A}_n, \mathbb{A}_{n'})$ if (for $s > 0$)

$$d_t(0, a_1) = d_t(0, b_1)$$
$$d_t(a_i, a_{i+1}) = d_t(b_i, b_{i+1}) \quad \text{for } 1 \leq i < s$$
$$d_t(a_s, n+1) = d_t(b_s, n'+1)$$

and $\emptyset \in I_t$ iff $d_t(0, n+1) = d_t(0, n'+1)$.

$(I_t)_{0 \leq t \leq m}$ satisfies the back-and-forth conditions and $\emptyset \in I_m$ whenever $n = n'$ or $n, n' \geq 2^m - 1$. Hence $\mathbb{A}_n \simeq_m \mathbb{A}_{n'}$ for $n, n' \geq 2^m - 1$. Putting $n := 2^m$, $n' := 2^m - 1$ Q is not closed under $\simeq_m$ $\square$

Variation: k variables

k pebbles

The k-variable fragment and k-pebble game

- all vocabularies are finite and purely relational
- $k \geq 2$ is arbitrary but fixed

<u>Definition</u>

(i) The k-variable fragment of first-order logic, $FO^k \subseteq FO$ consists of those FO-formulae in which only the variables $x_1, \dots, x_k$ are used (free or bound)

(ii) $\mathbb{A}, \bar{a}$ and $\mathbb{B}, \bar{b}$ (with $|\bar{a}| = |\bar{b}| \leq k$) are k-variable equivalent $\boxed{\mathbb{A}, \bar{a} \equiv^k \mathbb{B}, \bar{b}}$, if for all $\varphi(\bar{x}) \in FO^k$ we have $\mathbb{A} \models \varphi[\bar{a}] \Leftrightarrow \mathbb{B} \models \varphi[\bar{b}]$

Similarly, for $m \in \mathbb{N}$ $\boxed{\mathbb{A}, \bar{a} \equiv^k_m \mathbb{B}, \bar{b}}$ if they agree on all $\varphi(\bar{x}) \in FO^k$ with $qr(\varphi) \leq m$.

- often use $x, y, \dots$ (but only k distinct ones)
- atomic formulae in FO^k cannot use more than k distinct variables $\rightarrow$ consider $FO^k(\tau)$ in connection with relational vocabularies τ whose relation symbols have arities up to k at most, also consider parameter tuples of length k

<u>Example</u> (i) For $k=3$, the class ORD is definable in $FO^k(\{<\})$

(ii) Over $\mathbb{A} = (A, <^{\mathbb{A}}) \in ORD$ there are $FO^2(\{<\})$-formulae φ_n defining the subset consisting of the first n elements wrt $<$, for $n \geq 1$. Inductively

$\varphi_1(x_1) := \forall x_2 \neg x_2 < x_1$, $\varphi_{n+1}(x_1) := \forall x_2 (x_2 < x_1 \rightarrow \varphi_n(x_2))$ [$\varphi_n(x_2)$ is obtained from $\varphi_n(x_1)$ by swapping variables x_1 and x_2]

The k-pebble game

- obtained as a simple variation of the FO pebble game
- k pairs of pebbles numbered $1, \dots, k$

Positions in the k-pebble game over $\mathbb{A}$ and $\mathbb{B}$ are positions $(\mathbb{A}, \bar{a}; \mathbb{B}, \bar{b})$ with $\bar{a} \in A^k$, $\bar{b} \in B^k$.
(not initial phases)

A single round - challenge / response exchange

- In position $(\mathbb{A}, \bar{a}, \mathbb{B}, \bar{b})$ $\boxed{\forall}$ chooses one pebble in one of the structures and relocates it on any element of that structure (e.g. pebble i in $\mathbb{A}$ is moved to element $a \in A$)

- $\exists$ has to respond by moving the corresponding pebble in the opposite structure (in the example, moving pebble i in $\mathbb{B}$ to some element $b \in B$)
- Writing $\boxed{\bar{a}\frac{a}{i}}$ for the result of replacing the ith component of $\bar{a}$ by a, a round played with pebble i thus leads from a position $(\mathbb{A}, \bar{a}; \mathbb{B}, \bar{b})$ to a position $(\mathbb{A}, \bar{a}\frac{a}{i}; \mathbb{B}, \bar{b}\frac{b}{i})$

The constraints and winning conditions are strictly analogous to those for the FO game

Definition

The m-round k-pebble game $\boxed{G_m^k(\mathbb{A},\bar{a};\mathbb{B},\bar{b})}$ continues for m rounds starting from position $(\mathbb{A},\bar{a};\mathbb{B},\bar{b})$.

∃ wins ~~the~~ plays in which she maintains isomorphic pebble configurations through m rounds, and loses otherwise.

Definition

A back-and-forth system for G_m^k over $\mathbb{A}$ and $\mathbb{B}$ is a system $(I_i)_{0\le i\le m}$ such that $\emptyset \ne I_i \subseteq \{(\bar{a}\mapsto\bar{b}) \in \mathrm{Part}(\mathbb{A},\mathbb{B}) : \bar{a}\in A^k, \bar{b}\in B^k\}$, and for $1\le i\le m$, every $(\bar{a}\mapsto\bar{b})\in I_i$ has back-and-forth extensions in I_{i-1}:

forth $\quad \forall j\in\{1,\dots,k\}\ \forall a\in A\ \exists b\in B: \left(\bar{a}\tfrac{a}{j}\mapsto\bar{b}\tfrac{b}{j}\right)\in I_{i-1}$

back $\quad \forall j\in\{1,\dots,k\}\ \forall b\in B\ \exists a\in A: \left(\bar{a}\tfrac{a}{j}\mapsto\bar{b}\tfrac{b}{j}\right)\in I_{i-1}$

$(I_i)_{0\le i\le m}$ is a back-and-forth system for $G_m^k(\mathbb{A},\bar{a};\mathbb{B},\bar{b})$ if $(\bar{a}\mapsto\bar{b})\in I_m$.

We write $\boxed{(I_i)_{0\le i\le m} : \mathbb{A},\bar{a} \simeq_m^k \mathbb{B},\bar{b}}$ in this situation and say that $\mathbb{A},\bar{a}$ and $\mathbb{B},\bar{b}$ are k-pebble ~~m~~-equivalent.

Strictly analogous to the case of the m-round FO game:

Theorem
The following are equivalent for all $\mathbb{A}, \bar{a}; \mathbb{B}, \bar{b}$ and m:

(i) $\mathbb{A}, \bar{a} \simeq^k_m \mathbb{B}, \bar{b}$

(ii) $\exists$ wins $G^k_m(\mathbb{A}, \bar{a}; \mathbb{B}, \bar{b})$

(iii) $\mathbb{A}, \bar{a} \equiv^k_m \mathbb{B}, \bar{b}$

- Just as for the FO-game, one uses characteristic formulae $\chi^m_{\mathbb{A},\bar{a}} \in FO^k$, $qr(\chi^m_{\mathbb{A},\bar{a}}) = m$ that characterise the $\simeq^k_m$-class of $\mathbb{A}, \bar{a}$.
- For $m=0$, $\chi^m_{\mathbb{A},a}$ is the conjunction over all atomic and negated atomic FO^k-formulae that are true of $\bar{a}$ in $\mathbb{A}$. Inductively

$$\chi^{m+1}_{\mathbb{A},\bar{a}}(\bar{x}) := \chi^m_{\mathbb{A},\bar{a}} \wedge \underbrace{\bigwedge_{1\le j\le k} \bigwedge \{\exists x_j\, \chi^m_{\mathbb{A},\bar{a}\frac{a}{j}}(\bar{x}) : a \in A\}}_{\text{forth}} \wedge \underbrace{\bigwedge_{1\le j\le k} \forall x_j \bigvee \{\chi^m_{\mathbb{A},\bar{a}\frac{a}{j}}(\bar{x}) : a \in A\}}_{\text{back}}$$

- $\exists$ wins $G^k_m(\mathbb{A}, \bar{a}, \mathbb{B}, \bar{b})$ iff $\mathbb{B} \models \chi^m_{\mathbb{A},\bar{a}}[\bar{b}]$ (exercise)

The unbounded k-pebble game and k-variable types

Definition

The infinite or unbounded k-pebble game $\boxed{G_\infty^k(\mathbb{A}, \bar{a}, \mathbb{B}, \bar{b})}$

- starts from $(\mathbb{A}, \bar{a}, \mathbb{B}, \bar{b})$
- consists of an unending succession of rounds in which isomorphic pebble configurations are maintained
- or ends with a loss for $\exists$ when local isomorphism is violated
- $\exists$ wins the game $G_\infty^k(\mathbb{A}, \bar{a}, \mathbb{B}, \bar{b})$ if she has a strategy to maintain isomorphic pebble configurations indefinitely in any play starting from $(\mathbb{A}, \bar{a}, \mathbb{B}, \bar{b})$

- The game graph for G_∞^k over any two fixed finite $\mathbb{A}$ and $\mathbb{B}$ is finite (there are only $|A|^k \cdot |B|^k$ distinct positions)

→ The analysis of the game is therefore essentially finite
– any sufficiently long play must eventually repeat some configuration

Back-and-forth systems for infinite game

Definition

A back-and-forth system for G^k_∞ over $\mathbb{A}$ and $\mathbb{B}$ is a single set $I \subseteq \{(\bar{a} \mapsto \bar{b}) \in \mathrm{Part}(A, B) : \bar{a} \in A^k, \bar{b} \in B^k\}$, such that every $(\bar{a} \mapsto \bar{b}) \in I$ has back-and-forth extensions in I. I is back-and-for system for $G^k_\infty(\mathbb{A}, \bar{a}; \mathbb{B}, \bar{b})$ if $(a \mapsto b) \in I$.

We write $\boxed{I : \mathbb{A}, \bar{a} \simeq^k_\infty \mathbb{B}, \bar{b}}$ in this situation, and say that $\mathbb{A}, \bar{a}$ and $\mathbb{B}, \bar{b}$ are k-pebble equivalent

Analysis of k-variable types

Definition

For $\mathbb{A}$ and $\bar{a} \in A^k$ define

(i) The k-variable type $\boxed{tp^k(\mathbb{A}, \bar{a}) := \{\varphi(\bar{x}) \in FO^k : \mathbb{A} \models \varphi[\bar{a}]\}}$

(ii) The rank m k-variable type $\boxed{tp^k_m(\mathbb{A}, \bar{a}) := tp^k(\mathbb{A}, \bar{a}) \cap \{\varphi \in FO^k : qr(\varphi) \leq m\}}$

By the Ehrenfeucht-Fraïssé theorem, the rank m k-variable types $tp^k_m(\mathbb{A}, \bar{a})$ exactly specify the $\simeq^k_m$-equivalence class of $\mathbb{A}, \bar{a}$.

It is therefore also determined by the characteristic formula $\varphi^m_{\mathbb{A}, \bar{a}}$, which is itself member of this type.

which is itself member of this type.

Inductive refinement

- Consider an individual $\mathbb{A} \in \mathrm{FIN}(\tau)$
- An inductive refinement generates $\simeq^k_i$ and as their limit, $\simeq^k_\infty$, as equivalence relations on A^k.
- Let $\sim_0$ be the equivalence relation corresponding to qfr-rank 0 equivalence:

$$\boxed{\bar a \sim_0 \bar a'} \quad \text{iff} \quad \boxed{\mathbb{A}, \bar a \simeq^k_0 \mathbb{A}, \bar a'} \quad \text{iff} \quad \boxed{tp_0^k(\mathbb{A}, \bar a) = tp_0^k(\mathbb{A}, \bar a')} \quad \text{iff} \quad \boxed{\mathbb{A}\restriction \bar a, \bar a \simeq \mathbb{A}\restriction \bar a', \bar a'}$$

- Suppose the equivalence relation $\sim_i$ on A^k is given such that

$$\boxed{\bar a \sim_i \bar a'} \quad \text{iff} \quad \boxed{\mathbb{A}, \bar a \simeq^k_i \mathbb{A}, \bar a'} \quad \text{iff} \quad \boxed{tp_i^k(\mathbb{A}, \bar a) = tp_i^k(\mathbb{A}, \bar a')}$$

- For any $1 \le j \le k$, any $\sim_i$ equivalence class $\alpha \in A^k/\sim_i$ and $\bar a \in A^k$ define

$$\iota_{j,\alpha}(\bar a) := \begin{cases} 1 & \text{if } \exists a \in A \left(\bar a \tfrac{a}{j} \in \alpha\right) \\ 0 & \text{else} \end{cases}$$

Now put $\boxed{\bar a \sim_{i+1} \bar a'}$ iff $\boxed{\bar a \sim_i \bar a' \text{ and } \forall j \forall \alpha : \iota_{j,\alpha}(\bar a) = \iota_{j,\alpha}(\bar a')}$

i.e. for $\bar a \simeq^k_i \bar a'$ we put $\bar a \sim_{i+1} \bar a'$ if, and only if, $\bar a \mapsto \bar a'$ has back-and-forth extensions which maintain $\simeq^k_i$ equivalence

- It follows, as desired, $\sim_{i+1}$ coincides with $\simeq^k_{i+1}$ (and $\equiv^k_{i+1}$) on $\mathbb{A}$.

- the sequence $(\sim_i)_{i \geq 0}$ is a monotone sequence of successively refined equivalence relation on the finite set A^k
- for some $r \leq |A|^k$ we must have $\sim_r = \sim_{r+1} = \sim_{r+s}$ for all $s \in \mathbb{N}$
  the minimal such r ist called <u>k-rank of $\mathbb{A}$, k-rank$(\mathbb{A})$</u>

<u>Lemma</u>

(i) For all $i \in \mathbb{N}$: $\boxed{\bar{a} \sim_i \bar{a}'}$ iff $\boxed{\mathbb{A}, \bar{a} \simeq_i^k \mathbb{A}, \bar{a}'}$

(ii) For <u>$r = k\text{-rank}(\mathbb{A})$</u>: $\boxed{\bar{a} \sim_r \bar{a}'}$ iff $\boxed{\mathbb{A}, \bar{a} \simeq_\infty^k \mathbb{A}, \bar{a}'}$

<u>Proof</u>

(i) from the definition of the $\sim_i$ by induction on i

(ii)
- consider $I := \{\bar{a} \mapsto \bar{a}' : \bar{a} \sim_r \bar{a}'\}$
  - We claim that I has back-and-forth extensions, so that $I : \mathbb{A}, \bar{a} \simeq_\infty^k \mathbb{A}, \bar{a}'$ for any $(\bar{a} \mapsto \bar{a}') \in I$.
  - Let $(\bar{a} \mapsto a') \in I$, $0 \leq j \leq k$ and $a \in A$. As $\sim_r = \sim_{r+1}$, $\bar{a} \sim_{r+1} \bar{a}'$.
    Hence $\mathbb{A}, \bar{a} \simeq_{r+1}^k \mathbb{A}, \bar{a}'$, which guarantees the existence of some $a' \in A$ fur which $\mathbb{A}, \bar{a}\frac{a}{j} \simeq_r^k \mathbb{A}, \bar{a}'\frac{a'}{j}$.
  - It follows that $(\bar{a}\frac{a}{j} \mapsto \bar{a}'\frac{a'}{j}) \in I$ is as desired

$\square$

- for every $\mathbb{A}$ and $\bar{a} \in A^k$ there is also a single FO^k-formula that characterises the $\simeq^k_\infty$ class of $\mathbb{A}, \bar{a}$.

- Let $r = k\text{-rank}(\mathbb{A})$ and put

$$\boxed{\chi_{\mathbb{A}} := \bigwedge_{\bar{a} \in A^k} \exists \bar{x}\, \chi^r_{\mathbb{A},\bar{a}} \wedge \forall \bar{x} \bigvee_{\bar{a} \in A^k} \chi^r_{\mathbb{A},\bar{a}} \wedge \bigwedge_{1 \le j \le k} \bigwedge_{\bar{a} \in A^k} \forall \bar{x} \Big[\chi^r_{\mathbb{A},\bar{a}} \to \Big(\bigwedge_{a \in A} \exists x_j\, \chi^r_{\mathbb{A},\bar{a}\frac{a}{j}} \wedge \forall x_j \bigvee_{a \in A} \chi^r_{\mathbb{A},\bar{a}\frac{a}{j}}\Big)\Big]}$$

- the first two conjuncts - precisely the rank r k-types of $\mathbb{A}$ are realised
- the third conjunct implies that $\sim_{r+1} = \sim_r$, whence rank r types determine the $\simeq^k_\infty$ types

- For $\bar{a} \in A^k$ put $\boxed{\chi_{\mathbb{A},\bar{a}}(\bar{x}) := \chi_{\mathbb{A}} \wedge \chi^r_{\mathbb{A},\bar{a}}(\bar{x})}$

- $qr(\chi_{\mathbb{A},a}) = r + k + 1$
- Suppose $\mathbb{B} \models \chi_{\mathbb{A}}$. Then $\mathbb{B}$ has exactly the same rank r types as $\mathbb{A}$

  and $k\text{-rank}(\mathbb{B}) = r$.
  - The following system has back-and-forth extensions

$$I := \{\bar{a} \mapsto \bar{b} : tp^k_r(\mathbb{B}, \bar{b}) = tp^k_r(\mathbb{A}, a)\}$$

- If $\mathbb{B} \models \chi_{\mathbb{A},\bar{a}}[\bar{b}]$, then also $tp^k_r(\mathbb{B}, b) = tp^k_r(\mathbb{A}, \bar{a})$ and $\mathbb{A}, \bar{a} \simeq^k_\infty \mathbb{B}, b$ via I.

Theorem

Let $\mathbb{A} \in \mathrm{FIN}(\tau)$, $r := k\text{-rank}(\mathbb{A})$, $\bar{a} \in A^k$. Then the following are equivalent for any $\mathbb{B} \in \mathrm{FIN}(\tau)$ and $\bar{b} \in B^k$:

(i) $\boxed{\mathbb{A}, \bar{a} \simeq^k_\infty \mathbb{B}, \bar{b}}$

(ii) $\boxed{\mathbb{A}, \bar{a} \simeq^k_{r+k+1} \mathbb{B}, \bar{b}}$

(iii) $\boxed{\mathbb{A}, \bar{a} \equiv^k_{r+k+1} \mathbb{B}, \bar{b}}$

(iv) $\boxed{\mathbb{A}, \bar{a} \equiv^k \mathbb{B}, \bar{b}}$

Proof

- (ii) $\Leftrightarrow$ (iii) from the EF theorem for k-pebble game
- $\simeq^k_\infty$ equivalence implies $\simeq^k_i$ equivalence for all i $\leadsto$ we always have (i) $\Rightarrow$ (ii)
- (i) $\Rightarrow$ (iv) also from the theorem
- (iv) $\Rightarrow$ (iii) trivial
- (iii) implies $\mathbb{B} \models \chi_{\mathbb{A},\bar{a}}[\bar{b}]$, also (iii) $\Rightarrow$ (i) from the consideration above

$\square$

A global preordering w.r.t. k-variable types

- upgrade the inductive refinement that separated out the different $\simeq^k_\infty$-types over an individual finite structure $\mathcal{A}$
- so that it provides a linear ordering of these types
  - ie a linear ordering of $A^k/\simeq^k_\infty$
- in stages
- now generate
  - transitive
  - reflexive
  - total preordering

relations $\precsim_i$ on A^k such that

$$\bar{a} \sim_i \bar{a}' \iff (\bar{a} \precsim_i \bar{a}' \text{ and } \bar{a}' \precsim_i \bar{a}) \qquad (*)$$

- It follows that $A^k/\sim_i$ is linearly ordered (in the sense of $\leq$) by $\precsim_i$.

- at level $i=0$ use some arbitrary but fixed linear ordering of the (finitely many) qfr-free k-variable types
- inductively assume that $\precsim_i$ satisfies $(*)$ and hence induces a linear ordering on $A^k/\sim_i$.
  - consider the values of boolean functions $b_{j,\alpha}$ on A^k, for $\bar a \sim_i \bar a'$ we have $\bar a \sim_{i+1} \bar a'$ iff $b_{j,\alpha}(\bar a) = b_{j,\alpha}(\bar a')$ for all j,α
  - consider the boolean tuple listing the $b_{j,\alpha}$-values in order of increasing j, and within the same j, increasing w.r.t. α in the sense of $\precsim_i$.

    The set of all such tuples carries a natural lexicographic ordering, based on the first position where the two tuples differ (if not equal)

We now put

$$\bar a \precsim_{i+1} \bar a' \quad \text{iff} \quad \left(\bar a \precsim \bar a' \text{ and } (b_{j,\alpha}(\bar a))_{j,\alpha} \leq_{lex} (b_{j,\alpha}(\bar a'))_{j,\alpha}\right)$$

- Then $\precsim_{i+1}$ is again transitive, reflexive and total and provides a linear ordering of the $\sim_{i+1}$-classes according to $(*)$
- $\precsim_{i+1}$ is uniformly FO-definable in terms of $\precsim_i$ over $\mathfrak{A}$. Let $\precsim$ be the limit $\precsim^{\mathfrak{A}} = \precsim^{\mathfrak{A}}_r$ for $r = k\text{-rank}(\mathfrak{A})$. Then $\precsim^{\mathfrak{A}}$ is a linear ordering (in the sense of $\leq$) on $A^k/\sim^k_\infty$

Lemma

There is a global relation $\preccurlyeq$ of arity $2k$, uniformly definable by an inductive iteration of a first-order definable operation, such that $\preccurlyeq^{\mathbb{A}}$ is a pre-ordering on A^k which linearly orders w.r.t. k-variable types.

The linear ordering introduced by $\preccurlyeq$ on $A^k / \simeq^k_\infty$ is completely determined by the k-variable types in $\mathbb{A}$ and hence only depends on the $\equiv^k$-class of $\mathbb{A}$.

The same is true of the information about the qfr-free formulae in each $\simeq^k_\infty$ type $\quad P_\theta := \{ \alpha \in A^k/\simeq^k_\infty : \mathbb{A} \models \theta[\bar{a}] \text{ for } \bar{a} \in \alpha \}$ for each qfr-free $\theta \in FO^k(\tau)$

the incidence between $\simeq^k_\infty$-types w.r.t. the relations describing moves of the j-th pebble in the game

$$E_j := \{ (\alpha, \alpha') : \exists a \, (\bar{a}\tfrac{a}{j} \in \alpha') \text{ for } \bar{a} \in \alpha \} \quad \text{for } j = 1, \dots, k$$

For a distinguished tuple $a \in A^k$, we may also identify its $\simeq^k_\infty$ class $[\bar{a}]_{\simeq^k_\infty}$ as a distinguished element in the quotient $A^k / \simeq^k_\infty$.

Definition

The k-variable invariant of $\mathbb{A}, \bar{a}$ is defined to be the linearly ordered quotient structure $\mathcal{Y}^k(\mathbb{A}, \bar{a}) := (A^k/\simeq^k_\infty, \leq, (P_\theta), (E_j), [a]_{\simeq^k_\infty})$

Proposition

For $\mathbb{A}, \bar{a}$ and $\mathbb{B}, \bar{b}$: $\mathbb{A}, \bar{a} \simeq^k_\infty \mathbb{B}, \bar{b}$ iff $\mathcal{Y}^k(\mathbb{A}, \bar{a}) \simeq \mathcal{Y}^k(\mathbb{B}, b)$

Proof

($\Leftarrow$) from the definition of $\mathcal{Y}^k$

($\Rightarrow$) the actual k-variable types realised in $\mathbb{A}$ and in particular the k-variable type of $\bar{a}$ in $\mathbb{A}$ can be identified from just $\mathcal{Y}^k(\mathbb{A}, \bar{a})$

For this one determines $tp^k_m(\mathbb{A}, \bar{a}')$ for $\bar{a}' \in \alpha \in \mathcal{Y}^k$ by induction on m $\square$

- Invariants are polynomial time computable
- isomorphism between finite linearly ordered structures is trivially decidable in polynomial time

Corollary k-variable equivalence can be decided by a polynomial time algorithm.

Thank you

for your attention

<u>Sources</u>:

W. Hodges: Shorter model theory (chpt 3)

M. Otto: Finite Model Theory - lecture notes (sec 2.1, 2.4)