We use the notation from Sec.31.2.

In the proof of Lemma 31.2.1 we pick by averaging e s.t. at least a fraction of $\delta\frac{1}{(3m)^k}$ more inputs u to C (and f) yield a sample $a(u,e) \in W$ whose trace is exactly $\bar{\imath}$ than those which do yield $a(u,e) \in W$ whose trace properly contains $\bar{\imath}$ (Claims 1 and 2). The error in the argument for Claim 3 is that we have no control over the number of u for which $a(u,e) \notin W$ but its trace contains $\bar{\imath}$, i.e. of the size of the set $U \setminus W$.

However, if we knew that the size of the complement of W is at most e.g.

$$w_c \ := \ \frac{1}{2} \cdot 2^{n^{1/3}} \cdot \frac{1}{(3m)^c}$$

then the argument works: w_c bounds the number of bad u and the algorithm constructed in Claim 3 gets the advantage at least (we ignore δ now)

$$\frac{1}{(3m)^k} - \frac{1}{2} \cdot \frac{1}{(3m)^c} \ \geq \ \frac{1}{2} \cdot \frac{1}{(3m)^k}$$

and the rest of the proof (bottom p.212, top p.213) remains the same.

Hence what is established in Sec.31.2 is the following statement.

Lemma A: *Under the same hypothesis as in Lemma 31.2.1, the number of samples $\omega \in \Omega$ for which $\alpha(\omega)$ is defined is at least*

$$w_c \ := \ \frac{1}{2} \cdot 2^{n^{1/3}} \cdot \frac{1}{(3m)^c}$$

where c bounds the number of queries α can ask on any sample.

Note that w_c is a nonstandard number for any $m \in \mathcal{M}_n$ and any standard c.

We would like to use Lemma A to establish

Lemma B: *There exists an infinite set $\Omega^* \subseteq \Omega_b$, $\Omega^* \in \mathcal{M}$, such that each $\alpha \in F_b$ is defined on all but an infinitesimal fraction of samples from Ω^*.*

Taking F_b^*, the family of random variables defined as F_b but restricted to Ω^*, determines model $K(F_b^*)$ for which the analogous statement to Lemma 31.2.1 (Lemma B) holds and it can be used in place of $K(F_b)$.

Lemma B can be derived by a combinatorial argument for small $m > n$ but here we shall give a model-theoretic argument which has the advantage of being much simpler and working for any m, using a smaller set of random variables.

Namely, for any string $w \in \mathcal{M}_n$ let $F_{b,w}^{unif}$ be the family of partial random variables on Ω_b defined as F_b but allowing the algorithms computing the random variables to use as an advice only the triple (A, b, w). This is perfectly sufficient for any application of the eventual model in Secs.31.3. and 31.4 (w can contain e.g. a proof of the τ-formula or a witness of the membership of b in an NP set R, etc.) and has the great advantage that the family $F_{b,w}^{unif}$ is now countable.

Lemma C: *Let $w \in \mathcal{M}_n$ be arbitrary. Then there exists an infinite set $\Omega^* \subseteq \Omega_b$, $\Omega^* \in \mathcal{M}$, such that each $\alpha \in F_{b,w}^{unif}$ is defined on all samples from Ω^*.*

Proof:

Enumerate $\alpha_1, \alpha_2, \ldots$ the set $F_{b,w}^{unif}$ in such a way that the algorithm defining α_k runs in time $\leq m^k$ and ask at most k queries, for all $k \geq 1$.

Let $(\alpha_i)_{i<t} \in \mathcal{M}$ be its non-standard extension obtained via the $\aleph_1$-saturation (see p.9).

If we take $\alpha_1, \ldots, \alpha_k$ we can compose the programs defining the αs by first running α_1, if it is not aborted then instead of outputting a value run α_2, etc. , and output (arbitrary) values only at the end, if the computation is not aborted earlier. The resulting function is computed in time $O(km^k)$ using at most $k(k+1)/2 \leq k^2$ queries. Hence by Lemma A it is defined on at least w_{k^2} samples from Ω_b. This yields the following

Claim: For each standard $k \geq 1$ there exists definable subset $\Omega^k \subseteq \Omega_b$ of size at least w_{k^2} such that all $\alpha_1, \ldots, \alpha_k$ are defined on all samples from Ω^k.

By Overspill the statement of the Claim holds also for the sequence $(\alpha_i)_{i<s}$ for some non-standard $s < t$, and we can take s small enough (but still non-standard) such that $\Omega^* := \Omega^s$ satisfies the statement of the lemma.

q.e.d.