The necessity of λ [illegible], we could ask whether for a countable T we

For example, if $\aleph_0 < \lambda < 2^{\aleph_0}$, we could ask whether for a countable T we can omit a family of λ nonisolated types. This turns out to depend on set theoretic assumptions (see Exercise 4.5.14).

We give one concrete application of the Omitting Types Theorem. Let $\mathcal{L} = \{+, \cdot, <, 0, 1\}$, and let PA be the axioms for Peano arithmetic. Suppose that $\mathcal{M}, \mathcal{N} \models$ PA. We say that $\mathcal{N}$ is an *end extension* of $\mathcal{M}$ if $N \supset M$ and $a < b$ for all $a \in M$ and $b \in N \setminus M$.

Theorem 4.2.5 *If $\mathcal{M}$ is a countable model of PA, then there is $\mathcal{M} \prec \mathcal{N}$ such that $\mathcal{N}$ is a proper end extension of $\mathcal{M}$.*

Proof Consider the language $\mathcal{L}^*$ where we have constant symbols for all elements of M and a new constant symbol c. Let $T = \text{Diag}_{\text{el}}(\mathcal{M}) \cup \{c > m : m \in M\}$, and for $a \in M \setminus \mathbb{N}$ let p_a be the type $\{v < a, v \neq m : m \in M\}$. Any $\mathcal{N} \models T$ is a proper elementary extension of $\mathcal{M}$. If $\mathcal{N}$ omits each p_a, then $\mathcal{N}$ is an end extension of $\mathcal{M}$. By Theorem 4.2.4, it suffices to show that each p_a is nonisolated.

Suppose that $\phi(v)$ is an $\mathcal{L}^*$ formula isolating p_a. Let $\phi(v) = \theta(v, c)$, where θ is an $\mathcal{L}_M$-formula. Then

$$T \cup \theta(v, c) \models v < a.$$

Because $T \cup \{\theta(v, c)\}$ is satisfiable,

$$\mathcal{M} \models \forall x \exists y > x \exists v < a\ \theta(v, y).$$

The Pigeonhole Principle is provable in Peano arithmetic. Thus

$$\mathcal{M} \models [\forall x \exists y > x \exists v < a\ \theta(v, y)] \rightarrow \exists v < a \forall x \exists y > x\ \theta(v, y). \qquad (**)$$

Thus, there is $m < a$ such that

$$\mathcal{M} \models \forall x \exists y > x\ \theta(m, y).$$

We claim that $T \cup \{\theta(m, c)\}$ is satisfiable. If not, there is $n \in M$ such that

$$\text{Diag}_{\text{el}}(\mathcal{M}) + c > n \models \neg\theta(m, c)$$

contradicting $(**)$. Thus, $\phi(v)$ does not isolate p_a, a contradiction.