Mutual interpretability of PA and "finite" ZFC

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"finite" ZFC (= fZFC): as ZFC but with the axiom of infinity:

$$\exists y,\ \emptyset \in y \wedge (\forall x \in y,\ succ(x) \in y)$$

where $succ(x) := x \cup \{x\}$, replaced by its negation.

interpretability: T in a relational lang. L_T is interpretable in S iff

- each symbol $R(\bar{x}) \in L_T$ can be "interpreted" by a L_S-fla $\varphi_R(\bar{x})$, and
- quantifiers $\exists/\forall$ relativized to some L_S-fla $\varphi_{dom}(x)$, s.t.
- all axioms of T translate into L_S-sentences provable in S.

Remarks: (1) I.e., we have uniform way of defining in each model of S a model of T.

(2) There are many (!) variants of this concept.

PA interprets in fZFC

- domain of $\exists/\forall$: $\varphi_{dom}(x) \equiv$ "x is an ordinal"
- ~~[illegible]~~ interpret. of L_{PA}:

$$x = y \to x = y$$
$$x \leq y \to (x \in y \vee x = y)$$
$$0 \to \emptyset$$
$$1 \to \mathrm{succ}(\emptyset)$$

and $(x+y)$ and $(x \cdot y)$ are defined by talking about cardinalities of the disjoint union $x \cup y$ and $x \times y$.

- Non-induction ax's of PA translate into properties of finite cardinalities proved by the LAP (i.e. the well-ordering of On). One needs to prove that x and $\mathrm{succ}(x)$ have different cardinalities, similarly as we prove below IND.
- IND: Assume an L_{PA}-fla translates into $\varphi(x)$ violating (the translation of) IND:

$$\varphi(\emptyset) \wedge \forall x \in On(\varphi(x) \to \varphi(\mathrm{succ}(x))) \wedge \exists y \in On, \neg\varphi(y).$$

The $\{x \in y \mid \varphi(x)\}$ exist by the comprehension ax. (y is a witness for $\exists$) and it witnesses the ax. of ∞: that is a contradiction as we assume in fZFC its negation.

$\square$

fZFC interprets in PA

For $u, v \in \mathbb{N}$ define L_{PA} fla:

$u E v$ $\iff$ "2^u occurs in the (unique) expression of v as a sum of powers of 2"

Formally:

$\exists a,b,c \le v : a < b \wedge$ "$2^u = b$" $\wedge\ 2b \mid c \wedge a+b+c = v$

To define b as L_{PA}-fla we use Gödel's lemma (cf. Chpt. 5):

$\exists$ sequence $s = (w_0, \dots, w_u) : w_0 = 1 \wedge (\forall i < u\ w_{i+1} = 2 \cdot w_i)$

$\wedge\ w_u = b$

If we translate $x \in y$ by $x E y$ (and $=$ by $=$) then PA proves all (translations of) fZFC ax's.

(1) – first prove (by ind.) that $\forall x\, \exists \max y,\ y E x$,

(2) – and then use ind. on $\max x$ to show that ax's like pairing $\exists\{x,y\}$, union $\exists \bigcup x$, power set $\exists P(x)$, comprehen $\exists\{y \in x \mid \varphi(y)\}$, etc. are satisfied.

(3) – The ~~...~~ negation of the ax. of ∞: follows from (1).

$\square$

Remark: ax. of choice (i.e. WO principle) follows as $\forall x\, \exists z \subseteq x^2$ the $<$ on x, where $u E v \iff u < v$, and $<$ is w.o. (by the LNP, i.e. by ind.).