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Horizon and stages in applications of stochastic programming in finance

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Abstract To solve a decision problem under uncertainty via stochastic programming means to choose or to build a suitable stochastic programming model taking into account the nature of the real-life problem, character of input data, availability of software and computer technology. In applications of multistage stochastic programs additional rather complicated modeling issues come to the fore. They concern the choice of the horizon, stages, methods for generating scenario trees, etc. We shall discuss briefly the ways of selecting horizon and stages in financial applications. In our numerical studies, we focus on alternative choices of stages and their impact on optimal first-stage solutions of bond portfolio optimization problems.

Keywords Stochastic dynamic optimization · Horizon · Stages · Bond portfolio management

AMS Subject classification 90C15 · 92B28

1. Introduction

In applications of stochastic programming for portfolio optimization one can exploit the standardized types of stochastic programming models (e.g., two-stage and multistage stochastic programs with recourse, models with individual and joint probabilistic constraints, integer stochastic programs) and the relevant software systems available or in progress; we refer to

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recent textbooks and monographs on stochastic programming (Birge and Louveaux, 1997; Kall and Wallace, 1994; Prékopa, 1995) (see also Preface of (Wets and Ziemba, 1999) for a list of books and collections of papers on stochastic programming and its applications), to introductory surveys on applications of stochastic programming in finance, e.g., (Mulvey, Rosenbaum, and Shetty, 1997; Mulvey and Ziemba, 1995) and to Part II of Dupačová, Hurt, and Štěpán (2002).

An important task is an adequate reflection of uncertainty and of the dynamic aspects, including further development of tractable numerical approaches. The probability distribution P is rarely known completely and/or it has to be approximated so that one mostly solves an approximate stochastic program instead of the underlying "true" decision problem. The task is to generate the required input, i.e., to approximate P bearing in mind the chosen type of the model; see e.g. (Dupačová, Consigli, and Wallace, 2000). The dynamic features are partly reflected by multiperiod stochastic programs, e.g. (Brodt, 1984; Dupačová and Bertocchi, 2001; Dupačová, Bertocchi, and Moriggia, 1998; Golub et al., 1995; Kusy and Ziemba, 1986; Nielsen and Zenios, 1996). Multistage models try to capture an additional important feature of the decision processes: One is allowed to use only the available (past) information and updates the decision when an additional information gets revealed. This nonanticipativity condition brings along further complexity, cf. (Dupačová, 2002): Besides the formulation of goals and constraints and identification of the driving random process $\omega = (\omega_1, \omega_2, \ldots)$, building a scenario-based multistage stochastic program requires specification of the horizon, stages and generation of the input in the form of scenario tree. In a majority of cases, the horizon and the stages are declared as given.

In financial models, the horizon may be often determined by the nature of the decision problem. It may be tied to a fixed date, e.g., to the end of the fiscal year, to a date related with the annual Board of Directors' meeting, or to the deadline for repayment of the debt (Dempster, and Ireland, 1988). Another possibility is to consider the horizon connected with a time interval of a fixed (possibly even infinite) length, given for instance by the periodicity of the underlying random process, such as the yearly cycle of a pension plan. We refer to the discussion in Mulvey and Ziemba (1995).

The problem may be solved repeatedly, with new horizons, taking into account just the already achieved state of the system, e.g. (Bradley and Crane, 1980; Cariño, Myers, and Ziemba, 1998). To guarantee the possibility of such continuation, the models are usually extended for additional constraints and/or terms in the objective function to reduce the end effects, with or without reference to an additional, auxiliary stage.

Rolling forward after the T-stage problem has been solved, a first-stage decision accepted and a new information obtained means to solve a subsequent $T-1$-stage stochastic program with a reduced number of stages or another T-stage problem with the initial state of the system determined by the applied first-stage decision and by an observation of ω_1, and using process ω shifted in time.

Also selection of stages results sometimes from the problem formulation (e.g., the dates of maturity of bonds (Frauendorfer and Marohn, 1998) or expiration dates of options) but more frequently, stages are fixed ad hoc, by application of heuristic rules and/or experience and regarding software and computer facilities.

Hence, for an already chosen horizon, the crucial step is to relate the time instants and stages. To use multiperiod two-stage model or to assign one stage to each of discretization points are two extreme cases. There are only a few papers which compare the optimal first-stage decisions in dependence on the *number of stages*. We refer to Kusy and Ziemba (1986) for a comparison of the two-stage model results with those for parallel three stage one, to Nielsen and Zenios (1996) comparing results for 1–5 stages formulation and to the recent paper (Nielsen and Poulsen, 2004). Some recommendations are common for financial applications (Cariño, Myers, and Ziemba, 1998; Mulvey and Ziemba, 1995): Accept unequal lengths of time periods between subsequent stages, starting with a short first period. Together with repeated rolling of the model over time, this may replace well the full dynamics of the decision process even for problems with a few stages. Another, general suggestion (Grinold, 1986) is to break the problem with a long (possibly infinite) horizon into three phases: To use the scenario tree structure for $1 \leq t \leq T_1$, to design just one descendant from each node for $T_1 + 1 \leq t \leq T$ (i.e., the horse-tail structure) and to aggregate the rest of the process into one additional stationary stage.

Starting with a given initial structure of the problem, one generates the input accordingly. This includes designing the *branching scheme* of the scenario tree, cf. (Dupačová, Consigli, and Wallace, 2000) and references ibidem. Evidently, one has to accept compromises between the size of the resulting problem and the desired precision of the results. A detailed analysis of the origin and of the initial structure of the solved problem may be exploited to aggregate the stages, may help to prune the tree or to extend it for other out-of-sample scenarios or branches. It is even possible to test the influence of including additional stages, e.g., using the contamination approach. We refer to Dupačová (1995, 1999); Dupačová, Bertocchi, and Moriggia (1998), to further papers in Ziemba and Mulvey (1999) and to various numerical studies.

The basic information on modeling horizon and stages in financial applications is summarized in Table 1, in the last column, R1 is used for rolling forward with a fixed horizon, R2 with a shorter horizon, YFF for end effect treatment according to Grinold (1986).

Before formulating further guidelines, more experience about influence of the chosen horizon and/or of its discretization into stages has to be collected using simulations, comparisons

Table 1 Survey of multistage models applied in finance

Paper	Problem	T	Stages–steplength	Comparisons	End effect
(Ainassari, Kallio, and Ranne, 1998)	Pension company	30Y	2Y, 3Y, 5Y, 2×10Y	Various rules	
(Bradley and Crane, 1980)	Bond portfolio	3Y	yearly	Ladder, barbell	R1
(Cariño, Myers, and Ziemba, 1998)	ALM - insurance	5Y	1Q, 3×1Q, 1Y, 3Y	Markowitz	YFF, R1
(Carino and Turner, 1998)	ALM	5Y	1Y, 2Y, 2Y	Fixed-mix	Bounds
(Consigli and Dempster, 1998)	Pension plan	10Y	Yearly		
(Dempster, and Ireland, 1988)	Debt financing	5Y	4× 1Q, 4× 1Y		
(Dert, 1998)	Pension plan	10Y	Yearly	Static, myopic	Bounds
(Fleten, Høyland, and Wallace, 2002)	ALM-insurance	3Y	Yearly	Fixed-mix	
(Kouwenberg, 1998)	Pension fund	5Y	Yearly	Fixed-mix	R2
(Nielsen and Poulsen, 2004)	Mortgagor's plan	30Y	Various, 3-8 stages		
(Zenios et al., 1998)	Financial product	3Y	18M, 18M	One-stage	

and validation of the obtained results. In this paper we focus on alternative choices of stages and their impact on optimal first-stage solutions of the bond portfolio problem we analyzed in our earlier papers such as (Bertocchi, Dupačová, and Moriggia, 2000; Dupačová and Bertocchi, 2001; Dupačová, Bertocchi, and Moriggia, 1998). The problem will be briefly introduced in the next Section and selected numerical results presented in Section 3.

2. The bond portfolio management model

Consider a bond portfolio of a risk neutral or risk averse institutional investor who wants to maximize the expected performance of the portfolio over time. In this paper, neither liabilities nor external cashflows are taken into account and the interest rate evolution is assumed to be the only factor which drives the prices of the considered default free bonds. It means that given a sequence of equilibrium future short rates r_t valid for the time interval $(t, t+1]$, $t = 0, \ldots, \tau - 1$, the fair price of the jth bond at time t just after the coupon was paid equals the value at t of the cashflows f_{jl} generated at $l = t+1, \ldots, \tau$:

$$P_{jt}(\boldsymbol{r}) = \sum_{l=1}^{\tau} f_{jl} \prod_{h=t}^{l-1} (1 + r_h)^{-1}, \tag{2.1}$$

where τ is greater or equal to the time to maturity. As the future interest rates are not known with certainty, we accept that they are random. We assume that their probability distribution is well approximated by a discrete probability distribution carried by a finite number of scenarios $\boldsymbol{r}^s$, $s = 1, \ldots, S$. We denote

$j = 1, \ldots, J$ indices of the bonds and T_j the dates of their maturities; $\tau = \max_j T_j$.
$t = 0, \ldots, T$ the discretization of the planning horizon;
$b_j \geq 0$ the initial holdings of bond j;
b_0 the initial holding in riskless asset;
f_{jt}^s cashflow generated under scenario s from the unit quantity of bond j at time t;
ξ_{jt}^s and ζ_{jt}^s are the selling and purchasing prices of bond j at time t for scenario s obtained from the corresponding fair prices (2.1) adding the accrued interest A_{jt}^s and subtracting or adding scenario independent transaction costs and spread; the initial prices ξ_{j0} and ζ_{j0} are known constants, i.e., scenario independent;
x_j/y_j are face values of bond j purchased/sold at the beginning of the planning period, i.e., at $t = 0$;
z_{j0} is the quantity of bond j held in portfolio after the initial decisions x_j, y_j have been made;
$x_{jt}^s, y_{jt}^s, z_{jt}^s$ are the corresponding values for period t under scenario s.

The first-stage decision variables x_j, y_j, z_{j0} are nonnegative,

$$y_j + z_{j0} = b_j + x_j \; \forall j \tag{2.2}$$

$$y_0^+ + \sum_j \zeta_{j0} x_j = b_0 + \sum_j \xi_{j0} y_j \tag{2.3}$$

where the nonnegative variable y_0^+ denotes the surplus. We assume a *positive market value of the initial portfolio*, $b_0 + \sum_j \xi_{j0} b_j > 0$ which implies that the set of the feasible first-stage solutions is nonempty and bounded.

The second-stage decisions on rebalancing the portfolio and reinvestment depend on individual scenarios. They have to fulfil constraints on conservation of holdings in each bond at each time period and for each of scenarios

$$z^s_{jt} + y^s_{jt} = z^s_{j,t-1} + x^s_{jt} \quad \forall j, s,\ t \geq 1 \tag{2.4}$$

$$\sum_j \xi^s_{jt} y^s_{jt} + \sum_j f^s_{jt} z^s_{j,t-1} + \left(1 - \delta + r^s_{t-1}\right) y^{+s}_{t-1} = \sum_j \zeta^s_{jt} x^s_{jt} + y^{+s}_t \quad \forall s, t \tag{2.5}$$

with $y^{+s}_0 = y^+_0$, $z^s_{j0} = z_{j0}\,\forall j, s$. Positive values of δ account for the difference between the returns for bonds and cash (y^{+s}_t).

The problem is maximizing the expected utility of the final wealth at the planning horizon T

$$\sum_s p_s U\left(W^s_T\right) \tag{2.6}$$

subject to (2.2)–(2.5) and nonnegativity constraints on all variables and with

$$W^s_T := \sum_j \xi^s_{jT} z^s_{jT} + y^{s+}_T. \tag{2.7}$$

Provided that an initial trading strategy determined by feasible, scenario independent first-stage decisions $x_j, y_j, \forall j$ has been accepted, the second-stage scenario dependent decisions must be made in an optimal way regarding the goal of the model – to maximize the final wealth subject to constraints on conservation of holdings, with possible rebalancing the portfolio:

$$\text{maximize } W^s_T := \sum_j \xi^s_{jT} z^s_{jT} + y^{s+}_T \tag{2.8}$$

subject to (2.4), (2.5) and nonnegativity constraints. Because of the possibility of reinvestments, the second-stage problem has always a feasible solution. Hence, this basic model is a *multiperiod two-stage stochastic programming model with random relatively complete recourse*. The main outcome is the optimal value of the objective function and the optimal values of the first-stage variables x_j, y_j (and y^+_0, z_{j0}) for all j. The optimal values of the second stage variables contribute to evaluation of the objective function (2.6). In a dynamic setting, the first-stage decision is applied at the beginning of the first period, and at the beginning of the next period, the model is solved again for the changed input information on holdings and on scenarios of interest rates; this is the *rolling forward* approach in the context of two-stage stochastic programming models, see (Kusy and Ziemba, 1986) for a detailed explanation of this idea.

The model may be extended for further constraints, such as allocation group restrictions (box constraints on holdings in specified assets categories – cash, medium term bonds, long bonds, etc.) or benchmark targets concerning wealth and/or rate of return (Carino and Turner, 1998). Another suggestion, see (Messina and Mitra, 1996), is to limit the gain or loss from intertemporal rebalancing by constraints

$$\sum_j \left|\xi^s_{j,t-1} - \xi^s_{jt}\right| y^s_{jt} \leq V_t, \quad \forall s,\ t \tag{2.9}$$

or to limit the portfolio duration, e.g., by

$$D^* - \Delta \cdot D^* \le \sum_{j=1}^{J} z_{0j} D_j \le D^* + \Delta \cdot D^* \tag{2.10}$$

where

$$D_j = \sum_{t=1}^{T} \frac{t f_{jt}}{(1+\hat{r}_j)^{t+1}}, \quad j = 1, \dots J \tag{2.11}$$

is the dollar duration of bond j, $\hat{r}_j$ its yield to maturity,

$$D^* = \sum_{j=1}^{J} b_{0j} D_j, \tag{2.12}$$

is the initial portfolio duration and Δ is the chosen tolerance parameter.

Another possibility is to design a *multistage model*. This means that another *true* rebalancing based on a newly revealed information is allowed in one or more latter periods. Such information can be the recognition of increasing or decreasing interest rates, the outcome concerning an exercise of option, etc. Then it is necessary to design also the input in the form of a scenario tree along with the path and arc probabilities and to incorporate the nonanticipativity constraints. With reference to Dupačová (2000) we shall discuss now one of relevant multistage reformulations.

Example . Assume that r_0 is the known interest rate for the first period $(0, 1]$, $r_1^s, 1 \le s \le S$, the considered realizations of the interest rate valid for the period $(1, 2]$, with probabilities $p_s > 0$, $\sum_s p_s = 1$ and denote $\mathcal{D}(s)$ the set of descendants of scenario s – the set of rates for the subsequent time periods $r_t^{s\sigma}$ with conditional (arc) probabilities $\pi_\sigma > 0, \sigma \in \mathcal{D}(s)$, $\sum_{\sigma \in \mathcal{D}(s)} \pi_\sigma = 1$. The expected optimal outcome of the second rebalancing strategy $\boldsymbol{x}_1^s, \boldsymbol{y}_1^s$ $(\boldsymbol{z}_1^s, y_1^{+s})$ (in its dependence on a feasible first-stage decision $\boldsymbol{x}, \boldsymbol{y}$) is evaluated along the branch of scenarios $r_t^{s\sigma}$, $2 \le t \le T-1, \sigma \in \mathcal{D}(s)$ emanating from r_1^s and equals the *optimal value* $W_T^s(\boldsymbol{x}, \boldsymbol{y}, z_0, y_0^+)$ of the following two-stage stochastic linear program (compare with (2.2)–(2.7)):

$$W_T^s\big(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}_0, y_0^+\big) := \max \sum_{\sigma \in \mathcal{D}(s)} \pi_\sigma \left[\sum_j \xi_{jT}^{s\sigma} z_{jT}^{s\sigma} + y_T^{s\sigma} \right] \tag{2.13}$$

subject to

$$y_{j1}^s + z_{j1}^s = z_{j0} + x_{j1}^s \, \forall j \tag{2.14}$$

$$y_1^{+s} + \sum_j \zeta_{j1}^s x_{j1}^s = (1 - \delta + r_0) y_0^+ + \sum_j \xi_{j1}^s y_{j1}^s + \sum_j f_{j1} z_{j0} \tag{2.15}$$

$$z_{jt}^{s\sigma} + y_{jt}^{s\sigma} = z_{j,t-1}^{s\sigma} + x_{jt}^{s\sigma} \quad \forall j, 2 \le t \le T, \sigma \in \mathcal{D}(s), \tag{2.16}$$

$$\sum_j \xi_{jt}^{s\sigma} y_{jt}^{s\sigma} + \sum_j f_{jt} z_{j,t-1}^{s\sigma} + \big(1 - \delta + r_{t-1}^{s\sigma}\big) y_{t-1}^{+s\sigma}$$

$$= \sum_j \zeta_{jt}^{s\sigma} x_{jt}^{s\sigma} + y_t^{+s}, \quad 2 \le t \le T, \sigma \in \mathcal{D}(s), \tag{2.17}$$

with $r_1^{s\sigma} = r_1^s$, $y_1^{+s\sigma} = y_1^{+s}$, $z_{j1}^{s\sigma} = z_{j1}^s \; \forall j, \sigma \in \mathcal{D}(s)$ and under nonnegativity of all variables.

The full stochastic program can be written in the form

$$\text{maximize} \sum_s p_s U\big(W_T^s\big(\boldsymbol{x}, \boldsymbol{y}, z_0, y_0^+\big)\big) \tag{2.18}$$

subject to (2.2), (2.3) and nonnegativity constraints.

The prices $\xi_{jt}^{s\sigma}$, $\zeta_{jt}^{s\sigma}$ are obtained from the fair prices $P_{jt}(\boldsymbol{r}^{s\sigma})$ computed according to (2.1) for $t \geq 2$ and $\sigma \in \mathcal{D}(s)$. The fair prices P_{j1} follow by the expectation hypothesis: They are equal to the discounted expected value of the sum of cashflows f_{j2} due at $t = 2$ and fair prices $P_{j2}(\boldsymbol{r}^{s\sigma})$:

$$P_{j1}^s = (1 + r_1^s)^{-1} \sum_{\sigma \in \mathcal{D}(s)} \pi_\sigma (f_{j2} + P_{j2}(\boldsymbol{r}^{s\sigma})). \tag{2.19}$$

Notice, that this is the way how to get the required tree structure of the model coefficients. In comparison with the initial two-stage problem (2.2)–(2.3), (2.6) with W_T^s defined by (2.7) using the above arborescent form of the three stage problem results in a decrease of the size of the problem as to the number of variables. Based on the tree structure of coefficients, the three stage problem (2.2)–(2.3), (2.18) with W_T^s defined by (2.13)–(2.17) can be also written in the split variable form which formally corresponds to the two-stage formulation complemented for explicit nonanticipativity constraints. This formulation evidently means an extension of the size of the deterministic problem to be solved.

Similar formulas for prices apply also for other instances of scenario tree structures. However if the branchings appear after several time periods one has to apply specific rules for the interstage portfolio management. This is mostly the buy-and-hold strategy with accumulation of cash and coupons according to interest rates. In our case, no bond matures within the time horizon, so only coupons are transfered to cash account. Hence, the buy-and-hold strategy applied between stages, say, between $t = t_1$ and $t = t_2$ means that for all scenarios $x_{jt}^s = 0$, $y_{jt}^s = 0$, $t_1 \leq t \leq t_2 \; \forall j$, so that the holdings are fixed, $z_{jt}^s = z_{j1}^s \; \forall s,\ j$ and $t_1 \leq t \leq t_2$ and prices ξ_{jt}^s, ζ_{jt}^s, $t_1 \leq t \leq t_2$ are irrelevant. Another possibility would be to allow for reinvesting coupons into the security which provided them.

3. Input data and numerical experiments

In this section, numerical experiments are reported. The model is implemented in GAMS v. 19.4, using CPLEX as the linear programming solver on a Pentium 400 MHz machine with 128 Mbytes of RAM running under Windows NT 4.0 SP6. In our experiments we start with an initial portfolio designed by experts, see Table 2, and we use the planning horizon of 1 year. Initial portfolio market value is equal to 10484.55 and the initial portfolio dollar duration is 36407.7.

The presented results exploit the Black, Derman, and Toy (1990) method for generation interest rate scenarios; this means that the interest rates r_t^s valid in the interval $(t, t+1]$, $t > 0$ are expressed as

$$r_t^s = r_{t0} k_t^{i_t(s)} \tag{3.1}$$

Table 2 Portfolio Composition on October 3rd 1994. Initial portfolio market value: 10484.55, Initial portfolio dollar duration: 36407.7

Bonds	Qt	Coupon	Payment dates	Exercise	Redemp	Maturity
BTP36658	10	3.9375	01Apr & 01Oct		100.187	01Oct96
BTP36631	20	5.0312	01Mar & 01Sep		99.531	01Mar98
BTP12687	15	5.2500	01Jan & 01Jul		99.231	01Jan02
BTP36693	10	3.7187	01Aug & 01Feb		99.387	01Aug04
BTP36665	5	3.9375	01May & 01Nov		99.218	01Nov23
CTO13212	20	5.2500	20Jan & 20Jul	20Jan95	100.000	20Jan98
CTO36608	20	5.2500	19May & 19Nov	19May95	99.950	19May98

where $i_t(s) = \sum_{\tau=1}^{t} w_\tau^s$ is the scenario dependent position on the lattice (see Figure 1) that equals the number of up moves, coded by $w_\tau^s = 1$, for the given scenario s which occur at time points $\tau = 1, \ldots, t$.

The scenario independent quantities r_{t0}, the lowest short rate that may occur for period $(t, t+1]$, and the lattice volatilities k_t are obtained by calibration of the Black-Derman-Toy (BDT) model by the (estimated) market structure of October 3, 1994. All these steps lead to the fitted binomial lattice which provides different 2^T scenario of interest rates where T is the longest bond maturity in the portfolio. A smaller, manageable number of scenarios have to be selected or sampled from this large set.

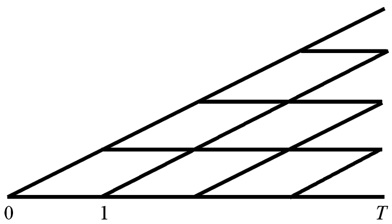
The sampling strategy was chosen between different alternatives:

– use of Zenios and Shtilman (1993) non random sampling strategy with different length in covering fully the beginning of the lattice, and proceeding with alternative up-down movements; the acronyms are ZS (No. of scenarios);
– use of 8 particular scenarios along the planning horizon–12 months as reported in Figure 2 and proceed with alternative up-down movements; the acronym is Part (8).

Except for the initial market price of the bonds, which is observable, the fair price of the jth bond at time t just after the coupon was paid and under scenario s follows for each scenario according formula (2.1).

Several papers were devoted to the model performance wrt. transaction costs, spread, etc., to sensitivity of results on selection of representative scenarios of interest rates and out-of-sample behavior, influence of errors due to input data and robustness of the first-stage decision wrt. to perturbations of the estimated term structure, (see e.g. Bertocchi, Dupačová, and Moriggia, 2000; Dupačová, 2000; Dupačová and Bertocchi, 2001; Dupačová, Bertocchi, and Moriggia, 1998). In this paper we focus on comparisons of the two-stage and the multistage version of the model introduced in Section 2.

Fig. 1 Full Lattice



	Investment time horizon													
Scenario	1	2	3	4	5	6	7	8	9	10	11	12	13	continue
0	down	down	down	down	down	down	down	down	down	down	down	down	down	up&down
1	down	down	down	up	down	down	down	down	down	up	down	down	down	up&down
2	down	up	down	down	down	up	down	up	down	down	down	up	down	up&down
3	up	down	up	down	up	down	up	down	up	down	up	down	up	up&down
4	up	down	up	down	up	up	down	up	up	down	up	down	up	up&down
5	up	down	up	up	up	down	up	down	up	up	up	down	up	up&down
6	up	up	up	down	up	up	up	up	up	down	up	up	up	up&down
7	up	up	up	up	up	up	up	up	up	up	up	up	up	up&down

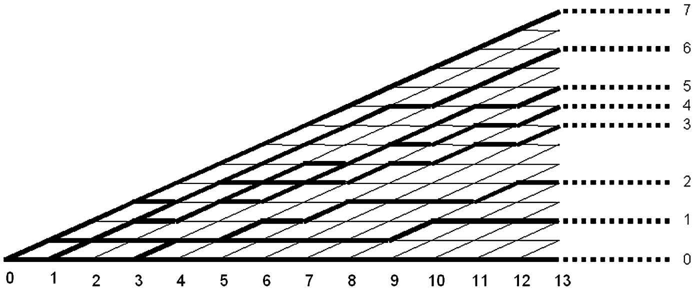


Fig. 2 Part (8) sampling strategy

3.1. Selected results for multiperiod two-stage formulation with monthly step

The first numerical experiments are related to monthly discretization on a 12 months horizon.

We observe that for BDT lattice with monthly step the full problem is based on $2^{12} = 4096$ scenarios (not counting additional scenarios used for evaluation of fair prices (2.1) at the horizon).This gives more than one million of variables and almost 400 000 equations in (2.2)–(2.5). The optimal value for *linear* utility function is 11436.56, the optimal first-stage solution is to keep the shortest (BTP36658) and the medium bonds (BTP12687) and to invest in CTO 13212 (76.66). The computing time is more than 3 days.

We solved several problems of reduced size using different strategies for choosing *subsets of scenarios*.

a) Using sample Part(8) of 8 scenarios leads to the optimal value 11531 and the optimal first-stage decision is to invest all (100.9) in the puttable bond CTO 13212. Moreover, this portfolio persists up to the 4th period under all considered scenarios.
b) Using sample ZS(8) of 8 scenarios leads to the optimal value 11512 and the optimal first-stage decision is to invest all (101.25) in to BTP12687. Moreover, this portfolio persists for $t = 1, 2$ under all considered scenarios. The reason of such difference between the cases a) and b) comes from the fact that for ZS(8) the embedded put option of the puttable bond, CTO 13212, is exercised only for one scenario while for PART(8) this happens for four scenarios.
c) Using sample ZS(16) of 16 scenarios leads to the optimal value 11437 and the optimal first-stage decision keeps BTP36658, BTP12687, CTO 13212 and cash 5884 which is similar to results for the full problem. For the second period there are differences in strategies between the up scenario and the down scenario groups. See the difference between scenarios 2 and 6 (down scenarios), 11 and 13 (up scenarios) in Figure 3.

The expected values of perfect information (EVPI) and the value of stochastic solution (VSS) for the different cases of sampling are illustrated in Table 3.

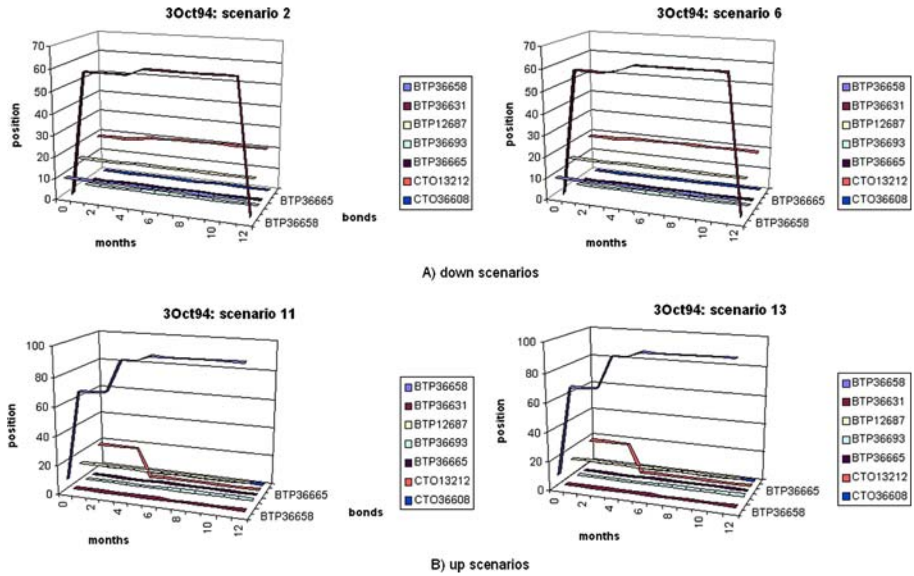


Fig. 3 Different strategies within up/down scenarios

The results suggest, as already pointed out in Bertocchi, Dupačová, and Moriggia (2000), that Part(8) sampling strategy seems to represent better the uncertainty in comparison with the remaining two sampling strategies.

d) Having in mind the results of Shapiro, Homem-de-Mello, and Kim (2002), we explored also the possibility of solving empirical stochastic programs based on randomly sampled scenarios from the full binomial lattice. According to Example 2 in Dupačová (1999) or Application 1 in Dupačová (2001), if the (true) first-stage optimal solution based on the *full* BDT lattice is unique and the sample size N is large enough, the obtained first-stage empirical optimal solutions are equal with probability 1 to the true optimal solution. Moreover, with increasing N, the convergence is exponentially fast and the required sample size is related to the condition number of the stochastic program in question, cf. (Shapiro, Homem-de-Mello, and Kim, 2002). To estimate it for our problem we applied the two-stage procedure of Shapiro, Homem-de-Mello, and Kim (2002) as follows: By solving the problem for randomly sampled $N_0 = 64$ scenarios among 4096 for $R = 100$ times, the most frequent optimal first-stage solution (the candidate true solution)–to invest all in BTP12687–was obtained in $R_0 = 79$ cases. This provides an estimate $\alpha_0 = 1 - R_0/R = 0.21$ of the probability that the optimal empirical solution differs from the true solution. With z^*, the solution of $z + \log z = \log \frac{1}{2\pi{\alpha_0}^2}$, we get an estimate of the condition number of our problem as $\hat{k} = \frac{N_0}{z^*} = 90.45$. This implies that, if we want a confidence level of $1 - \alpha = 99\%$ that the true solution is obtained by solving an empirical problem, it is necessary to use (in the worst case) the number of randomly sampled scenarios

Table 3 EVPI and VSS values for the two-stage problem; monthly discretization

Uncertainty measures	Part(8)	ZS(8)	ZS(16)
EVPI	529	47	79
VSS	2290	1706	2753

$N \geq z_1^* \cdot \hat{k} = 249$ where z_1^* is the solution of $z_1 + \log z_1 = \log \frac{1}{2\pi\alpha^2}$. This number is quite far away from 4096 which may be considered as a good sign that our problem is not ill-conditioned.

3.2. Selected results for multiperiod two-stage problem with quarterly steps

Another possibility to reduce the problem dimension is by using time aggregation. It looks reasonable to use quarterly steps such that the BDT lattice is based now only on 2^4 scenarios (i.e., a substantial *time aggregation*) and the size of the problem allows for various experiments with the full problem: linear versus nonlinear utility function, additional constraints on cash holdings and on duration.

We mainly check our model behavior in the following five cases.

1. For the *linear case* with no additional constraint, the optimal value is 11440 and the optimal first-stage solution is to sell all bonds and to keep cash 10474. This changes by the second period decision in a different way for different scenarios, namely investments are in the puttable bond CTO 13212 or in BTP36631.

   For the subsequent periods, we observed multiple optimal solutions. The number of variables is 1257 and the number of constraints is 867, computing time is .51 secs under GAMS-MINOS solver. Notice that with this small dimension it is absolutely indifferent using MINOS or CPLEX solver.
2. For the linear case with the additional bound 1000 on maximal cash holdings the optimal value decreases to 11397 and the optimal first-stage decision keeps BTP36658 at the initial level, increases holding of CTO 13212 to 81.95, sells other bonds to reach the limit 1000 for cash holdings. The optimal decisions for the subsequent periods depend on scenarios.
3. For the linear case we also introduced additional constraint on portfolio dollar duration for the first-stage decision variables following formulas (2.10)–(2.12). Considering our portfolio, we had to take into account that computing the duration of a puttable bond implies two questions: which yield to maturity and which cashflows have to be used? It seems to be reasonable to restrict to the worst case that the holder of puttable bond has to deal with, i.e., to use the yield to maturity as if the option would be exercised at the exercise time and to use the cashflows as if the option would never be exercised.

   The results with $\Delta = 5\%$ lead to an optimal value of 11378.09; the first-stage optimal decisions are to invest in CTO 13212 (98.7) and in the longest BTP36665 (2.87). Increasing Δ to 10%, the optimal value changes to 11393 and the optimal-first stage decisions stay in CTO 13212 (99.63) and in cash 132.35.
4. Several experiments with various utility functions were done; the results differ from those of the linear case and depend on the choice of the utility function, of course. However, there are still many questions to be investigated, such as the choice of the initial starting point, or possibility of unfeasible solutions and local maxima.
5. We adopt two approaches for rolling forward technique with duration constraints.

   (a) Having a horizon of 12 months corresponding to 4 quarters, we solved a two-stage quarterly model on October 3, 1994 over the whole planning horizon (meaning 16 scenarios). Then we solved a two-stage quarterly model over 9 months periods (meaning 8 scenarios) starting with the new rebalanced portfolio and for interest rates from the BDT lattice fitted to the market data of January 3, 1995. The optimal value of the portfolio at the planning horizon 11398 was achieved for everything invested in CTO 13212 (100.72).

(b) We also solved on October 3, 1994 a two-stage quarterly model on 9 months period (meaning 8 scenarios). With the new rebalanced portfolio and for interest rates from the BDT lattice fitted to the market data of January 3, 1995 we run again the model with 9 months horizon. The optimal value of the portfolio at the planning horizon 11375 came again for everything invested in CTO 13212 (100.53).

We observe that in both cases new information obtained after the first quarter influenced the optimal rebalancing of portfolio in a similar way in spite of the fact that a different number of scenarios (16 in case a), 8 in case b)) was used.

Let us consider the solution of the problem with 4096 scenarios as the "true" solution and compare the other results. None of the sampling strategies gives a solution sufficiently close to the true solution. However, aggregation in time, i.e., the quarterly step, gives an optimal function value sufficiently close to the true optimal value. Introducing additional ad hoc constraints also creates some dynamics in the portfolio choice.

3.3. Numerical results for three stage model

In the case of the BDT lattice, there is a natural branching of the interest rate process in the first period, namely to the "up" scenarios and to the "down" scenarios, (see Figure 4). With one month step it suggests to consider rebalancing after the 1st month, whereas with 3 months step rebalancing would take place only after 3 months. If further rebalancings take place after the same step length, the lattice is acceptable. If rebalancings occur only at some multiples of the step length, it is necessary to include a trading strategy (buy-and-hold, or fix-mix) for the intermediate time steps. The last part of the tree is the "horse tail", i.e., after the last branching up to the horizon a (possibly multiperiod) two-stage problem remains.

Even in this case, the whole input of all coefficients has to be adjusted to the chosen tree structure. This means that at branching points, expectation hypothesis is used to get uniquelly defined selling and purchasing prices by averaging the prices obtained by discounting along scenarios belonging into the given set of descendants, recall (2.19).

Notice, that with interest rate scenarios for the two-stage stochastic program simulated from a continuous time model, for instance (Cox, Ingersoll, and Ross, 1985), the choice of the first branching point cannot be based only on the input scenarios.

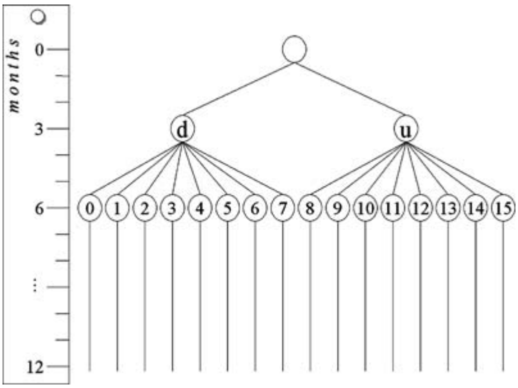


Fig. 4 Scenarios of three-stage model

3.4. Full lattice with monthly steps and branching after the first month

Comparing the results with those obtained for the multiperiod two-stage formulation with monthly step, we observe that number of variables is decreased below one million: 995395 and also the number of equations decreases to 364569. Consequently, the CPU time decreased to approximately two days. The optimal value and the first-stage optimal decisions are the same as in the case of the full two-stage problem with monthly step. Hence, there is a robustness

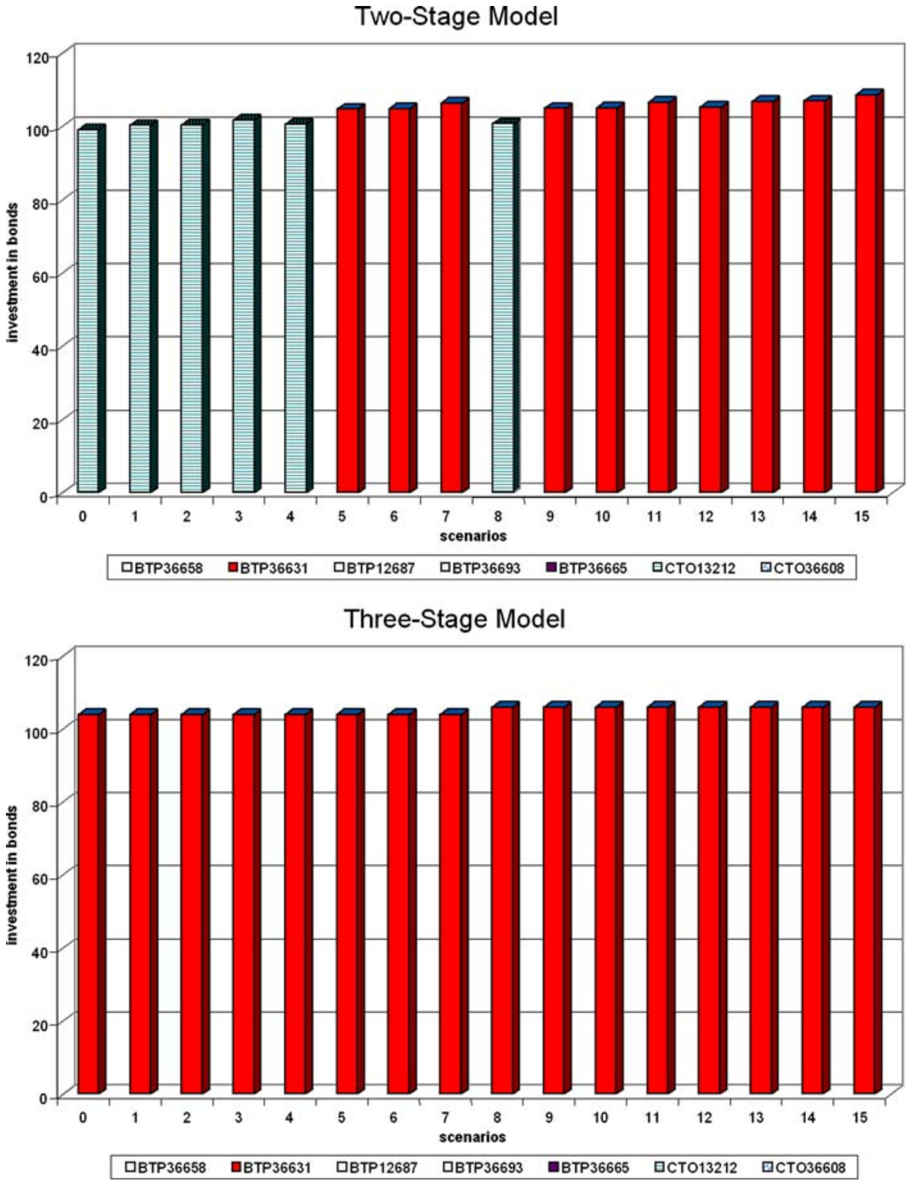


Fig. 5 Second stage decisions variables

of trading decisions for these two different stage models, the optimal final wealth remains stable, the optimal first-stage decisions are identical.

3.5. Full lattice with quarterly steps and branching after the first 3 months period

The optimal value decreases to 11440, the optimal first-stage solution does not change with respect to the multiperiod two-stage model with quarterly steps, but the subsequent trading strategy is robust, grouping in a natural way, with all investments going to the BTP36631 (different holdings for the two groups of scenarios). This was not the case for the two-stage model, (see Figure 5).

In presence of the cash constraint, both the optimal value and the optimal first-stage decision are kept, and the character of the subsequent decisions does not change either.

4. Conclusions

There is a natural aspect of dynamics in many real-life problems (see Table 1). To reflect it means to fix not only the horizon but also the rebalancing points (branching points) and to *create* the scenario tree for all scenario dependent coefficients accordingly. Moreover, when using the arborescent form of the multistage problem, the number of variables and constraints gets reduced. Nevertheless, to set an appropriate SP model for financial applications, it is necessary to analyze carefully the output for differently chosen horizons and/or different discretizations into stages and to investigate the use of non linear utility functions.

In our problem, the three stage model seems to reduce volatility of trading strategies–close (identical) partial scenarios of interest rates (prices) result in similar trading strategies. This is not the case for variants of the two-stage model ZS(16) with monthly step or of the two-stage model with quarterly step. Additional constraints force more realistic, diversified decisions for a small decrease of portfolio value; moreover, the duration criterion is in agreement with the common techniques of risk management.

The illustrative application of the two rolling forward techniques shows the impact of new information and of number of scenarios; the technique captures dynamic aspects for a reduced computational effort.

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References

Ainassari, K., M. Kallio, and A. Ranne. (1998). "Selecting an Optimal Investment Portfolio for a Pension Insurance Company." Papers of the 8th AFIR Colloquium, pp. 7–23.

Bertocchi, M., J. Dupačová and V. Moriggia. (2000). "Sensitivity of Bond Portfolio's Behavior with Respect to Random Movements in Yield Curve: A Simulation Study." *Ann. Oper. Res.* 99, 267–286.

Birge, J.R. and F. Louveaux. (1997). *Introduction to Stochastic Programming*, Springer, New York.

Black, F., E. Derman, and W. Toy. (1990). "A One-Factor Model of Interest Rates and Its Application to Treasury Bond Options." *A Financial Analysts Journal*, pp. 33–39.

Bradley, S.P. and D.B. Crane. (1980). "Managing a Bank Portfolio Over Time." In M.A.H. Dempster (ed.), pp. 449–471.

Brodt, A.I. (1984). "Intertemporal Bank Asset and Liability Management." *J. Bank. Res.* 15, 82–94.

Cariño, D.R., D.H. Myers, and W.T. Ziemba. (1998). "Concepts, Technical Issues, and Uses of the Russell-Yasuda Kasai Financial Planning Model." *Oper. Res.* 46, 450–462.

Carino, D.R. and A.L. Turner. (1998). "Multiperiod Asset Allocation with Derivative Assets." In W.T. Ziemba and J. Mulvey (eds.), pp. 182–204.

Consigli, G. and M.A.H. Dempster. (1998). "Dynamic Stochastic Programming for Asset-Liability Management." *Ann. Oper. Res.* 81, 131–161.

Cox, J.C., J.E. Jr. Ingersoll, and S.A. Ross. (1985). "A Theory of Term Structure of Interest Rates," *Econometrica* 53, 385–407.

Dempster, M.A.H. (ed.), (1980). *Stochastic Programming*, Academic Press, London.

Dempster, M.A.H. and A.M. Ireland. (1988). "A Financial Expert Decision Support System." In *Mathematical Models for Decision Support*, G. Mitra (ed.), NATO ASI Series 48, pp. 415–440.

Dert, C.L. (1998). "A Dynamic Model for Asset Liability Management for Defined Benefit Pension Funds." In W.T. Ziemba and J. Mulvey (eds.), pp. 501–536.

Dupačová, J. (1995). "Postoptimality for Multistage Stochastic Linear Programs." *Ann. Oper. Res.* 56, 65–78.

Dupačová, J. (1999). "Portfolio Optimization Via Stochastic Programming: Methods of Output Analysis." *MMOR* 50, 245–270.

Dupačová, J. (2000). "Stability Properties of a Bond Portfolio Management Problem." *Ann. Oper. Res.* 99, 251–265.

Dupačová, J. (2000). "Horizon and Stages in Applications of Stochastic Programming in Finance." WP 30, University of Bergamo.

Dupačová, J. (2001). "Output Analysis for Approximated Stochastic Programs." In *Stochastic Optimization: Algorithms and Applications*, S. Uryasev and P. M. Pardalos (eds.), Kluwer Acad. Publ., pp. 1–29.

Dupačová, J. (2002). "Applications of Stochastic Programming: Achievements and Questions." *European J. Oper. Res.* 140, 281–290.

Dupačová, J. and M. Bertocchi. (2001). "From Data to Model and Back to Data: A Bond Portfolio Management Problem." *European J. Oper. Res.* 134, 261–278.

Dupačová, J., M. Bertocchi, and V. Moriggia. (1998). "Postoptimality for Scenario Based Financial Models with an Application to Bond Portfolio Management." In W.T. Ziemba and J. Mulvey (eds.), pp. 263–285.

Dupačová, J., G. Consigli, and S.W. Wallace. (2000). "Scenarios for Multistage Stochastic Programs." *Ann. Oper. Res* 100, 25–53.

Dupačová, J., J. Hurt, and J. Štěpán. (2002). *Stochastic Modeling in Economics and Finance*, Kluwer Acad. Publ., Dordrecht.

Fleten, S.E., K. Høyland, and S.W. Wallace. (2002). "The Performance of Stochastic Dynamic and Fixed Mix Portfolio Models." *European J. Oper. Res.* 140, 37–49.

Frauendorfer, K. and Ch. Marohn. (1998). "Refinement Issues in Stochastic Multistage Linear Programming." *Stochastic Programming Methods and Technical Applications*, In K. Marti and P. Kall (eds.), LNEMS 458, pp. 305–328.

Golub, B. et al. (1995). "Stochastic Programming Models for Portfolio Optimization with Mortgage-Backed Securities." *European J. Oper. Res.* 82, 282–296.

Grinold, R.C. (1986). "Infinite Horizon Stochastic Programs." *SIAM J. Control Optim.* 24, 1246–1260.

Kall, P. and S.W. Wallace. (1994). *Stochastic Programming*, Wiley, Chichester.

Kouwenberg, R.R.P. (1998). "Scenario Generation and Stochastic Programming Models for Asset Liability Management." *European J. Oper. Res.* 134, 279–292.

Kusy, M.I. and W.T. Ziemba. (1986). "A Bank Asset and Liability Management Model." *Oper. Res.* 34, 356–376.

Messina, E. and G. Mitra. (1996). "Modelling and Analysis of Multistage Stochastic Programmnig Problems: A Software Environment." *European J. Oper. Res.* 101, 343–359.

Mulvey, J.M., D.P. Rosenbaum, and B. Shetty. (1997). "Strategic Financial Risk Management and Operations Research." *European J. Oper. Res.* 97, 1–16.

Mulvey, J.M. and W.T. Ziemba. (1995). "Asset and Liability Allocation in a Global Environment." Chapter 15 *Handbooks in OR& MS* 9, In R. Jarrow et al. (eds.), Elsevier.

Nielsen, S.S. and R. Poulsen. (2004). "A Two-Factor, Stochastic Programming Model of Danish Mortgage-Backed Securities, *J. Econ. Dynamics and Control* 28(7), 1267–1289, Elsevier.

Nielsen, S.S. and S.A. Zenios. (1996). "A Stochastic Programming Model for Funding Single Premium Deferred Securities." *Math. Programming* 75, 177–200.

Prékopa, A. (1995). *Stochastic Programming*, Kluwer, Dordrecht and Académiai Kiadó, Budapest.

Shapiro, A., T. Homem-de-Mello, and J. Kim. (2002). "Conditioning of Convex Piecewise Linear Stochastic Programs." *Math. Programming* A94, 1–19.

Wets, R.J.-B. and W.T. Ziemba. (eds.) (1999). *Stochastic Programming. State of the Art, 1998. Ann. Oper. Res.* 85.

Zenios, S.A. and M.S. Shtilman. (1993). "Constructing Optimal Samples from a Binomial Lattice." *J. Inform. Optim. Sci.* 14, 125–147.

Zenios, S.A. et al. (1998). "Dynamic Models for Fixed-Income Portfolio Management under Uncertainty." *J. Econ. Dynamics Control* 22, 1517–1541.

Ziemba, W.T. and J. Mulvey. (eds.) (1998). *World Wide Asset and Liability Modeling*, Cambridge Univ. Press.