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STABILITY IN STOCHASTIC PROGRAMMING WITH RECOURSE - ESTIMATED PARAMETERS

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In this paper, stability of the optimal solution of stochastic programs with recourse with respect to parameters of the given distribution of random coefficients is studied. Provided that the set of admissible solutions is defined by equality constraints only, asymptotical normality of the optimal solution follows by standard methods. If nonnegativity constraints are taken into account the problem is solved under assumption of strict complementarity known from the theory of nonlinear programming (Theorem 1). The general results are applied to the simple recourse problem with random right-hand sides under various assumptions on the underlying distribution (Theorems 2–4).

Key words: Stochastic Programming, Estimation, Stability, Asymptotical Normality, Minimax Approach, Deterministic Equivalent, Simple Recourse Problem.

1. Introduction

When solving stochastic programs, complete knowledge of the distribution of random coefficients is usually supposed. In real-life situations, however, this assumption is hardly acceptable and the common procedures should be at least supplemented by proper stability studies.

Consider the following stochastic program with recourse:

$$\text{maximize} \quad E_F\{c^{\mathrm{T}}x - \varphi(x; A, b)\} \quad \text{on the set } \mathscr{X} \tag{1}$$

where $\mathscr{X}$ is a set of admissible solutions. An example of (1) is when a linear program

$$\text{maximize} \quad c^{\mathrm{T}}x$$

$$\text{subject to} \quad Ax \leq b, \quad x \geq 0,$$

has some of components of the m-vector b, n-vector c or (m, n)-matrix A random.

Assume

(i) For fixed A, b, $\varphi(x; A, b)$ is a nonnegative convex function of x.
(ii) For arbitrary $x \in \mathscr{X}$, $\varphi(x; A, b)$ is a convex function of A, b.
(iii) $\mathscr{X} \subset \mathbb{R}^n$ is a nonempty closed convex set.

Provided that the joint distribution F of random coefficients is known, (1) is in principle reducible to a nonlinear deterministic program. Such programs have been studied by many authors from many different viewpoints (see e.g. [11, 14]). Their explicit form as well as their optimal solution depend on the given distribution F. In this paper uncertainty with respect to the distribution F will be taken into account.

A first idea could be to study stability of the optimal solution of program (1) with respect to the underlying distribution directly. To a certain extent, it can be done using empirical distributions [15] or the concept of ε-contamination (see [6, 7, 8]). In this paper, stability of the optimal solution of program (1) with respect to the parameters of the distribution F will be studied. Two alternatives will be considered:

I. The distribution F belongs to a given parametric family of distributions.

II. The distribution F belongs to a specified set of distributions defined by prescribed values of certain moments.

In Case I, stability of the optimal solution with respect to the parameters and related statistical problems were studied for the simple recourse problem with normally distributed right-hand sides b_i, $1 \leq i \leq m$. (See e.g. [6, 9, 16].) On the following numerical example ([4, 13]) dependence of the optimal solution on the parameter p of the symmetrical beta distribution is illustrated:

Example. The numerical data concern a two-stage stochastic production program with simple recourse. The four random right-hand sides are supposed to have marginal distributions $B(p, p)$ (with the same value of parameter p) on given intervals. The nonzero components of the optimal solution are given below.

Table 1

p	x_1	x_2	x_4	x_6	x_8	x_{10}	x_{12}	x_{13}	x_{14}
0	61.24	—	—	164.11	—	165.26	—	141.91	—
$\frac{1}{8}$	70.93	0.64	—	160.34	2.50	183.62	5.03	172.25	0.47
$\frac{1}{4}$	72.86	0.60	—	163.33	9.38	184.17	12.93	178.90	1.05
$\frac{1}{2}$	78.12	1.78	—	73.36	114.54	76.71	136.48	195.35	9.31
1	76.25	5.09	4.47	14.10	195.15	17.35	229.63	176.31	52.55
2	61.95	16.81	19.02	22.56	214.48	22.90	264.20	54.54	251.30
4	32.05	33.37	40.40	22.24	230.35	22.66	290.17	48.91	243.04
8	37.56	57.53	15.09	18.48	245.67	18.48	309.34	34.50	274.86

Let us summarize the problem we face in Case I. Our aim is to solve the program

$$\text{maximize} \quad f(x; \eta) \quad \text{on the set } \mathscr{X} \tag{2}$$

where

$$f(x; \eta) = E_{F_\eta}\{c^{\mathrm{T}}x - \varphi(x; A, b)\} \tag{3}$$

and η is the true parameter vector of the distribution F. If η is not known precisely, it is substituted by an estimate, say y, and the substitute program

$$\text{maximize} \quad f(x; y) \quad \text{on the set } \mathscr{X} \tag{4}$$

is solved instead of (2).

In Case II, one admits that the knowledge of the distribution F is not complete but limited to the fact that the distribution F belongs to a given set $\mathscr{F}$ of distributions. One approach is via minimax [17]; any optimal solution of

$$\underset{x\in\mathscr{X}}{\text{maximize}} \quad f(x)=\min_{F\in\mathscr{F}} E_F\{c^{\mathrm{T}}x-\varphi(x;A,b)\} \tag{5}$$

will be called the minimax solution of stochastic program (1).

For solving problem (5), general results concerned with the moment problem can be used provided that the set $\mathscr{F}$ is defined by prescribed values, η, of certain moments of distributions $F\in\mathscr{F}$:

Let $Z\subset\mathbb{R}^l$ and $g=(g_1,\dots,g_k):Z\to\mathbb{R}^k$, $h:Z\to\mathbb{R}^1$ be a Borel measurable mapping and function, respectively. Denote $g(Z)$ the image of the set Z in mapping g, $Y=\operatorname{conv} g(Z)$ and assume that $\operatorname{int} Y\neq\emptyset$. For $\eta\in\operatorname{int} Y$ denote by $\mathscr{F}_\eta$ the set of distributions of a random vector z on $(Z,\mathscr{B}_Z)$ such that $g_1,\dots,g_k, h$ are integrable with respect to all $F\in\mathscr{F}_\eta$ and

$$E_F g(z)=\eta \quad \forall F\in\mathscr{F}_\eta. \tag{6}$$

The problem is

$$U(\eta)=\sup_{F\in\mathscr{F}_\eta} E_F h(z) \quad \text{or} \quad L(\eta)=\inf_{F\in\mathscr{F}_\eta} E_F h(z). \tag{7}$$

In many important cases, the suprema or infima in (7) are assumed by a discrete distribution $F^*\in\mathscr{F}_\eta$ and, correspondingly, for $\mathscr{F}=\mathscr{F}_\eta$, the explicit form of the objective function in (5) can be found (see [4, 5, 17]). As a result, one gets relatively easily computable interval estimates for the optimal value of program (1):

$$\begin{aligned}\max_{x\in\mathscr{X}}\min_{F\in\mathscr{F}_\eta} E_F\{c^{\mathrm{T}}x-\varphi(x;A,b)\} &\le \max_{x\in\mathscr{X}} E_F\{c^{\mathrm{T}}x-\varphi(x;A,b)\}\\ &\le \max_{x\in\mathscr{X}}\max_{F\in\mathscr{F}_\eta} E_F\{c^{\mathrm{T}}x-\varphi(x;A,b)\} \quad \forall F\in\mathscr{F}_\eta.\end{aligned}$$

The explicit form of the objective function

$$f(x;\eta)=\min_{F\in\mathscr{F}_\eta} E_F\{c^{\mathrm{T}}x-\varphi(x;A,b)\}$$

and the optimal solution of (5) depend on the parameter vector η. When the prescribed values η of the moments are not known precisely enough, which is often the case, the problem of stability of the minimax solution comes to the fore. Similarly as in Case I, one substitutes η by an estimate y and solves the substitute program

$$\max_{x\in\mathscr{X}} f(x;y)=\min_{F\in\mathscr{F}_y} E_F\{c^{\mathrm{T}}x-\varphi(x;A,b)\} \tag{8}$$

instead of $\max_{x\in\mathscr{X}} f(x;\eta)$.

Leaving aside the deterministic stability concepts such as the global and local stability (for a result concerning Case II see [7, Theorem 6]) we shall aim to prove asymptotical normality of the optimal solution $\hat{x}(y)$ of the substitute programs (4) and (8). Provided that the set $\mathscr{X}$ of admissible solutions is defined by equality constraints only, asymptotical normality of the optimal solution follows by standard methods of mathematical statistics (see similar results for the case of maximum likelihood estimates [1]). Inequality constraints, however, bring along additional problems. In this paper, nonnegativity constraints are taken into account.

2. General theorem

Let $Y \subset \mathbb{R}^m$ be an open set, $\eta \in Y$ and $f: \mathbb{R}^n \times Y \to \mathbb{R}^1$. Let the set of admissible solutions

$$\mathscr{X} = \{x \in \mathbb{R}^n : Px = p, x \geq 0\} \tag{1}$$

where P-(r, n) and $p \in \mathbb{R}^r$ are a given matrix and vector, $r(P) = r$. For any $y \in Y$, let $\hat{x}(y)$ denote the optimal solution of the program

$$\text{maximize} \quad f(x; y) \quad \text{on the set } \mathscr{X}. \tag{2}$$

For the optimal solution $\hat{x}(\eta)$ of the program

$$\text{maximize} \quad f(x; \eta) \quad \text{on the set } \mathscr{X}, \tag{3}$$

denote

$$J = \{j : \hat{x}_j(\eta) > 0\}, \quad \text{card } J = s, \quad \hat{x}_J(\eta) = (\hat{x}_j(\eta), j \in J), \quad P_J = (p_{kj})_{\substack{1 \leq k \leq r,\\ j \in J}}$$

$$C_J = \left(\frac{\partial^2 f(\hat{x}(\eta); \eta)}{\partial x_j \, \partial x_l}\right)_{j,l \in J}, \quad B_J = \left(\frac{\partial^2 f(\hat{x}(\eta); \eta)}{\partial x_j \, \partial y_i}\right)_{\substack{j \in J\\ 1 \leq i \leq m}}. \tag{4}$$

Theorem 1. *Assume*:

(i) *For any $y \in Y$, $f(\cdot\,; y)$ is a concave function on $\mathbb{R}^n$ such that the second order derivatives*

$$\frac{\partial^2 f}{\partial x_j \, \partial x_l}, \qquad \frac{\partial^2 f}{\partial x_j \, \partial y_i}, \qquad 1 \leq j, l \leq n, 1 \leq i \leq m,$$

exist and are continuous in a neighborhood of the point $[\hat{x}(\eta), \eta]$ and the matrix C_J is nonsingular.

(ii) *y^N is an asymptotically normally distributed estimate of the vector of true parameters, $\eta \in Y$:*

$$\sqrt{N}(y^N - \eta) \sim N(0, \Sigma)$$

with a known nonsingular matrix Σ.

(iii) *The set of admissible solutions* (1) *is a nonempty convex polyhedron with nondegenerated vertices.*

(iv) *A strict complementarity condition holds true for the components of the optimal solution* $\hat{x}(\eta)$ *and of the corresponding vector* $\hat{\pi}(\eta)$ *of multipliers*:

$$\hat{x}_j(\eta) > 0 \Leftrightarrow \frac{\partial f(\hat{x}(\eta); \eta)}{\partial x_j} + \sum_{k=1}^{r} p_{kj}\hat{\pi}_k(\eta) = 0 \quad \forall_j. \tag{5}$$

Then asymptotically

$$\sqrt{N}(\hat{x}_j(y^N) - \hat{x}_j(\eta),\ 1 \leq j \leq n) \sim N(0, V_1) \tag{6}$$

with the variance matrix

$$V_1 = \left(\frac{\partial \hat{x}(\eta)}{\partial y}\right) \Sigma \left(\frac{\partial \hat{x}(\eta)}{\partial y}\right)^{\mathrm{T}};$$

the submatrix $(\partial \hat{x}_J(\eta)/\partial y) = (\partial \hat{x}_j(\eta)/\partial y_i)_{\substack{j \in J \\ 1 \leq i \leq m}}$ *of the matrix* $(\partial \hat{x}(\eta)/\partial y)$ *is given by*

$$\left(\frac{\partial \hat{x}_J(\eta)}{\partial y}\right) = -[I - C_J^{-1} P_J^{\mathrm{T}} (P_J C_J^{-1} P_J^{\mathrm{T}})^{-1} P_J] C_J^{-1} B_J \tag{7}$$

and

$$\frac{\partial \hat{x}_j(\eta)}{\partial y_i} = 0 \quad \textit{for } j \notin J,\ 1 \leq i \leq m. \tag{8}$$

The rank of the distribution (6) *is determined by* $r(V_1)$.

Proof. (a) According to (i), (iii) for arbitrary fixed $y \in Y$ there is an optimal solution $\hat{x}(y)$ of program (2) which together with the corresponding r-vector $\hat{\pi}(y)$ of multipliers fulfils the local Kuhn–Tucker conditions

$$\nabla_x f(\hat{x}(y); y) + P^{\mathrm{T}}\hat{\pi}(y) \leq 0, \tag{9}$$

$$P\hat{x}(y), \qquad -p = 0, \tag{10}$$

$$\hat{x}(y) \geq 0, \tag{11}$$

$$\hat{x}(y)^{\mathrm{T}}[\nabla_x f(\hat{x}(y); y) + P^{\mathrm{T}}\hat{\pi}(y)] = 0. \tag{12}$$

Using (iv), these conditions can be for $y = \eta$ rewritten in an equivalent form

$$\frac{\partial f(\hat{x}(\eta); \eta)}{\partial x_j} + \sum_k p_{kj}\hat{\pi}_k(\eta) = 0, \quad j \in J, \tag{13}$$

$$\frac{\partial f(\hat{x}(\eta); \eta)}{\partial x_j} + \sum_k p_{kj}\hat{\pi}_k(\eta) < 0, \quad j \notin J, \tag{14}$$

$$\sum_{j \in J} p_{kj}\hat{x}_j(\eta), \qquad -p_k = 0, \quad 1 \leq k \leq r, \tag{15}$$

$$\hat{x}_j(\eta) > 0,\ j \in J \quad \text{and} \quad \hat{x}_j(\eta) = 0,\ j \notin J. \tag{16}$$

Denote by $f_J : \mathbb{R}^s \times Y \to \mathbb{R}^1$ the function for which

$$f_J(x; y) = f(\tilde{x}; y),$$

where $\tilde{x}_j = x_j$ for $j \in J$, and $\tilde{x}_j = 0$ for $j \notin J$. Let $\tilde{h}_j : \mathbb{R}^s \times \mathbb{R}^r \times Y \to \mathbb{R}^1$, $\tilde{l}_j : \mathbb{R}^s \times \mathbb{R}^r \times Y \to \mathbb{R}^1$ be defined by

$$\tilde{h}_j(x_J, \pi; y) = \frac{\partial f_J(x; y)}{\partial x_j} + \sum_{k=1}^{r} p_{kj}\pi_k, \quad j \in J,$$

$$\tilde{l}_k(x_J, \pi; y) = \sum_{j \in J} p_{kj}x_j - p_k, \quad 1 \leq k \leq r.$$

In view of (13)–(16), the system of equations

$$\tilde{h}_j(x_J, \pi; y) = 0, \quad j \in J, \tag{17}$$

$$\tilde{l}_k(x_J, \pi; y) = 0, \quad 1 \leq k \leq r, \tag{18}$$

has a solution, namely, $\hat{x}_J(\eta) > 0$, $\hat{\pi}(\eta)$, η. Functions $\tilde{h}_j$, $j \in J$, $\tilde{l}_k$, $1 \leq k \leq r$, are continuously differentiable with respect to x_j, $j \in J$ and π_k, $1 \leq k \leq r$. The matrix

$$\begin{pmatrix} C_{J,y} & P_J^{\mathrm{T}} \\ P_J & 0 \end{pmatrix} \quad \text{where } C_{J,y} = \left(\frac{\partial^2 f_J(\hat{x}_J(y); y)}{\partial x_j\, \partial x_l}\right)_{j,l \in J},$$

of their derivatives with respect to x_j, $j \in J$ and π is nonsingular. According to the implicit functions theorem, there is a neighborhood $O_1(\eta)$ such that for arbitrary $y \in O_1(\eta)$, the system (17), (18) has a unique solution $\hat{x}_J(y)$, $\hat{\pi}(y)$, y and the components of $\hat{x}_J(y)$, $\hat{\pi}(y)$ are continuously differentiable functions of y. Their derivatives are given by

$$\begin{pmatrix} \dfrac{\partial \hat{x}_J}{\partial y} \\[2mm] \dfrac{\partial \hat{\pi}}{\partial y} \end{pmatrix} = \begin{pmatrix} C_{J,y} & P_J^{\mathrm{T}} \\ P_J & 0 \end{pmatrix}^{-1} \begin{pmatrix} -B_{J,y} \\ 0 \end{pmatrix} \tag{19}$$

with

$$B_{J,y} = \left(\frac{\partial^2 f_J(\hat{x}_J(y); y)}{\partial x_j\, \partial y_i}\right)_{\substack{j \in J \\ 1 \leq i \leq m}}.$$

Assumption (iv) together with the continuity of both $\hat{x}_J(y)$ and $\hat{\pi}(y)$ implies that there is an open neighborhood $O_2(\eta) \subset O_1(\eta)$ such that, for arbitrary $y \in O_2(\eta)$, the inequalities

$$\hat{x}_j(y) > 0, \qquad j \in J, \tag{20}$$

$$\frac{\partial f_J(\hat{x}_J(y); y)}{\partial x_j} + \sum_{k=1}^{r} p_{kj}\hat{\pi}_k(y) < 0, \qquad j \notin J, \tag{21}$$

hold true.

For $y \in O_2(\eta)$, the local Kuhn–Tucker conditions (9)–(12) are evidently satisfied by the vector $\hat{x}(y)$ consisting of components $\hat{x}_j(y)$, $j \in J$, and zeros for $j \notin J$ and by the vector $\hat{\pi}(y)$. The index set J of nonzero components of the optimal solution $\hat{x}(y)$ remains thus fixed for all $y \in O_2(\eta)$. The matrix of the first order derivatives $(\partial\hat{x}/\partial y)$ contains the submatrix $(\partial\hat{x}_J/\partial y)$ defined by (19) and zero elements for $j \notin J$, $1 \leqslant i \leqslant m$.

(b) According to (ii) and [12, p. 388],

$$\sqrt{N}(\hat{x}_j(y^N) - \hat{x}_j(\eta), j \in J) \sim N\left(0, \left(\frac{\partial\hat{x}_J(\eta)}{\partial y}\right)\Sigma\left(\frac{\partial\hat{x}_J(\eta)}{\partial y}\right)^{\mathrm{T}}\right) \tag{22}$$

where the explicit form of $(\partial\hat{x}_J(\eta)/\partial y)$ follows from (19) by formula

$$\begin{pmatrix} C_J & P_J^{\mathrm{T}} \\ P_J & 0 \end{pmatrix}^{-1} = \begin{pmatrix} [I - C_J^{-1}P_J^{\mathrm{T}}(P_JC_J^{-1}P_J^{\mathrm{T}})^{-1}P_J]C_J^{-1} & C_J^{-1}P_J^{\mathrm{T}}(P_JC_J^{-1}P_J^{\mathrm{T}})^{-1} \\ (P_JC_J^{-1}P_J^{\mathrm{T}})^{-1}P_JC_J^{-1} & -(P_JC_J^{-1}P_J^{\mathrm{T}})^{-1} \end{pmatrix}.$$

The assertion of Theorem 1 follows from these arguments and from the form of $(\partial\hat{x}(\eta)/\partial y)$ obtained in the part (a) of the proof.

Remark 1. All elements of $(\partial\hat{x}/\partial y)$ are continuous on a neighborhood of η, so that the asymptotical distribution (6) can be substituted by

$$N\left(0, \left(\frac{\partial\hat{x}(y^N)}{\partial y}\right)\Sigma\left(\frac{\partial\hat{x}(y^N)}{\partial y}\right)^{\mathrm{T}}\right)$$

(see [12, p. 388]).

Remark 2. Let's denote by $\tilde{x}(y)$ the optimal solution of the problem

$$\text{maximize} \quad f(x; y) \quad \text{on the set } \{x \in \mathbb{R}^n\colon Px = p\}. \tag{23}$$

Condition (5) means that for $y = \eta$ the optimal solution of (23) does not belong to the boundary of the nonnegative orthant $\mathbb{R}_+^n$.

To simplify the matter we shall discuss condition (5) under assumptions that, in addition to (i)–(iv), y^N is a strongly consistent estimate of η and $f(\cdot\,; y)$ is strictly concave on $\mathbb{R}^n$ for all y belonging to a neighborhood $O(\eta)$.

If $\tilde{x}(\eta)$ is not a boundary point of $\mathbb{R}_+^n$ then with probability 1 for N large enough $\tilde{x}(y^N)$ is not a boundary point of $\mathbb{R}_+^n$ and vice versa (due to continuity of $\tilde{x}(y)$ on $O(\eta)$ and to strong consistency of y^N). The fact that the strict complementarity condition holds true for $\hat{x}(y^N)$ thus indicates that the condition (5) is fulfilled.

Let us study the error of approximation in (6). Introducing higher-order moment assumptions, the Berry–Esséen theorem can be made use of:

In addition to assumptions (i)–(iv) of Theorem 1, let y^ν, $1 \leqslant \nu \leqslant N$, be a sequence of i.i.d. m-dimensional random vectors such that

$$Ey^1 = \eta, \quad \operatorname{var} y^1 = \Sigma, \quad E|y_i^1|^3 < \infty, \ 1 \leqslant i \leqslant m. \tag{24}$$

The arithmetical mean

$$\bar{y}^N = \frac{1}{N} \sum_{\nu=1}^{N} y^\nu$$

fulfils evidently assumption (ii) of Theorem 1.

Let the function f have bounded and continuous derivatives

$$\frac{\partial^3 f}{\partial x_j\, \partial x_l\, \partial x_h}, \quad \frac{\partial^3 f}{\partial x_j\, \partial x_l\, \partial y_k}, \quad \frac{\partial^3 f}{\partial x_j\, \partial y_i\, \partial y_k}, \qquad 1 \leq j, l, h \leq n,\ 1 \leq i, k \leq m \tag{25}$$

in some neighborhood U of $[\hat{x}(\eta), \eta]$. Then we have (see e.g. [2])

$$\sup_{u \in \mathbb{R}^s} \left| P\{\sqrt{N}(\hat{x}_j(\bar{y}^N) - \hat{x}_j(\eta)) \leq u_j, j \in J\} - \int_{M^N(u)} \phi_{V_j}(z)\, \mathrm{d}x \right| \leq \mathbb{R} N^{-1/2} \tag{26}$$

where ϕ_{V_j} is the probability density function of (22),

$$M^N(u) = \left\{ z \in \mathbb{R}^m : \sqrt{N}\left[\hat{x}_j\left(\eta + \frac{z}{\sqrt{N}}\right) - \hat{x}_j(\eta)\right] \leq u_j, j \in J \right\}$$

and d depends only on the moments of y^1 or orders three and less and on the first order derivatives $(\partial \hat{x}/\partial y)$ on U.

3. Special cases

The general result of Section 2 will be applied now to the simple recourse problem with random right-hand sides only, under special assumptions on the underlying family of distributions.

Theorem 2. *Assume*

$$\text{(i)} \quad f(x; y) = c^{\mathrm{T}} x - E_{G_y}\left[\sum_{i=1}^{m} q_i (X_i - b_i)^+ \right] \tag{1}$$

with

$$X_i = \sum_{j=1}^{n} a_{ij} x_j, \quad 1 \leq i \leq m,$$

$A = (a_{ij}),\quad 1 \leq i \leq m, 1 \leq j \leq n,\quad$ *of the full column rank,*

$q_i > 0,\quad 1 \leq i \leq m.$

(ii) b_i, $1 \leq i \leq m$, *are random variables with given continuous marginal distributions that depend on location parameters* η_i, $1 \leq i \leq m$, *respectively. The corresponding densities are denoted by* $g_i(\cdot\,;\eta_i)$, $1 \leq i \leq m$, *and the mean values* Eb_i, $1 \leq i \leq m$, *are supposed to exist.*

(iii) y^N *is an asymptotically normally distributed estimate of the true parameter vector* η, *i.e.*,

$$\sqrt{N}(y_i^N - \eta_i, 1 \leq i \leq m) \sim N(0, \Sigma).$$

(iv) *The set* $\mathscr{X}$ *of admissible solutions satisfies assumptions* (iii), (iv) *of Theorem* 1.

(v) *For the optimal solution* $\hat{x}(\eta) \in \arg\max_{x \in \mathscr{X}} f(x;\eta)$ *corresponding to the true parameter vector* η, g_i *is continuous in a neighborhood of the point*

$$\left[\sum_{j=1}^{n} a_{ij}\hat{x}_j(\eta);\, \eta\right]$$

and

$$g_i\left(\sum_{j=1}^{n} a_{ij}\hat{x}_j(\eta);\, \eta\right) > 0, \quad 1 \leq i \leq m.$$

Then asymptotically

$$\sqrt{N}(\hat{x}_j(y^N) - \hat{x}_j(\eta), 1 \leq j \leq n) \sim N(0, V_2)$$

where, for the variance matrix

$$V_2 = \left(\frac{\partial \hat{x}(\eta)}{\partial y}\right) \Sigma \left(\frac{\partial \hat{x}(\eta)}{\partial y}\right)^{\mathrm{T}},$$

we substitute

$$C = -A^{\mathrm{T}}QA, \qquad B = A^{\mathrm{T}}Q$$

with (2)

$$Q = \operatorname{diag}\left(q_i g_i\left(\sum_{j=1}^{n} a_{ij}\hat{x}_j(\eta);\, \eta_i\right), 1 \leq i \leq m\right)$$

in (2.4), (2.7).

Proof. The proof follows from Theorem 1 by direct computation of matrices

$$C = \left(\frac{\partial^2 f(x;y)}{\partial x_j\, \partial x_l}\right) \quad \text{and} \quad B = \left(\frac{\partial^2 f(x;y)}{\partial x_j\, \partial y_i}\right).$$

For computation of B, the assumption (ii) is taken into account: The marginal distribution function $G_i(X_i; y_i)$ can be written as $\tilde{G}_i(X_i - y_i)$ where $\tilde{G}_i$, again, is a distribution function.

An analogical result for $b_i \sim N(\mu_i, \sigma_i^2)$, $1 \leq i \leq m$, with estimated location parameters μ_i is given in [6] under more limiting assumptions than those considered here.

In case of unknown scale parameters we have

Theorem 3. *Let the assumptions* (i), (iii), (iv), (v) *of Theorem* 2 *be fulfilled. Let* b_i, $1 \leq i \leq m$, *be random variables with known continuous marginal distributions that depend on scale parameters* $\eta_i > 0$, *respectively, and let the mean values* Eb_i, $1 \leq i \leq m$, *exist. Then asymptotically*

$$\sqrt{N}\{\hat{x}_j(y^N) - \hat{x}_j(\eta),\ 1 \leq j \leq n\} \sim N(0, V_3)$$

where, for the variance matrix V_3,

$$c = -A^{\mathrm{T}}QA,\ B = A^{\mathrm{T}}Q \operatorname{diag}\left[\frac{1}{\eta_i}\sum_{j=1}^{n} a_{ij}\hat{x}_j(\eta),\ 1 \leq i \leq m\right]$$

with Q given by (2) *is substituted in* (2.4), (2.7).

Proof. The proof again follows by direct computation of matrices C, B. The marginal distribution function $G_i(X_i; y_i)$ depending on a scale parameter y_i can be written as $G_i(X_i/y_i)$, where G_i is a distribution function.

As the last application, the case of minimax solution will be studied. The set of distributions under consideration will be specified through prescribed mean values and variances:

$$\mathscr{F}_{\eta,\sigma^2} = \{F : E_F b_i = \eta_i,\ \operatorname{var}_F b_i = \sigma_i^2 > 0,\ 1 \leq i \leq m\}.$$

The objective function (1.5) has the form (see [5, 4, 10])

$$\begin{aligned} f(x; \eta, \sigma^2) &= \min_{F \in \mathscr{F}_{\eta,\sigma^2}} E_F\left\{c^{\mathrm{T}}x - \sum_{i=1}^{m} q_i(X_i - b_i)^+\right\} \\ &= c^{\mathrm{T}}x - \sum_{i=1}^{m} \tfrac{1}{2}q_i\left(\sum_{j=1}^{n} a_{ij}x_j - \eta_i\right) - \sum_{i=1}^{m} \tfrac{1}{2}q_i\left[\sigma_i^2 + \left(\eta_i - \sum_{j=1}^{n} a_{ij}x_j\right)^2\right]^{1/2}. \end{aligned}$$

Provided that the true mean values η_i, $1 \leq i \leq m$, have been estimated and that their (vector) estimate y^N is asymptotically normally distributed, the asymptotical normality of the optimal solution $\hat{x}(y^N)$ of the substitute program

$$\max_{x \in \mathscr{X}} f(x; y^N)$$

with

$$f(x;y)=c^{\mathrm{T}}x-\sum_{i=1}^{m}\tfrac{1}{2}q_i\left(\sum_{j=1}^{n}a_{ij}x_j-y_i\right)$$
$$-\sum_{i=1}^{m}\tfrac{1}{2}q_i\left[\sigma_i^2+\left(y_i-\sum_{j=1}^{n}a_{ij}x_j\right)^2\right]^{1/2}$$

again follows directly from Theorem 1.

To summarize, we have

Theorem 4. *Assume*

(i) $$f(x;y)=\min_{F\in\mathscr{F}_y}E_F\left\{c^{\mathrm{T}}x-\sum_{i=1}^{m}q_i(X_i-b_i)^+\right\}$$

with

$$X_i=\sum_{j=1}^{n}a_{ij}x_j,\quad 1\leqslant i\leqslant m,$$

$A=(a_{ij}),\quad 1\leqslant i\leqslant m, 1\leqslant j\leqslant n,$ *of the full column rank,*

$$q_i>0,\quad 1\leqslant i\leqslant m,$$

$$\mathscr{F}_y=\{F\colon E_Fb_i=y_i, \operatorname{var}_F b_i=\sigma_i^2>0, 1\leqslant i\leqslant m\}.$$

(ii) y^N *is an asymptotically normally distributed estimate of the true parameter vector* η, *i.e.*,

$$\sqrt{N}(y_i^N-\eta_i, 1\leqslant i\leqslant m)\sim N(0,\Sigma).$$

(iii) *The set* $\mathscr{X}$ *satisfies assumptions* (iii), (iv) *of Theorem* 1.
Then asymptotically

$$\sqrt{N}(\hat{x}_j(y^N)-\hat{x}_j(\eta), 1\leqslant j\leqslant n)\sim N(0,V_4)\tag{3}$$

where, for the variance matrix V_4, *we substitute*

$$c=-A^{\mathrm{T}}KA,\qquad B=A^{\mathrm{T}}K$$

with (4)

$$K=\operatorname{diag}\left\{\tfrac{1}{2}q_i\sigma_i^2\left[\sigma_i^2+\left(\eta_i-\sum_{k=1}^{n}a_{ik}\hat{x}_k(\eta)\right)^2\right]^{-3/2}, 1\leqslant i\leqslant m\right\}$$

in (2.4), (2.7).

Remark 3. Instead of Q given by (2) or K given by (4), matrices

$$Q^N=\operatorname{diag}\left\{q_ig_i\left(\sum_{j=1}^{n}a_{ij}\hat{x}_j(y^N);y_i^N\right), 1\leqslant i\leqslant m\right\}$$

or

$$K^N = \text{diag}\left\{\tfrac{1}{2}q_i\sigma_i^2\left[\sigma_i^2 + \left(y_i^N - \sum_{k=1}^{n} a_{ik}\hat{x}_k(y^N)\right)^2\right]^{-3/2}, 1 \leq i \leq m\right\}$$

can be used in Theorems 2, 3 or in Theorem 4, respectively. The reasoning is similar to that used in Remark 1.

Remark 4. In this case, assumptions (2.25) are fulfilled. Assuming the existence of the third absolute moments of b_i, $1 \leq i \leq m$, we have the rate of convergence $O(N^{-1/2})$ in (3) for the case that the true parameter vector η has been estimated by the arithmetical mean $\bar{y}^N$ (see (2.26)).

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