APPROXIMATIONS AND CONTAMINATION BOUNDS FOR PROBABILISTIC PROGRAMS[1]

Martin BRANDA and Jitka DUPAČOVÁ

Charles University in Prague, Faculty of Mathematics and Physics, Department of Probability and Mathematical Statistics, Czech Republic

ABSTRACT:

In this paper we aim at output analysis with respect to changes of the probability distribution for problems with probabilistic (chance) constraints. The perturbations are modeled via contamination of the initial probability distribution. Dependence of the set of solutions on the probability distribution rules out the straightforward construction of the convexity-based global contamination bounds for the perturbed optimal value function whereas local results can be still obtained. To get global bounds we shall explore several approximations and reformulations of stochastic programs with probabilistic constraints by stochastic programs with suitably chosen recourse or penalty-type objectives and fixed constraints.

Key words: Stochastic programs with probabilistic constraints, output analysis, contamination technique

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1 Modeling issues

Classical stochastic programming (SP) models aim at hedging against consequences of possible realizations of random parameters — *scenarios* — so that the expected final outcome or position is the best possible.
Modeling part of realistic applications consists of a clear declaration of random factors to be taken into account, of distinguishing between hard and soft constraints and of a choice of a sensible optimality criterion. The starting point may be formulation of a deterministic problem which would be solved if no randomness is considered, e.g.

$$\min \{f(x) \,:\, x \in \mathcal{X}, g_k(x) \le 0,\, k = 1, \ldots, m\}\,.$$

Taking into account the presence of a random factor ω and the fact that a decision x has to be chosen before ω occurs, a reformulation of the minimization problem is needed. Two prevailing approaches have been used to this purpose:
• static expected penalty models,
• probabilistic programs.
For penalty type models, $\mathcal{X}$ is defined by hard constraints plus some other conditions that guarantee plausible properties of the model, whereas soft constraints, such as $g_k(x, \omega) \le 0$, are reflected by penalties included into the random objective function. For probabilistic programs, probabilistic reliability-type constraints are introduced.

1.1 Stochastic programming models with penalties

The basic SP model with penalties is of the form

$$\min_{x \in \mathcal{X}} E_P f(x, \omega). \tag{1}$$

It is identified by

- a known probability distribution P of random parameter ω whose support Ω is a closed subset of $\mathbb{R}^s$; E_P denotes the corresponding expectation. In the sequel, the same character ω will be used both for the random vector and its realization.

- a given, nonempty, closed set $\mathcal{X} \subset \mathbb{R}^n$ of decisions x,

- a preselected random objective f from $\mathcal{X} \times \Omega$ to the extended reals — a loss or a cost caused by the decision x when scenario ω occurs. As a function of ω, f is measurable for each fixed $x \in \mathcal{X}$ and such that its expectation $E_P f(x,\omega)$ is well defined. The structure of f may be quite complicated e.g. for multistage problems. For convex $\mathcal{X}$, a frequent assumption is that f is lower semicontinuous and convex with respect to x, i.e. f is *convex normal integrand*.

An example of (1) is the two-stage stochastic linear program with fixed recourse where $\mathcal{X}$ is convex polyhedral and the random objective function $f(x,\omega) = c^\top x + q(x,\omega)$ involves the second-stage (recourse) function q defined as

$$q(x,\omega) = \min_y \{q^\top y \,:\, Wy = b(\omega) - T(\omega)x,\ y \in \mathbb{R}^r_+\}. \tag{2}$$

Vector $q \in \mathbb{R}^r$ and recourse matrix $W(m,r)$ are fixed, $b(\omega)$, $T(\omega)$ are of consistent dimensions with components affine linear in ω.

Our knowledge of probability distribution P is often merely a hypothesis. Hence, we want to know, how sensitive to its violation our solution of (1) is. To this purpose we shall consider $F(x,P) := E_P f(x,\omega)$ as a function from $\mathcal{X} \times \mathcal{P}$ to extended reals, where $\mathcal{P}$ is a set of probability distributions on Ω. This implies that in (1) $\mathcal{X}$ is independent of P.

1.2 Probabilistic constraints

Instead of (1) one may consider stochastic programs

$$\min_{x \in \mathcal{X}(P)} F(x,P) := E_P f(x,\omega) \tag{3}$$

in which the set of feasible solutions $\mathcal{X}(P) \subset \mathbb{R}^n$ depends on the probability distribution P.
Formally, the independence of the set of feasible solutions of P can be achieved by means of the extended real indicator functions. Problem (3) can be e.g. written as

$$\min_{x \in \mathcal{X}} \left[F(x,P) + \mathrm{ind}_{\mathcal{X}(P)}(x)\right] \tag{4}$$

with $\mathcal{X}(P) \subset \mathcal{X} \subseteq \mathbb{R}^n$, $\mathcal{X}$ a fixed closed set independent of P, and the indicator function $\mathrm{ind}_{\mathcal{X}(P)}(x) := 0$ for $x \in \mathcal{X}(P)$ and $+\infty$ otherwise. However, the resulting extended real objective function in (4) is then very likely to lose the convenient properties of the original objective function $F(x,P)$ in (3).

A special type of (3) is *probabilistic programming* obtained when $\mathcal{X}(P) = \mathcal{X} \cap \mathcal{X}_\varepsilon(P)$ with $\mathcal{X}_\varepsilon(P)$ defined e.g. by the *joint probabilistic constraint*

$$\mathcal{X}_\varepsilon(P) := \{x \in \mathbb{R}^n : P(g(x,\omega) \le 0) \ge 1 - \varepsilon)\} \tag{5}$$

with $g : \mathbb{R}^n \times \Omega \to \mathbb{R}^m$ and $\varepsilon \in (0,1)$ fixed, chosen by the decision maker. It is a reliability type constraint which can be written as

$$H(x,P) := P(\max_{k=1,\ldots,m} g_k(x,\omega) \le 0) \ge 1 - \varepsilon. \tag{6}$$

We make use of the following convention: If $V(\omega)$ is a predicate on ω, we write $P(V(\omega))$ instead of $P(\{\omega \in \Omega : V(\omega)\})$.

Individual, separate probability constraints are a special type of probabilistic constraints which treat constraints $g_k(x,\omega) \le 0$ separately: Given probability thresholds $\varepsilon_1, \ldots, \varepsilon_m$ the feasible solutions are $x \in \mathcal{X}$ that fulfil m individual (separate) probabilistic constraints

$$P(g_k(x,\omega) \le 0) \ge 1 - \varepsilon_k,\ k = 1, \ldots, m. \tag{7}$$

This is a relatively easy structure of problem, namely, if ω_k are separated being the right-hand sides of constraints, i.e. $g_k(x,\omega) = \omega_k - g_k(x)\, \forall k$.
Denote $u_{1-\varepsilon_k}(P_k)$ quantile of marginal probability distribution P_k of ω_k. Then (7) can be reformulated as

$$g_k(x) \ge u_{1-\varepsilon_k}(P_k),\ k = 1, \ldots, m.$$

For concave $g_k(x)\, \forall k$ the set of feasible decisions is convex and for linear objective function and linear $g_k(x)\, \forall k$ the resulting problem is a linear program.

Such results are no more valid for joint probability constraints. Even for random right-hand sides only, special requirements on the probability distribution P are needed; cf. log-concave or quasiconcave probability distributions [28]. These seminal results on convexity properties of the set $\mathcal{X}_\varepsilon(P)$ and of the related function $H(x,P)$ are due to Prékopa; see e.g. [27, 28]. They have been reported and further extended in various monographs and collections devoted to stochastic programming, e.g. [2, 22].
Semicontinuity properties of the corresponding indicator functions, see e.g. [21], can be obtained under various sets of assumptions about the function g and/or its components g_k and about the probability distribution P. These

play an important role in qualitative and quantitative stability analysis with respect to changes of the probability measure; see [30].

Klein Haneveld [26] suggested to replace probability constraints (5) and (7) by *Integrated Chance Constraints,* ICC

$$E_P(\max_k [g_k(x,\omega)]^+) \le \beta \text{ and } E_P([g_k(x,\omega)]^+) \le \beta_k \,\forall k, \tag{8}$$

respectively, with fixed nonnegative values β, β_k. Mathematical properties of ICC are much nicer and from the modeling point of view, it is convenient that integrated chance constraints *quantify the size* of unfeasibilities.

The two prevailing types of static stochastic programs — with penalties and with probabilistic constraints — are not competitive but rather complementary. Contrary to penalty models, probabilistic programs capture the reliability requirements or risk restrictions even in cases which do not allow for reasonably accurate evaluation of penalties, e.g. [15]. A suggestion of [28] is to apply probabilistic constraints (5), (6) or (7) and at the same time, to extend the objective function for an expected penalty term which is active whenever the original constraints $g_k(x,\omega) \le 0$ are not fulfilled:

> Prékopa [28] "...we are convinced that the best way of operating a stochastic system is to operate it with a prescribed (high) reliability and at the same time use penalties to punish discrepancies."

In addition, ideas of multiobjective programming can be used if the choice of the penalty function is not clear and multiple penalty functions are therefore considered; cf. [29].
Another suggestion of [28] was to assign a probabilistic constraint on the second-stage variables y in the two-stage stochastic program as a way how to restrict a possible unfeasibility of the second-stage constraints in incomplete recourse problems. Integrated chance constraints (8) of [26] aim at a similar goal.

Such extensions may be useful when modeling real problems. An example of a "mixed" model and its properties was recently presented in [3]; see Example 3.

In this paper we focus on potential applications of the contamination technique to quantitative stability results for the optimal value of probabilistic programs with respect to changes in probability distribution P. The brief presentation of the contamination technique in Section 2 reveals that its exploitation to construction of global contamination bounds for probabilistic programs is limited. Therefore their reformulations to expected penalty-type problems are suggested in Section 3. As illustrated in Section 4, such reformulations open a possibility to construct approximate global contamination bounds for the probabilistic programs in question.

2 Contamination technique

In real applications, full knowledge of the probability distribution P can hardly be expected. Assume instead that the SP of interest, e.g. (9) was solved for a suitably chosen probability distribution P which is considered a sensible choice regarding the problem to be solved and the available input data. The first issue of interest is then the sensitivity of the obtained optimal solutions with respect to perturbances of the probability distribution. It may be quantified by bounds on the "error" in the perturbed optimal value. The background of relevant techniques is based on advanced results of parametric programming, e.g. [30].
We shall exploit parametric stability and sensitivity results, ideas of multi-objective optimization and the contamination technique to derive bounds for the perturbed optimal value function focusing on probabilistic programs and related problems. See [16] for discussion of the first results in this direction developed for stress testing of the risk measure VaR.

Contamination technique was initiated in mathematical statistics as one of the tools for analysis of robustness of estimators with respect to deviations from the assumed probability distribution and/or its parameters. It goes back to von Mises and the concepts are briefly described e.g. in [32]. In stochastic programming, it was developed in a series of papers, see e.g. [7, 10] for results applicable to two-stage stochastic linear programs with recourse. Describing perturbation by means of contamination occurs also in sensitivity analysis of worst-case probabilistic objectives with respect to deviations from the assumed type of probability distributions, cf. [25].
For derivation of contamination bounds one mostly assumes that the stochastic program is reformulated as

$$\min_{x \in \mathcal{X}} F(x, P) := E_P f(x, \omega) = \int_\Omega f(x, \omega) P(d\omega) \tag{9}$$

where P is a fixed fully specified probability distribution of the random parameter $\omega \in \Omega \subset \mathbb{R}^s$, $\mathcal{X} \subset \mathbb{R}^n$ is nonempty, closed, *independent* of P and $f : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}$ such that the expectation E_P is well defined.
Denote $\varphi(P)$ the optimal value and $\mathcal{X}^*(P)$ the set of optimal solutions of (9). Possible changes or perturbations of probability distribution P are modeled using contaminated distributions P_λ,

$$P_\lambda := (1-\lambda)P + \lambda Q,\ \lambda \in [0,1] \tag{10}$$

with Q another *fixed* probability distribution. Limiting thus the analysis to a selected direction $Q - P$ only, the results are directly applicable but they are less general than quantitative stability results with respect to arbitrary (but small) changes in P summarized e.g. in [30].
Via contamination, robustness analysis with respect to changes in probability distribution P gets reduced to a much simpler analysis with respect to a scalar parameter λ : The objective function in (9) is linear in P, hence the perturbed objective

$$F(x,\lambda) := \int_\Omega f(x,\omega)P_\lambda(d\omega) = (1-\lambda)F(x,P) + \lambda F(x,Q)$$

is linear in λ. Suppose for simplicity that stochastic program (9) has an optimal solution for all considered distributions P_λ, $0 \le \lambda \le 1$ of the form (10). Then the optimal value function

$$\varphi_{PQ}(\lambda) := \min_{x\in\mathcal{X}} F(x, P_\lambda)$$

is concave on $[0,1]$ which implies its continuity and existence of directional derivatives on $(0,1)$. Continuity at the point $\lambda = 0$ is a property related with stability results for the stochastic program in question. In general, one needs a nonempty, bounded set of optimal solutions $\mathcal{X}^*(P)$ of the initial stochastic program (9). This assumption together with stationarity of derivatives $\frac{dF(x,\lambda)}{d\lambda} = F(x,Q) - F(x,P)$ is used to derive the form of the directional derivative

$$\varphi'_{PQ}(0^+) = \min_{x\in\mathcal{X}^*(P)} \frac{dF(x,\lambda)}{d\lambda}\Big|_{\lambda=0^+} = \min_{x\in\mathcal{X}^*(P)} F(x,Q) - \varphi(P) \tag{11}$$

which enters the upper bound for the optimal value function $\varphi_{PQ}(\lambda) = \varphi(P_\lambda)$:

$$\varphi(P) + \lambda\varphi'_{PQ}(0^+) \ge \varphi_{PQ}(\lambda) \ge (1-\lambda)\varphi(P) + \lambda\varphi(Q),\ \lambda \in [0,1]; \tag{12}$$

for details see [7, 10] and references therein. Formula (11) follows e.g. by application of Theorem I. in Chapter III. of [5] provided that $\mathcal{X}$ is compact and the both objectives $F(x,P)$, $F(x,Q)$ are continuous in x.
If $x^*(P)$ is the *unique* optimal solution of (9), $\varphi'_{PQ}(0^+) = F(x^*(P),Q) - \varphi(0)$, i.e. the *local change of the optimal value function caused by a small change of P in direction $Q-P$ is the same as that of the objective function at* $x^*(P)$. If there are multiple optimal solutions, each of them leads to an upper bound $\varphi'_{PQ}(0^+) \leq F(x(P),Q) - \varphi(P)$, $x(P) \in \mathcal{X}^*(P)$. Relaxed contamination bounds can be then written as

$$(1-\lambda)\varphi(P) + \lambda F(x(P),Q) \geq \varphi_{PQ}(\lambda) \geq (1-\lambda)\varphi(P) + \lambda\varphi(Q) \qquad (13)$$

valid for an arbitrary optimal solution $x(P) \in \mathcal{X}^*(P)$ and for all $\lambda \in [0,1]$.
The contaminated probability distribution P_λ may be also understood as a result of contaminating Q by P. Provided that the set of optimal solutions $x(Q)$ of the problem $\min_{x\in\mathcal{X}} F(x,Q)$ is nonempty and bounded, an alternative upper bound may be constructed in a similar way. Together with the original upper bound from (13) one may use a tighter upper bound

$$\min\{(1-\lambda)\varphi(P) + \lambda F(x(P),Q), \lambda\varphi(Q) + (1-\lambda)F(x(Q),P)\} \qquad (14)$$

for $\varphi(P_\lambda)$.

For problems with $F(x,P)$ *concave* in P and $\mathcal{X}$ *independent* of P concavity of the optimal value function $\varphi(\lambda)$ is preserved. Under additional assumptions one may then apply general results by [5, 19] and others to get the existence and the form of the directional derivative

$$\varphi'_{PQ}(0^+) = \min_{x\in\mathcal{X}^*(P)} \frac{d}{d\lambda} F(x,P_\lambda)\Big|_{\lambda=0+} \qquad (15)$$

which enters contamination bounds (12); see [10, 12, 13].

Contamination bounds (12), (13) help to *quantify* the change in the optimal value due to the considered perturbations of (9). They exploit the optimal value $\varphi(Q)$ of the problem solved under the alternative probability distribution Q and the expected performance $F(x(P),Q)$ of the optimal solution $x(P)$ obtained for the original probability distribution P in situation that Q applies. Notice that both of these values appear under heading of *stress testing* methods.

Contamination bounds have been derived also for mixed integer SLP with recourse [6], multistage SLP and SP with risk objectives [11, 13]. For their applications see e.g. [12, 14, 16]. There exist results on local stability of optimal solutions of contaminated stochastic programs and also differentiability results for the case that the set $\mathcal{X}$ depends on P, see [7, 8, 33].

In the present paper we shall discuss the role of contamination bounds in output analysis for stochastic programs with probabilistic constraints and related SP problem formulations. As the set of feasible solutions depends on P, a direct application of the contamination technique will be successful only exceptionally.

EXAMPLE 1
As the first example consider the stochastic linear program with individual probabilistic constraints and random right-hand sides ω_k

$$\min_{x \in \mathcal{X}} \{c^\top x \,:\, P(\omega_k - T_k x \le 0) \ge 1 - \varepsilon_k,\, k = 1, \dots, m\}.$$

(We assume for simplicity that $\mathcal{X} = \mathbb{R}^n$.) It reduces to a linear program whose right-hand sides are the corresponding quantiles $u_{1-\varepsilon_k}(P_k)$ of the marginal probability distributions P_k :

$$\varphi(P) := \min\{c^\top x \,:\, T_k x \ge u_{1-\varepsilon_k}(P_k),\, k = 1, \dots, m\}. \tag{16}$$

Assume that the optimal value $\varphi(P)$ of (16) is finite. Using duality theory for linear programming, the optimal value can be expressed as

$$\varphi(P) = \max_z \{\sum_k z_k u_{1-\varepsilon_k}(P_k) \,:\, T^\top z = c,\, z \in \mathbb{R}^m_+\} \tag{17}$$

where T is the (m, n) matrix composed of rows T_k, $k = 1, \dots, m$. This is a problem whose set of feasible solutions is fixed and only the objective function depends, in a nonlinear way, on probability distribution P; hence, it seems to be in the form suitable for construction of contamination bounds for optimal value $\varphi_{PQ}(\lambda) := \varphi(P_\lambda)$. Denote $z^*(P)$ an optimal solution of (17).
For one-dimensional probability distribution P and under assumptions about existence and continuity of its positive density p on a neighborhood of the quantile $u_\alpha(P)$ we get derivatives of quantiles $u_\alpha(P_\lambda)$ of contaminated probability distribution P_λ at $\lambda = 0^+$, cf. [32]: Let Γ denote the distribution function of the contaminating probability distribution Q. Then

$$\frac{d}{d\lambda} u_\alpha((1-\lambda)P + \lambda Q)\Big|_{\lambda=0^+} = \frac{\alpha - \Gamma(u_\alpha(P))}{p(u_\alpha(P))}. \tag{18}$$

Hence, if the objective function $\sum_k z_k u_{1-\varepsilon_k}((1-\lambda)P_k+\lambda Q_k)$ of (17) is convex in λ, we get (maximization type) contamination bounds with

$$\varphi'_{PQ}(0^+) = \sum_k z_k^*(P)\frac{1-\varepsilon_k-\Gamma_k(u_{1-\varepsilon_k}(P_k))}{p_k(u_{1-\varepsilon_k}(P_k))}$$

where Γ_k denote marginal distribution functions of probability distribution Q.

The main obstacle is that convexity with respect to λ of quantiles $u_\alpha(P_\lambda)$ cannot be guaranteed. It means that $\sum_k z_k u_{1-\varepsilon_k}((1-\lambda)P_k+\lambda Q_k)$, the objective function of the contaminated dual (maximization) linear program, need not be convex in λ either. Its convexity is obtained, for example, if for all k, ε_k are small enough and P_k, Q_k are normal probability distributions which differ only in their variances. A conjecture is that convexity of contaminated quantiles may be valid for unimodal P_k, Q_k and sufficiently small $\varepsilon_k\,\forall k$. Indeed, using classical stability results for linearly perturbed nonlinear programs, cf. Corollary 3.4.5 of [18], one can derive sufficient conditions for convexity assuming in addition continuously differentiable densities.

To overcome the difficulties, let us follow the suggestion of [8]. Assume that the optimal solution $x^*(P)$ of (16) is unique and nondegenerated, the marginal densities p_k are for all k continuous and positive at the points $T_k x^*(P)$ and the marginal distribution functions of the contaminating probability distribution Q have continuous derivatives on the neighborhoods of the points $T_k x^*(P)$. Using (18), approximate the right-hand sides of (16) linearly

$$u_k(\lambda) := u_{1-\varepsilon_k}((1-\lambda)P_k+\lambda Q_k) \approx u_{1-\varepsilon_k}(P_k) + \lambda \frac{du_k(\lambda)}{d\lambda}\Big|_{\lambda=0^+}.$$

Approximate the optimal solution $x^*(P_\lambda)$ of the contaminated program

$$\min_{x\in\mathcal{X}}\{c^\top x \,:\, T_k x \geq u_{1-\varepsilon_k}((1-\lambda)P_k+\lambda Q_k),\ k=1,\dots,m\} \tag{19}$$

by an optimal solution $\hat{x}(P_\lambda)$ of parametric linear program

$$\min_{x\in\mathcal{X}}\left\{c^\top x \,: T_k x \geq u_{1-\varepsilon_k}(P_k) + \lambda\frac{1-\varepsilon_k-\Gamma_k(u_{1-\varepsilon_k}(P_k))}{p_k(u_{1-\varepsilon_k}(P_k))},\ k=1,\dots,m\right\} \tag{20}$$

whose properties are well known; namely, the optimal value function $\hat{\varphi}(\lambda)$ of (20) is convex piecewise linear in λ. This allows to construct the convexity based contamination bounds. Moreover, if the noncontaminated problem (with $\lambda = 0$) has a unique nondegenerated optimal solution, there is a $\lambda_0 > 0$ such that $\hat{\varphi}(\lambda)$ is linear on $[0, \lambda_0]$ and the optimal basis B of the linear program dual to (20) stays fixed. Namely, $\hat{x}(P) = (B^\top)^{-1} \left[u_{1-\varepsilon_i}(P_i)\right]_{i \in I}$ for I denoting the set of active constraints of (16). In virtue of our assumptions, there is $\lambda_0 > 0$ such that for $0 \le \lambda \le \lambda_0$, the set of active constraints remains fixed, B is optimal basis of (20) and the optimal solution is

$$\begin{aligned} \hat{x}(P_\lambda) &= x^*(P) + \lambda (B^\top)^{-1} \left[\frac{1 - \varepsilon_k - \Gamma_k(u_{1-\varepsilon_k}(P_k))}{p_k(u_{1-\varepsilon_k}(P_k))} \right]_{k \in I} \\ &= x^*(P) + \lambda \mathrm{d} x^*(P; Q - P). \end{aligned}$$

For $0 \le \lambda \le \lambda_0$ the optimal value equals

$$\hat{\varphi}(\lambda) = \varphi(P) + \lambda c^\top (B^\top)^{-1} \left[\frac{1 - \varepsilon_k - \Gamma_k(u_{1-\varepsilon_k}(P_k))}{p_k(u_{1-\varepsilon_k}(P_k))} \right]_{k \in I}$$

where the second term determines the slope of the lower contamination bound. The upper contamination bound follows by convexity:

$$\hat{\varphi}(\lambda) \le (1 - \lambda)\varphi(P) + \lambda \varphi(Q).$$

This approximation was applied in [8] to real data for a water resources planning problem and similar ideas were used also for contamination of empirical VaR; cf. [16].

In the sequel, we shall explore a construction of approximate global contamination bounds based on ideas of multiobjective programming and on a relationship between optimal solutions of stochastic programs with probabilistic constraints and optimal solutions of stochastic programs with suitably chosen recourse or penalty type objectives and fixed constraints.

3 Probabilistic programs and their recourse reformulations

Linear problems with convex polyhedral set $\mathcal{X}$, with nonrandom matrix T composed of rows T_k, $k = 1, \dots, m$, and with individual probabilistic constraints on random right-hand sides ω_k

$$\min_{x \in \mathcal{X}} \{c^\top x \,:\, P(\omega_k - T_k x \le 0) \ge 1 - \varepsilon_k,\ k = 1, \dots, m\}$$

can be qualified as an easy case. They reduce to linear programs whose right-hand sides are the corresponding quantiles $u_{1-\varepsilon_k}(P_k)$ of the marginal probability distributions P_k, cf. (16). Moreover, under additional assumptions, contamination bounds can be constructed, see Example 1.

However, except for (16), even individual linear probabilistic programs (7) with $g_k(x, \omega) := b_k(\omega) - T_k(\omega)x$ and right-hand sides $b_k(\omega)$ and rows $T_k(\omega)$ affine linear in ω, need not be easy to solve and approximations by easier problems are of interest. Of course, for joint probabilistic constraints tractable approximations are of even greater importance.

The idea proposed in [24] is to construct another problem with a fixed set $\mathcal{X}$ of feasible solutions related to the probabilistic program

$$\min\{E_P f(x, \omega) \,:\, x \in \mathcal{X} \cap \mathcal{X}_\varepsilon(P)\}, \tag{21}$$

with $\mathcal{X}_\varepsilon(P)$ defined by (5): One assigns penalties $N[g_k(x, \omega)]^+$ with positive penalty coefficients and solves

$$\min_{x \in \mathcal{X}} \left[E_P f(x, \omega) + N \sum_{k=1}^{m} E_P [g_k(x, \omega)]^+ \right] \tag{22}$$

instead of (21).

ILLUSTRATIVE NUMERICAL EXAMPLE

Consider the example of probabilistic program with one linear joint probabilistic constraint taken from [28]:

$$\begin{aligned} &\min 3x_1 + 2x_2 \\ &\text{subject to} \\ &x_1 + 4x_2 \ge 4,\ 5x_1 + x_2 \ge 5,\ x_1 \ge 0,\ x_2 \ge 0, \\ &P(x_1 + x_2 - 3 \ge \omega_1, 2x_1 + x_2 - 4 \ge \omega_2) \ge 1 - \varepsilon. \end{aligned} \tag{23}$$

The random components (ω_1, ω_2) have bivariate normal distribution with $E[\omega_1] = E[\omega_2] = 0$, $E[\omega_1^2] = E[\omega_2^2] = 1$, and $E[\omega_1 \omega_2] = 0.2$. The corresponding simple recourse model may be formulated as follows.

$$\begin{aligned} \min \quad & 3x_1 + 2x_2 + N \cdot E\Big[(\omega_1 - x_1 - x_2 + 3)^+ + (\omega_2 - 2x_1 - x_2 + 4)^+\Big] \\ & \text{subject to} \\ & x_1 + 4x_2 \geq 4,\ 5x_1 + x_2 \geq 5,\ x_1 \geq 0,\ x_2 \geq 0. \end{aligned} \tag{24}$$

We used SLP-IOR, see [23], and the solver PROCON for solving the problem (23) with joint probabilistic constraint for decreasing levels ε; the solver SRAPPROX was used for simple recourse models (24) with increasing N.

Table 1: Optimal values and solutions for simple recourse model.

N	First-stage objective value	Recourse objective value	Optimal solution $\hat{x}_1^N$	$\hat{x}_2^N$
1	4.1053	2.9596	0.8421	0.7895
10	9.5596	0.9285	1.0000	3.2798
100	11.6185	0.5887	1.0000	4.3092
1000	12.9373	0.2680	1.0000	4.9686
10000	13.4786	0.0494	1.0000	5.2393
100000	13.5703	0.0052	1.0000	5.2851
1000000	13.5800	0.0004	1.0000	5.2900

Table 2: Optimal values and solutions for probabilistic program.

ε	Objective value	Optimal solution $\hat{x}_1^\varepsilon$	$\hat{x}_2^\varepsilon$
$0,2$	9.4438	1.0000	3.2219
$0,1$	10.2349	1.0000	3.6174
$0,05$	10.8899	1.0000	3.9450
$0,01$	12.1382	1.0000	4.5691
$0,001$	13.5754	1.0000	5.2877

When comparing results from tables 1 and 2, we observe that optimal solutions and optimal values of (23) and (24) behave similarly with increasing N and decreasing ε, respectively. A question is if there is a quantitative relation between the outputs of the two models in dependence on the choice of parameters N and ε.

The two problems (21) and (22) are not equivalent. However, by intuition one may expect that for a given $\varepsilon > 0$ there exists N large enough such that the obtained optimal solution, say $x_N(P)$, of (22) satisfies the probabilistic constraint (5). Then the corresponding value of $E_P f(x_N(P), \omega)$ may serve as an upper bound for the optimal value of (21). This conjecture is supported by the analysis of optimality conditions for the simple recourse problem with random right-hand sides; they provide a link between the simple-recourse penalty coefficients and the values of the probability thresholds for the corresponding problem with individual probabilistic constraints and random right-hand sides, cf. Section 3.2 of [2].
A more complicated form of penalty may be designed as the optimal value of the recourse function $q(x, \omega)$ of the second-stage linear program $\min_y \{q^\top y : Wy \geq b(\omega) - T(\omega)x, y \geq 0\}$ with some $q \in \mathbb{R}^m_+$ and a fixed recourse matrix W; problem (22) corresponds to $q_k = N \, \forall k$ and the simple recourse matrix $W = I$. For a numerical evidence see also [15] where a piecewise linear nonseparable penalty function $N[\max_j (x_j - \omega_j)]^+$ was applied and the results compared with those obtained for joint probabilistic constraint of the type $P\{x_j \leq \omega_j, \, j = 1, \dots, n\} \geq 1 - \varepsilon$.
Also convexity preserving penalty functions, say $\vartheta : \mathbb{R}^m \to \mathbb{R}_+$ which are continuous, nondecreasing in their components and are equal to 0 on $\mathbb{R}^m_-$ and positive otherwise seem to be a suitable choice. This idea stems from the connection between (21) and (22) that can be recognized within the framework of *convex* multiobjective stochastic programming. We shall firstly illustrate it for one of the already considered probabilistic programs. Indeed, the optimal solutions of (21) with $\mathcal{X}_\varepsilon(P)$ given by (5) can be viewed as efficient solutions of the *bi-criterial* problem

$$\text{“min”}_{x \in \mathcal{X}} [F(x, P), -H(x, P)]$$

obtained by the ϵ-constrained approach. Hence, for convex $\mathcal{X}$, $F(\cdot, P) = E_P f(\cdot, \omega)$ convex and $H(\cdot, P) := P(\omega : g(\cdot, \omega) \leq 0)$ concave there exists $N \geq 0$ such that these efficient solutions can be found as optimal solutions of the *parametric program*

$$\min_{x \in \mathcal{X}} [F(x, P) - N \cdot H(x, P)].$$

A similar relationship can be established for a whole family of stochastic programs of the form

$$\min_{x \in \mathcal{X}} \{E_P f(x, \omega) : E_P g_k(x, \omega) \le \epsilon_k,\ k = 1, \dots, m\} \tag{25}$$

with distribution dependent set of feasible solutions $\mathcal{X}_\varepsilon(P)$. Evidently, problems with probabilistic constraints and those with integrated chance constraints are special instances of (25). Again, for fixed thresholds ϵ_k, $k = 1, \dots, m$, the optimal solutions of (25) may be viewed as efficient solutions of the multiobjective problem

$$\text{"min"}_{x \in \mathcal{X}} \{E_P f(x, \omega),\ E_P g_k(x, \omega),\ k = 1, \dots, m\} \tag{26}$$

obtained by the ϵ-constrained approach.
If $\mathcal{X}$ is nonempty, convex, compact and the functions $E_P f(x, \omega), E_P g_k(x, \omega)$, $k = 1, \dots, m$, are *convex* in x on $\mathbb{R}^n$ then there exists a nonnegative parameter vector $t \in \mathbb{R}^m$, such that the efficient points can be obtained by solving a *scalar convex optimization problem*

$$\min_{x \in \mathcal{X}} [E_P f(x, \omega) + \sum_{k=1}^{m} t_k E_P g_k(x, \omega)]. \tag{27}$$

This scalarization is a special form of scalarization by a penalty function $\vartheta : \mathbb{R}^m \to \mathbb{R}$ which must be continuous and nondecreasing in its arguments to provide efficient solutions of (26):

$$\min_{x \in \mathcal{X}} \left[E_P f(x, \omega) + N \vartheta(E_P g_1(x, \omega), \dots, E_P g_m(x, \omega))\right]; \tag{28}$$

$N > 0$ is a parameter. (For relevant results on multiobjective optimization see e.g. [20].)

A rigorous proof of the relationship between optimal values and solutions of (21) and those of (22) for the penalty function $N \sum_{k=1}^{m} [g_k(x, \omega)]^+$ is due to Ermoliev, et. al. [17]. It is valid under modest assumptions on the nonlinear functions g_k, on continuity of the probability distribution P and on the structure of problem (21).

The approach by [17] can be further extended to a whole class of penalty functions ϑ. For functions $\vartheta : \mathbb{R}^m \to \mathbb{R}_+$ which are continuous nondecreasing in their components, equal to 0 on $\mathbb{R}^m_-$ and positive otherwise, it holds that

$$P(g_k(x,\omega) \leq 0,\ 1 \leq k \leq m) \geq 1 - \varepsilon \quad \Longleftrightarrow \quad P(\vartheta(g(x,\omega)) > 0) \leq \varepsilon.$$

The considered penalty function problem can be formulated as follows

$$\varphi_N(P) = \min_{x \in \mathcal{X}} \left[E_P f(x,\xi) + N \cdot E_P \vartheta(g(x,\omega)) \right] \tag{29}$$

with N a positive parameter. We denote $x_N(P)$ an optimal solution of (29) and $x_\varepsilon(P)$ an optimal solution of (21) with a level $\varepsilon \in (0,1)$.

Theorem 1 *For a fixed probability distribution P consider the two problems (21) and (29) and assume: $\mathcal{X} \neq \emptyset$ compact, $F(x,P) = E_P f(x,\omega)$ a continuous function of x,*
$\vartheta : \mathbb{R}^m \to \mathbb{R}_+$ a continuous function, nondecreasing in its components, which is equal to 0 on $\mathbb{R}^m_-$ and positive otherwise,

- *(i) $g_k(\cdot,\omega)\,\forall k$ are almost surely continuous;*
- *(ii) there exists a nonnegative random variable $C(\omega)$ with $E_P C^{1+\kappa}(\omega) < \infty$ for some $\kappa > 0$, such that $|\vartheta(g(x,\omega))| \leq C(\omega)$;*
- *(iii) $E_P \vartheta(g(x',\omega)) = 0$, for some $x' \in \mathcal{X}$;*
- *(iv) $P(g_k(x,\omega) = 0) = 0, \forall k$, for all $x \in \mathcal{X}$.*

Denote $\gamma = \kappa/2(1+\kappa)$, and for arbitrary $N > 0$ and $\varepsilon \in (0,1)$ put

$$\begin{aligned} \varepsilon(N) &= P(\vartheta(g(x_N(P),\omega)) > 0), \\ \alpha(N) &= N \cdot E_P \vartheta(g(x_N(P),\omega)), \\ \beta(\varepsilon) &= \varepsilon^{-\gamma} E_P \vartheta(g(x_\varepsilon(P),\omega)). \end{aligned}$$

THEN for any prescribed $\varepsilon \in (0,1)$ there always exists N large enough so that minimization (29) generates optimal solutions $x_N(P)$ which also satisfy the probabilistic constraints (5) with the given ε.

Moreover, bounds on the optimal value $\psi_\varepsilon(P)$ of (21) based on the optimal value $\varphi_N(P)$ of (29) and vice versa can be constructed:

$$\begin{array}{rcccl} \varphi_{1/\varepsilon^\gamma(N)}(P)-\beta(\varepsilon(N)) & \leq & \psi_{\varepsilon(N)}(P) & \leq & \varphi_N(P)-\alpha(N), \\ \psi_{\varepsilon(N)}(P)+\alpha(N) & \leq & \varphi_N(P) & \leq & \psi_{1/N^{1/\gamma}}(P)+\beta(1/N^{1/\gamma}), \end{array} \tag{30}$$

with

$$\lim_{N\to+\infty}\alpha(N)=\lim_{N\to+\infty}\varepsilon(N)=\lim_{\varepsilon\to 0_+}\beta(\varepsilon)=0.$$

Proof:
To simplify the notation for purposes of the proof, we shall drop the argument P used to indicate the dependence of optimal solutions on the probability distribution.
We denote

$$\delta_N=E_P\vartheta(g(x_N,\omega))$$

and assume that for some sequence x_N of optimal solutions of the problem (29) $\delta_N\to 0_+$. Our assumptions and general properties of the penalty function method, see [1, Theorem 9.2.2], ensure existence of such sequence of optimal solutions and also convergence of $\alpha(N)=N\delta_N\to 0$ for $N\to\infty$. By Chebyshev inequality

$$\begin{aligned} & P\Big(\vartheta(g(x_N,\omega))>0\Big)= \\ &= P\Big(0<\vartheta(g(x_N,\omega))\leq\sqrt{\delta_N}\Big)+P\Big(\vartheta(g(x_N,\omega))>\sqrt{\delta_N}\Big) \\ &\leq G_\vartheta(x_N,\sqrt{\delta_N})-G_\vartheta(x_N,0)+\frac{1}{\sqrt{\delta_N}}E_P\vartheta(g(x_N,\omega)) \\ &\leq G_\vartheta(x_N,\sqrt{\delta_N})-G_\vartheta(x_N,0)+\sqrt{\delta_N}\to 0 \text{ as } N\to\infty. \end{aligned}$$

Here for a fixed x, $G_\vartheta(x,\cdot)$ denotes the distribution function of $\vartheta(g(x,\omega))$ defined by

$$G_\vartheta(x,y)=P\Big(\vartheta(g(x,\omega))\leq y\Big).$$

Assumption (iii) implies that for every $\varepsilon>0$ there exists some $x_\varepsilon\in X$ such that

$$P(g_k(x_\varepsilon,\omega)\leq 0,\ 1\leq k\leq m)\geq 1-\varepsilon.$$

Then for any $\varepsilon > 0$ the following relations hold

$$
\begin{aligned}
& E_P\vartheta(g(x_\varepsilon,\omega)) = \\
&\quad \leq \int_\Omega C(\omega) I_{(\vartheta(g(x_\varepsilon,\omega))>0)} P(d\omega) \\
&\quad \leq \left(\int_\Omega C^{1+\kappa}(\omega)P(d\omega)\right)^{1/(1+\kappa)} \cdot \left(\int_\Omega I_{(\vartheta(g(x_\varepsilon,\omega))>0)} P(d\omega)\right)^{\kappa/(1+\kappa)} \\
&\quad \leq c \cdot P\Big(\vartheta(g(x_\varepsilon,\omega)) > 0\Big)^{\kappa/(1+\kappa)} \\
&\quad \leq c \cdot \varepsilon^{\kappa/(1+\kappa)},
\end{aligned}
$$

where $c := \left(\int_\Omega C^{1+\kappa}(\omega)P(d\omega)\right)^{1/(1+\kappa)}$, which is finite due to the assumption (ii). Accordingly, for $\varepsilon \to 0_+$

$$
0 \leq E_P\vartheta(g(x_\varepsilon,\omega)) \leq c \cdot \varepsilon^{\kappa/(1+\kappa)} \to 0.
$$

If we set

$$
\varepsilon(N) = P\Big(\vartheta(g(x_N,\omega)) > 0\Big),
$$

then the optimal solution x_N of the expected value problem is feasible for the probabilistic program with $\varepsilon = \varepsilon(N)$, because the following relations hold

$$
\begin{aligned}
& P(g_k(x_N,\omega) \leq 0,\ 1 \leq k \leq n) \geq 1 - \varepsilon(N) \\
&\qquad \Longleftrightarrow P\Big(\vartheta(g(x_N,\omega)) > 0\Big) \leq \varepsilon(N).
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
\varphi_N(P) &= F(x_N,P) + N \cdot E_P\vartheta(g(x_N,\omega)) \\
&\geq F(x_{\varepsilon(N)},P) + N \cdot E_P\vartheta(g(x_N,\omega)) \\
&= \psi_{\varepsilon(N)}(P) + \alpha(N).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\psi_\varepsilon(P) &= \Big(\psi_\varepsilon(P) + \varepsilon^{-\gamma} E_P\vartheta(g(x_\varepsilon,\omega))\Big) - \varepsilon^{-\gamma} E_P\vartheta(g(x_\varepsilon,\omega)) \\
&\geq \varphi_{\varepsilon^{-\gamma}}(P) - \varepsilon^{-\gamma} E_P\vartheta(g(x_\varepsilon,\omega)) \\
&= \varphi_{\varepsilon^{-\gamma}}(P) - \beta(\varepsilon).
\end{aligned}
$$

This completes the proof.

It means that under assumptions of the Theorem, the two problems (21), (29) are *asymptotically equivalent* which is a useful theoretical result. Notice, however, that when we want to evaluate one of the bounds in (30), we must be prepared to face some problems. We solve the penalty function problem (29) taking a sufficiently large $N > 0$ to get its optimal solution $x_N(P)$ and optimal value $\varphi_N(P)$. Then we are able to compute $\alpha(N)$, $\varepsilon(N)$, hence the upper bound for the optimal value $\psi_{\varepsilon(N)}(P)$ of the probabilistic program (5) with probability level $\varepsilon(N)$. But we are not able to compute $\beta(\varepsilon(N))$ without having the solution $x_{\varepsilon(N)}(P)$ which we do not want to find or even may not able to find. We can only solve the penalty function problem with $N = 1/\varepsilon^\gamma(N)$ getting its optimal solution $x_{1/\varepsilon^\gamma(N)}(P)$ and optimal value $\varphi_{1/\varepsilon^\gamma(N)}(P)$ which is only a part of the lower bound for the optimal value $\psi_{\varepsilon(N)}(P)$.

The bounds (30) and the terms $\alpha(N)$, $\varepsilon(N)$ and $\beta(\varepsilon)$ depend on the choice of the penalty function ϑ. Two special penalty functions are readily available: $\vartheta_1(u) = \sum_{k=1}^m [u_k]^+$ applied in [17] and $\vartheta_2(u) = \max_{1\le k\le m}[u_k]^+$. However, only ϑ_1 preserves convexity, whereas ϑ_2 may work better for joint probabilistic constraints.

The obtained problems with penalties and with a fixed set of feasible solutions are much simpler to solve and analyze then the probabilistic programs; namely, contamination technique can be used for output analysis of (29) provided that the expectation-type objective function is convex or continuous in x. A question for future research is how to choose the parameter N so that the probability level ε is ensured.

4 Approximate contamination bounds

As discussed in Section 2, to construct global bounds for the optimal value of stochastic programs under contamination of the probability distribution, i.e. with respect to the contamination parameter λ, we need that the optimal value function $\varphi(\lambda)$ is *concave* to get the lower bound in (12). To prove the existence and the form of the directional derivative in the upper bound of (12) one needs further results on *stability* of the initial program and differentiability of the contaminated objective function.
With the set of feasible solutions dependent on P, concavity of the optimal value function $\varphi_{PQ}(\lambda)$ cannot be guaranteed. Such problems may be ap-

proximated by penalty-type problems which possess a fixed set of feasible first-stage solutions. This in turn opens the possibility to construct approximate contamination bounds.

EXAMPLE 2.
For the general probabilistic program (21), let us accept the problem

$$\min_{x\in\mathcal{X}}\left[E_P f(x,\omega)+\sum_{k=1}^{m} t_k E_P[g_k(x,\omega)]^+\right] \tag{31}$$

with convex compact $\mathcal{X}\neq\emptyset$, convex functions $f(\cdot,\omega)$, $g_k(\cdot,\omega)$, $k=1,\dots,m$ and a fixed nonnegative parameter vector $t\in\mathbb{R}^m$ as an acceptable substitute for the probabilistic program (21)

$$\min\{E_P f(x,\omega)\,:\,x\in\mathcal{X}\cap\mathcal{X}_\varepsilon(P)\}$$

with $\mathcal{X}_\varepsilon(P)$ defined by (5). Denote the random objective in (31) as

$$\theta(x,\omega;t)=f(x,\omega)+\sum_{k=1}^{m} t_k[g_k(x,\omega)]^+.$$

Using this notation, we rewrite (31) as

$$\min_{x\in\mathcal{X}}\Theta(x,P;t):=E_P\theta(x,\omega;t).$$

The set of feasible solutions $\mathcal{X}$ does not depend on P and for fixed vectors t, the contamination bounds for the optimal value $\varphi_t(P_\lambda)$ of (31) for contaminated probability distribution (10) follow the usual pattern (13). They may serve as an indicator of robustness of the optimal value of (21) and may support decisions about the choice of weights t_k.

EXAMPLE 3.
In the model of [3], the second-stage variables y are supposed to satisfy with a prescribed probability $1-\delta$ certain constraints driven by another random factor, η, whose probability distribution is independent of P. The problem is

$$\min_{x\in\mathcal{X}} F(x,P,S):=E_P f(x,\omega,S)=c^\top x+\int_\Omega R(x,\omega,S)P(d\omega) \tag{32}$$

where

$$R(x,\omega,S)=\min_y\{q^\top y\,:\,Wy=h(\omega)-T(\omega)x,\;y\in\mathbb{R}^r_+,\;S(Ay\le\eta)\ge 1-\delta\}, \tag{33}$$

S denotes the probability distribution of s-dimensional random vector η and $A(s,r)$ is a deterministic matrix composed of rows A_j, $j = 1, \ldots, s$. With respect to P, (32) is an expectation type of stochastic program, with a fixed set $\mathcal{X}$ of feasible first-stage decisions and with an incomplete recourse; hence, there is a good chance to construct contamination bounds on the optimal value $\varphi(P_\lambda, S)$ of (32) for a *fixed* probability distribution S and for the contaminated probability distribution P_λ.

Assume that T is nonrandom with a full row rank, S is fixed and such that the set $\mathcal{Y}_S := \left\{ y \in \mathbb{R}^r_+ : \; S(Ay \leq \eta) \geq 1 - \delta \right\}$ is convex and bounded. Let the set $\mathcal{X}^*(P,S)$ of optimal solutions of (32) be nonempty and bounded. Under further assumptions which guarantee finiteness and convexity of the second stage value function $R(\cdot, \omega, S)$ for a fixed probability distribution S (hence finiteness and convexity of the random objective value function $f(\cdot, \omega, S)$), assumptions on existence of its expectation with respect to P and Q and assumptions on joint continuity of $F(\cdot, \cdot, S)$ in the decision vector x and the probability measure P, cf. [3], the directional derivative of the optimal value function $\varphi(\cdot, S)$ at P in the direction $Q - P$ exists and is of the form (11), i.e.

$$\min_{x \in \mathcal{X}^*(P,S)} F(x, Q, S) - \varphi(P, S). \tag{34}$$

Similar contamination bounds on the optimal value function can be derived even without assumptions which guarantee that the random objective function is convex in x. To this purpose, general contamination results of [9] and stability results of [3] may be exploited; cf. [4]. It is important, however, that some continuity properties hold true. This simplifies in case that the probability distribution P is absolutely continuous.

If S is a discrete probability distribution then for a fixed x and ω the penalty $R(x, \omega, S)$ can be evaluated by the approach of [29]. To solve the mixed program (32) and to construct contamination bounds based on (34) it would be necessary to modify the algorithm to the case of parametrized right-hand sides $h(\omega) - Tx$ of the second-stage problem (33).

To model perturbations with respect to S and to construct the corresponding contamination bounds is more demanding. A possibility is to replace the probabilistic constraint in (33) by an expected (with respect to S) penalty term added to $q^\top y$, as done e.g. in (22):

$R(x, \omega, S) \approx$

$$\min_y \{q^\top y + N \sum_{j=1}^{s} E_S[A_j y - \eta_j]^+ \,:\, Wy = h(\omega) - T(\omega)x,\; y \geq 0\} := \tilde{R}(x, \omega, S).$$

The approximate problem

$$\min_{x \in \mathcal{X}} \tilde{F}(x, P, S) := E_P \tilde{f}(x, \omega, S) = c^\top x + \int_\Omega \tilde{R}(x, \omega, S) P(d\omega)$$

is linear in P and concave in S, $\mathcal{X}$ is independent of P, S which allows to construct contamination bounds for contaminated probability distribution S, too. In this case, however, existence of the directional derivative (15) must be examined in detail.

5 Conclusions

Contamination technique is known to be a suitable tool to robustness analysis of penalty-type stochastic programs with respect to changes of probability distributions. It provides global bounds for the optimal value of stochastic program with respect to perturbations of the probability distribution modeled via contamination. As the set of feasible decisions of probabilistic programs depends on the probability distribution, the possibility of obtaining similar global results for probabilistic programs is limited to problems of a very special form and/or under special distributional assumptions; see Examples 1 and 3 for an illustration and discussion.

Reformulation of probabilistic programs by incorporating a suitably chosen penalty function into the objective helps to arrive at problems with a fixed set of feasible solutions which in turn opens the possibility of construction of the convexity-based contamination bounds. The recommended form of the penalty function follows the basic ideas of multiobjective programming and its suitable properties follow by generalization of results of [17].

For expectation-type objectives, conditions for differentiability with respect to the contamination parameter of the contaminated objective function reduce mostly to continuity or convexity of the two noncontaminated objectives. For general types of objectives, to derive the existence and the form of these directional derivatives the local qualitative stability results play an important role.

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