SCENARIO BASED STOCHASTIC PROGRAMS: RESISTANCE WITH RESPECT TO SAMPLE

Jitka Dupačová

Charles University, Prague, Czech Republic

Abstract. Contamination technique is presented as a numerically tractable tool to postoptimization and analysis of robustness of the optimal value of scenario based stochastic programs and of the expected value problems. Detailed applications of the method concern the two-stage stochastic linear programs with random recourse and the corresponding robust optimization problems.

Keywords: Contamination technique, SLP with random recourse, robust optimization, expected value problems, postpotimality, sensitivity analysis, worst case analysis.

1. Introduction

The numerical techniques designed for solving stochastic programming problems are mostly based on scenarios, i.e., they assume a given *discrete* distribution P concentrated in a finite number of points, say, $\omega_1, \dots, \omega_S$ with probabilities $p_s > 0 \quad \forall s, \quad \sum_{s=1}^S p_s = 1$ that enter the coefficients and the function values in a known way.

The origin of scenarios can be very diverse; they may be from a truly discrete known distribution, be obtained in the course of a discretization/approximation scheme or by a limited sample information, or come from attempts to model uncertainty by means of scenarios obtained by a preliminary analysis of the problem and with probabilities of their occurance that may reflect an ad hoc belief or a subjective opinion of an expert.

Naturally, one is interested in both the robustness of the obtained optimal solution and the optimal value of the objective function. The procedure should be robust in the sense that small perturbances of the input, i.e., of the chosen scenarios and of their probabilities, should alter the outcome only slightly so that the obtained results remain close to the unperturbed ones, and that somewhat larger perturbations do not cause a catastrophe. The importance of robust procedures increases with the complexity of the model and with its dimensionality.

Two types of numerical procedures can be distinguished: those based on a large and in principle *fixed* set of scenarios and those where proper sampling and generation of new scenarios becomes a part of the procedure, e.g., stochastic quasigradient methods [9] or stochastic decomposition [14]. We shall concentrate on the first of

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the mentioned groups where the concept of robustness is *resistance to changes with respect to sample*. That is, the output values are insensitive to small changes in the underlying sample, e. g., insensitive with respect to small changes in all values or large changes in a few values, insensitive with respect to changes of probabilities, etc.

The basic techniques used for study of resistance are

(i) miscellaneous approaches to stability, sensitivity and postoptimality for optimization, based, e.g., on results for linear programming [12], on duality for nonlinear programs [22], on stability concepts for general parametric programs [20] including input optimization [8];
(ii) the worst case analysis, e.g., [2], [8], [21];
(iii) contamination technique, cf. [7];
(iv) various simulation studies, e.g.,[4], [17], [23], the prevailing approach in real life applications.

We shall elaborate here the *contamination technique* which is, inter alia, suitable for analysis of influence of additional scenarios and for constructing error bounds. We refer to [5] for the first theoretical results and to [7] for the first application in the field of multistage stochastic linear programming with fixed complete recourse. In this paper, we shall extend the results to stochastic linear programs with random recourse and to problems in which the objective function is nonlinear in distribution P for to cover, e.g., the case of mean-variance criterion used in robust optimization models [18], [19].

We shall consider stochastic programs that can be put into the following form:

$$\text{Minimize} \quad f(\mathbf{x}, P) \quad \text{on the set} \quad \mathcal{X} \subset R^n \tag{1}$$

with

f convex in $\mathbf{x}$ and concave in P;

P the probability distribution of the random parameters $\omega \in \Omega$ that enter the problem; in the case of scenario based stochastic programs we consider here, P is a discrete probability distribution and for a given set of possible scenarios, this distribution is *fully determined* by the vector $\mathbf{p}$ of their probabilities.

$\mathcal{X}$ a closed, nonempty set that *does not depend on* P;

$\mathbf{x}$ the main decision variable, typically, the first stage decision.

Problems with $f(\mathbf{x}, \bullet)$ *linear* in P correspond, e.g., to minimization of the expected value of minus utility of the random outcome of decisions $\mathbf{x} \in \mathcal{X}$.

Example 1. Scenario based two-stage stochastic linear programs (SLP) with *random relatively complete recourse* appear in financial models that take into account random prices in connection with portfolio rebalancing or with conservation of cashflows, cf. [11]. They can be written in the familiar form:

Minimize

$$\mathbf{c}^\top \mathbf{x} + \sum_{s=1}^{S} p_s \mathbf{q}_s^\top \mathbf{y}_s \tag{2}$$

subject to

$$
\begin{array}{lllll}
\mathbf{Ax} & & & & = \mathbf{b} \\
\mathbf{T}_1\mathbf{x} + & \mathbf{W}_1\mathbf{y}_1 & & & = \mathbf{h}_1 \\
\mathbf{T}_2\mathbf{x} + & & \mathbf{W}_2\mathbf{y}_2 & & = \mathbf{h}_2 \\
\vdots & & & \ddots & \vdots \\
\mathbf{T}_S\mathbf{x} + & & \dots & + \ \mathbf{W}_S\mathbf{y}_S & = \mathbf{h}_S
\end{array} \tag{3}
$$

$$\mathbf{x} \geq 0, \mathbf{y}_s \geq 0, s = 1, \dots, S$$

where $\omega_s = [\mathbf{q}_s, \mathbf{T}_s, \mathbf{W}_s, \mathbf{h}_s], s = 1, \dots, S$ are scenarios or atoms at which the probability distribution P is concentrated and $p_s \geq 0, s = 1, \dots, S$ are their probabilities, $\sum_s p_s = 1$.

It is easy to reformulate (2), (3) into the form (1). We can put

$$\mathcal{X} = \{\mathbf{x} \in R^n | \mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq 0\}$$

and

$$f(\mathbf{x}, P) = \mathbf{c}^\top \mathbf{x} + \sum_{s=1}^{S} p_s q(\mathbf{x}, \omega_s)$$

where

$$q(\mathbf{x}, \omega_s) := \min_{\mathbf{y}_s} \left\{ q_s^\top \mathbf{y}_s | \mathbf{W}_s \mathbf{y}_s = \mathbf{h}_s - \mathbf{T}_s \mathbf{x}, \mathbf{y}_s \geq 0 \right\} \tag{4}$$

Provided that $\mathcal{X}$ is nonempty and that the second stage problems (4) have optimal solutions for all considered scenarios and for all $\mathbf{x} \in \mathcal{X}$, the basic assumptions on the the objective function $f(\mathbf{x}, P)$ and on the set of feasible first stage solutions $\mathcal{X}$ for problem (1) are evidently fulfilled.

Example 2. Robust optimization model for stochastic linear program (2), (3) as formulated in [18], [19] reads:

Minimize

$$\sum_{s=1}^{S} p_s \xi_s + \lambda \sum_{s=1}^{S} p_s \left[\xi_s - \sum_{j=1}^{S} p_j \xi_j \right]^2 \tag{5}$$

subject to

$$
\begin{array}{ll}
\mathbf{Ax} & = \mathbf{b} \\
\mathbf{T}_s\mathbf{x} + \mathbf{W}_s\mathbf{y}_s & = \mathbf{h}_s \\
\mathbf{c}^\top \mathbf{x} + \mathbf{q}_s^\top \mathbf{y}_s - \xi_s & = 0
\end{array} \tag{6}
$$

$$\mathbf{x} \geq 0, \mathbf{y}_s \geq 0, s = 1, \dots, S$$

The newly introduced variables ξ_s equal the cost of the decision $\mathbf{x}$ plus the corresponding cost of its compensation or of the recourse activity $\mathbf{y}_s$ if scenario ω_s occurs. The additional term in the objective function equals the variance of the random costs ξ and its weight in the objective function is expressed via a scalar parameter $\lambda \geq 0$.

Transformation of (5), (6) into the form (1) is not straighforward. The details will be given in Section 4.

Further examples that can be used to illustrate the general form of the considered problem (1) and to provide a motivation for our studies are scenario based multistage stochastic programs, see [7], expected utility models, tracking models [3], stochastic programs with piece-wise linear-quadratic recourse [16] or general nonlinear two-stage stochastic programs with recourse.

In all mentioned examples, we are interested in resistance of the obtained optimal decisions and of the optimal value with respect to the sample: For the already given universe of scenarios $\Omega = \{\omega_1, \ldots, \omega_S\}$ we want to study the influence of changes of input scenarios ω_s and of their probabilities, the influence of an additional scenario, etc., on the obtained outcome. We shall put aside problems of probability sampling or of generation of scenarios and we shall concentrate on numerically tractable stability and postoptimality results based on the contamination technique [5].

2. Contamination technique

We shall begin with a brief summary of the contamination technique (cf. [6], [7]) for the general form of stochastic programs (1) under assumptions that $\mathcal{X}$ is a given nonempty convex closed set of feasible solutions that does not depend on the probability distribution P and that the objective function f is convex in $\mathbf{x}$ and concave in P. We shall embed the problem (1) into a family of optimization problems parametrized by a *scalar* parameter t. This family results from contamination of the original probability distribution P by another *fixed* probability distribution Q, i. e., from using distributions P_t of the form

$$(7) \qquad P_t = (1-t)P + tQ \quad \text{with} \quad t \in (0,1)$$

in the objective function of (1) at the place of P. For fixed distributions P, Q, we denote

$$f_Q(\mathbf{x}, P_t) := f_Q(\mathbf{x}, t)$$

the corresponding objective function; it is evidently a convex - concave function on $R^n \times [0,1]$. We denote further

$$(8) \qquad \varphi_Q(t) = \inf_{\mathbf{x} \in \mathcal{X}} f_Q(\mathbf{x}, t) \quad \text{and} \quad \mathcal{X}_Q(t) = \arg\min_{\mathbf{x} \in \mathcal{X}} f_Q(\mathbf{x}, t)$$

the optimal value function and the set of optimal solutions of the perturbed stochastic program

$$(9) \qquad \text{minimize} \quad f(\mathbf{x}, P_t) := f_Q(\mathbf{x}, t) \quad \text{on the set} \quad \mathcal{X}$$

There are various statements about persistence, stability and sensitivity for parametric programs of the above type; see e. g. [10, §7]:

• Under the additional assumption that the set $\mathcal{X}(0) := \mathcal{X}_Q(0)$ of optimal solutions of the original problem (1) is nonempty and bounded and that $\mathcal{X}_Q(1) \neq \emptyset$, the function φ_Q is a finite concave function on $[0,1]$, continuous at $t=0$ (cf. [10], Theorem 15) and its value at $t=0$ equals the optimal value of (1):

$$\varphi_Q(0) = \min_{\mathbf{x}\in\mathcal{X}} f(\mathbf{x},P) := \varphi(0)$$

• If, moreover, the objective function f_Q is jointly continuous with respect to $\mathbf{x}$ and t, its derivative exists with respect to t at $t=0^+$ for all $\mathbf{x}$ from a neighborhood, say, $\mathcal{X}^*$ of $\mathcal{X}(0)$ and if the convergence of the difference quotients $1/t[f_Q(\mathbf{x},t) - f_Q(\mathbf{x},0)]$ for $t \to 0^+$ is uniform in $\mathbf{x}$ on $\mathcal{X}^*$, we can use a slight modification of Theorem 17 of [10] to get the marginal value of the perturbed program (9) at $t=0$:

$$\varphi'_Q(0^+) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi_Q(0^+) = \min_{\mathbf{x}\in\mathcal{X}(0)} \frac{\mathrm{d}}{\mathrm{d}t} f_Q(\mathbf{x},0^+) \tag{10}$$

In case of $f(\mathbf{x},P)$ *linear in* P,

$$f_Q(\mathbf{x},t) = (1-t)f(\mathbf{x},P) + tf(\mathbf{x},Q)$$

is a linear function in t and for an arbitrary fixed $\mathbf{x}$, the sequence of difference quotients is a stationary one. Accordingly, (10) reduces to

$$\varphi'_Q(0^+) = \min_{\mathbf{x}\in\mathcal{X}(0)} [f(\mathbf{x},Q) - f(\mathbf{x},P)] = \min_{\mathbf{x}\in\mathcal{X}(0)} f(\mathbf{x},Q) - \varphi(0) \tag{11}$$

It means that in this special but frequent case *the marginal value equals the difference between the minimal expected cost of an optimal decision based on the initial distribution P if $Q \neq P$ applies and the minimal expected costs under P.*

Using the marginal value and concavity of φ_Q on $[0,1]$ we can bound the considered perturbed optimal value function $\varphi_Q(t)$ as follows:

$$(1-t)\varphi(0) + t\varphi_Q(1) \le \varphi_Q(t) \le \varphi(0) + t\varphi'_Q(0^+) \quad \forall t \in [0,1] \tag{12}$$

and get bounds on the relative change of the perturbed optimal value due to contamination:

$$\varphi_Q(1) - \varphi(0) \le \frac{1}{t}\left[\varphi_Q(t) - \varphi(0)\right] \le \varphi'_Q(0^+) \quad \forall t \in [0,1] \tag{13}$$

It is important to realize that the bounds (12) and (13) are based on the assumed properties of the objective function $f(\mathbf{x},P)$ as a function of the probability distribution P *without any convexity assumptions concerning random coefficients* that enter the initial formulation of the analyzed stochastic program, such as (2), (3) or (5), (6).

The choice of a degenerated distribution $Q = \delta(\omega_*) := Q_*$ concentrated at $\omega_* \notin \Omega$ corresponds to an additional scenario and (10) or (11) provide an information about *the influence of including the additional scenario* ω_* on the optimal outcome. Similarly, a degenerated distribution $Q_* = \delta(\omega_*)$ with $\omega_* \in \Omega$ models the case of *increasing probability of scenario* ω_* and so on. The marginal values computed for degenerated contaminating distributions are related to the *influence curve* and they can be used to construct further characteristics of robustness acknowledged in robust statistics (cf. [13]) such as the *gross error sensitivity.* We shall reveal its role in *the worst case analysis* with respect to a whole set of scenarios.

Contamination by a distribution Q on Ω that gives the same expectation $E_Q\omega = E_P\omega$ is helpful in studying resistance with respect to changes of the sample in situations where the corresponding input information - the known fixed expectation of the random parameters ω - is to be preserved. Moreover, for the two stage SLP with P - the degenerated distribution concentrated at the expectation $E_P\omega$ that gives a unique optimal solution $\mathbf{x}(E)$ of the corresponding *expected value problem*, formula (11) provides a new interpretation of the difference between the expected total cost of the decision $\mathbf{x}(E)$ when a distribution Q applies and the optimal outcome of the expected value problem.

In the subsequent Sections, we shall apply the contamination technique to the two examples formulated in the Introduction.

3. Two-stage stochastic linear programs with random recourse

In this section we shall deal with the problem (2), (3) formulated in Example 1. As we know already, it is easy to link it with the general formulation (1): We consider the set of the feasible first stage solutions

$$\mathcal{X} = \{\mathbf{x}|\mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\} \tag{14}$$

and the objective function of the form

$$f(\mathbf{x}, P) = \mathbf{c}^\top \mathbf{x} + q(\mathbf{x}, P) \tag{15}$$

with

(16)

$$q(\mathbf{x}, P) = \min_{\mathbf{y}} \left\{ \sum\nolimits_{s=1}^{S} p_s \mathbf{q}_s^\top \mathbf{y}_s | \mathbf{W}_s \mathbf{y}_s = \mathbf{h}_s - \mathbf{T}_s \mathbf{x}, \quad \mathbf{y}_s \geq 0, s = 1, \dots, S \right\}$$
$$= \sum\nolimits_{s=1}^{S} p_s \min_{\mathbf{y}_s} \left\{ \mathbf{q}_s^\top \mathbf{y}_s | \mathbf{W}_s \mathbf{y}_s = \mathbf{h}_s - \mathbf{T}_s \mathbf{x}, \quad \mathbf{y}_s \geq 0 \right\} = \sum\nolimits_{s=1}^{S} p_s q(\mathbf{x}, \omega_s)$$

that is linear in P. We assume that $\mathcal{X} \neq \emptyset$ and that $q(\mathbf{x}, \omega_s)$ is finite on $\mathcal{X}$ for $s = 1, \dots, S$.

For the sake of simplicity we shall mostly assume that there is a *unique optimal first stage solution* $\mathbf{x}(0)$ of the problem (2), (3) for the initial discrete distribution P. In this case, (11) simplifies to

$$\varphi'_Q(0^+) = \frac{\mathrm{d}}{\mathrm{d}t} \varphi_Q(0^+) = f(\mathbf{x}(0), Q) - \varphi(0) \tag{17}$$

and *the additional numerical effort for computing the marginal value reduces to evaluation of the objective function for distribution Q at the already obtained first stage solution* $\mathbf{x}(0)$.

Application 1 - Sensitivity analysis with respect to probabilities. Assume first that the contaminating distribution Q is carried by scenarios ω belonging to the given universe of scenarios $\Omega = \{\omega_1, \ldots, \omega_S\}$ with probabilities $\pi_s \geq 0, s = 1, \ldots, S, \sum_s \pi_s = 1$. In this case, (17) assumes the following form:

$$\varphi'_Q(0^+) = \min \sum_s \pi_s \mathbf{q}_s^\top \mathbf{y}_s - \min \sum_s p_s \mathbf{q}_s^\top \mathbf{y}_s \tag{18}$$

with minimizations carried over all $\mathbf{y}_s, s = 1, \ldots, S$ belonging to the corresponding sets $\mathcal{Y}_s(\mathbf{x}(0))$ of feasible solutions of the systems

$$\mathbf{W}_s \mathbf{y}_s = \mathbf{h}_s - \mathbf{T}_s \mathbf{x}(0), \quad \mathbf{y}_s \geq 0 \tag{19}$$

for $s = 1, \ldots, S$; compare with (16). The optimal values of the two linear programs in (18) can be evidently rewritten as

$$\min_{\mathbf{y}_s \in \mathcal{Y}_s(\mathbf{x}(0)), s=1,\ldots,S} \sum_s \pi_s \mathbf{q}_s^\top \mathbf{y}_s = \sum_s \pi_s q(\mathbf{x}(0), \omega_s) \tag{20}$$

$$\min_{\mathbf{y}_s \in \mathcal{Y}_s(\mathbf{x}(0)), s=1,\ldots,S} \sum_s p_s \mathbf{q}_s^\top \mathbf{y}_s = \sum_s p_s q(\mathbf{x}(0), \omega_s) \tag{21}$$

with $q(\mathbf{x}(0), \omega_s)$ the minimal recourse costs attainable for the first stage solution $\mathbf{x}(0)$ and for the scenario ω_s:

$$q(\mathbf{x}(0), \omega_s) := \min_{\mathbf{y}_s} \left\{ \mathbf{q}_s^\top \mathbf{y}_s | \mathbf{W}_s \mathbf{y}_s = \mathbf{h}_s - \mathbf{T}_s \mathbf{x}(0), \mathbf{y}_s \geq 0 \right\} \tag{22}$$

Using (18) –(22) we get easily the upper and lower bounds for the marginal value $\varphi'_Q(0^+)$ over all considered contaminating distributions Q, in our case identified by probabilities π belonging to the set

$$\mathcal{Q} := \left\{ \pi \in R_+^S | \sum_{s=1}^S \pi_s = 1 \right\} \tag{23}$$

namely,

$$U_\mathcal{Q} := \max_{Q \in \mathcal{Q}} \varphi'_Q(0^+) = \mathbf{c}^\top \mathbf{x}(0) + \max_{s=1,\ldots,S} q(\mathbf{x}(0), \omega_s) - \varphi(0) \tag{24}$$

$$L_\mathcal{Q} := \min_{Q \in \mathcal{Q}} \varphi'_Q(0^+) = \mathbf{c}^\top \mathbf{x}(0) + \min_{s=1,\ldots,S} q(\mathbf{x}(0), \omega_s) - \varphi(0) \tag{25}$$

that are attained at extremal points of $\mathcal{Q}$, i. e., at the "most influential" individual scenarios. As the original probabilities $p \in \mathcal{Q}$, too, the maximum value in (24) is nonnegative and the minimum value in (25) is nonpositive. Using these bounds, we can easily get the *gross error sensitivity*

$$\gamma_{\mathcal{Q}} := \sup_{\omega\in\Omega} |\varphi'_{\delta(\omega)}(0^+)| = \max\{|L_{\mathcal{Q}}|, |U_{\mathcal{Q}}|\}$$

interpreted according to [13] as the worst approximate influence of a fixed amount of contamination by any degenerated distribution $Q = \delta(\omega)$ concentrated at $\omega \in \Omega$ on the optimal value. In our case, according to (24), (25), $\gamma_{\mathcal{Q}}$ provides the worst possible local influence of contamination by any discrete distribution Q on Ω, i. e., the worst possible influence of changing the initial probabilities p_s of the given scenarios $\omega_s, s = 1, \ldots, S$.

Additional assumptions about the set of contaminating measures $\mathcal{Q}$ lead to different extremal points that do not necessarily coincide with individual scenarios, nevertheless, the general methodology applies without any essential changes. For instance, according to results of [2], one can make use of a qualitative information of the type "scenario ω_i is at least as probable as scenario ω_j" that generates a partial order, say, $\succeq$ on the set Ω and to reformulate it into a system of linear constraints on the probabilities $\pi \in \mathcal{Q}$. For instance the set of probability distributions on Ω that is consistent with the simple partial order $\omega_i \succeq \omega_j$ mentioned above can be written as

$$\mathcal{Q} = \left\{\pi \in R_+^S | \pi_i \geq \pi_j, \sum_{s=1}^S \pi_s = 1\right\}$$

The extremal points of similar sets of probabilities were described for instance in [2]; in our example, the set of extremal points of $\mathcal{Q}$ consists of unit vectors of probabilities for individual scenarios with exception of ω_j and the vector π with nonzero components $\pi_i = \pi_j = 1/2$. Hence, the locally most influential changes of the initial probabilities $p_s, s = 1, \ldots, S$ do not necessarily correspond to contamination by a degenerated distribution.

Application 2 - Analysis of the expected value solution. Assume that the expectation $E\omega$ is fixed, considered as a known input. In this case, a proper choice of the set of contaminating distributions corresponds to probabilities belonging to

$$\mathcal{Q} = \left\{\pi \in R_+^S | \sum_{s=1}^S \pi_s = 1, \sum_{s=1}^S \pi_s \omega_s = E\omega\right\} \tag{26}$$

Assume further that the expected value $E\omega$ is a scenario in Ω and that the initial probability distribution P is degenerated, concentrated at this scenario $E\omega$. The corresponding optimal first stage solution $\mathbf{x}(0) = \mathbf{x}(E)$ is *the expected value solution* obtained by solving *the expected value linear program*:

Minimize

$$\mathbf{c}^\top \mathbf{x} + [E\mathbf{q}]^\top \mathbf{y} \tag{27}$$

subject to

$$\begin{aligned} \mathbf{A}\mathbf{x} \qquad\qquad &= \mathbf{b} \\ [E\mathbf{T}]\mathbf{x} + [E\mathbf{W}]\,\mathbf{y} &= [E\mathbf{h}] \end{aligned} \tag{28}$$

$$\mathbf{x} \geq 0, \mathbf{y} \geq 0$$

We assume that the $\mathbf{x}$– part $\mathbf{x}(E)$ of the optimal solution of (27), (28) is unique. Let $\mathbf{y}(\mathbf{x}(E))$ denote an optimal second stage solution. In this case, for any discrete distribution on Ω with the fixed prescribed expectation $E\omega$, i.e., for π belonging to the set (26), the marginal value

$$\varphi_Q'(0^+) = \sum_s \pi_s q(\mathbf{x}(E), \omega_s) - (E\mathbf{q})^\top \mathbf{y}(\mathbf{x}(E)) \tag{29}$$

equals the difference between the expected outcome of the optimal expected value solution $\mathbf{x}(E)$ evaluated under distribution Q belonging to the set (26) and the outcome of the expected value problem, i.e.,

$$\varphi_Q'(0^+) = E_Q EV - EV$$

in a slightly modified notation of [1]. The sign of the marginal value (29) gives an answer to a common question: Can we improve the result based solely on the expected value problem (27), (28) by using a non degenerated probability distribution with the same expected values? The known theoretical result based on Jensen's inequality holds true only for problems with a fixed recourse matrix $\mathbf{W}$ and fixed recourse costs $\mathbf{q}$ in which case,

$$\varphi_Q'(0^+) = E_Q EV - EV \geq 0$$

for all distributions Q from (16).

The worst case analysis with respect to all contaminating probability distributions on Ω with the given expected value $E\omega$ can be done again by solving a linear program, this time,

$$\text{maximize} \quad \sum_s \pi_s q(\mathbf{x}(E), \omega_s) \quad \text{on the set (26)} \tag{30}$$

(compare with (29)).

A similar approach can be designed for *postoptimality analysis of the optimal outcome based on another a priori selected scenario*, e.g., on the most probable scenario or on a "base case" scenario belonging to Ω.

Application 3 - Postoptimality analysis with respect to additional scenarios ■ corresponds to a contaminating distribution Q that is not carried by the given set of scenarios Ω but for which the two-stage stochastic linear program

$$\text{minimize} \quad f(\mathbf{x}, Q) \quad \text{on the set} \quad (14)$$

has an optimal solution. Contamination by a degenerated distribution $Q = Q_*$ concentrated at $\omega_* = [\mathbf{q}_*, \mathbf{T}_*, \mathbf{W}_*, \mathbf{h}_*]$ means inclusion of an additional scenario into the two-stage SLP (2), (3), an essential extension of the structure of this program that cannot be treaed efficiently by postoptimization techniques developed for linear programs.

The local change of the optimal value due to inclusion of scenario ω_* can be again described by means of the marginal value that is easy to compute provided that the first stage optimal solution $\mathbf{x}(0)$ of the original program is unique:

(31)
$$\varphi'_{Q_*}(0^+) = f(\mathbf{x}(0), Q_*) - \varphi(0) = \min_{\mathbf{y}_*} \{\mathbf{q}_*^\top \mathbf{y}_* | \mathbf{W}_* \mathbf{y}_* = \mathbf{h}_* - \mathbf{T}_* \mathbf{x}(0), \mathbf{y}_* \geq 0\} - q(\mathbf{x}(0), P)$$

As the contaminated optimal value $\varphi_{Q_*}(t)$ is finite and concave in t on the interval [0,1], we get immediately bounds (12) for the optimal value $\varphi_{Q_*}(t)$ that corresponds to the outcome of the inclusion of scenario ω_* with probability t and of proportional reduction of the original probabilities p_s of scenarios $\omega_s, s = 1, \ldots, S$ by the factor $1 - t$:

(32)
$$(1-t)\varphi(0) + t\varphi_{Q_*}(1) \leq \varphi_{Q_*}(t) \leq \varphi(0) + t\varphi'_{Q_*}(0^+), \quad \forall t \in [0,1]$$

where $\varphi_{Q_*}(1)$ is the optimal value of the linear program
minimize

$$\mathbf{c}^\top \mathbf{x} + \mathbf{q}_*^\top \mathbf{y}$$

subject to

(33)
$$\begin{aligned} \mathbf{A}\mathbf{x} \qquad &= \mathbf{b} \\ \mathbf{T}_*\mathbf{x} + \mathbf{W}_*\mathbf{y} &= \mathbf{h}_* \\ \mathbf{x} \geq 0, \mathbf{y} &\geq 0 \end{aligned}$$

that is based solely on the newly considered individual scenario ω_*. Using the formula (13) in our case, we can easily draw *the following conclusions*:

- if $f(\mathbf{x}(0), Q_*) \leq \varphi(0)$, inclusion of scenario ω_* improves the resulting outcome, i. e., its occurence contributes to decreasing the minimal total expected costs
- if $\varphi(0) \leq \varphi_{Q_*}(1)$, occurence of scenario ω_* causes a worse resulting outcome measured by minimal total expected costs.

In the same way, we can get bounds (32) for an arbitrary discrete distribution Q carried by "out-of-sample scenarios" that have not been included into the original set Ω. The optimal value $\varphi_Q(t)$ for the pooled sample can be estimated according to (10); the optimal values $\varphi(0)$ and $\varphi_Q(1)$ are computed for each group of scenarios separately, and evaluation of the marginal value means to compute in addition expected recourse costs for the optimal solution $\mathbf{x}(0)$ under new distribution Q. An application of this result to equiprobable scenarios has been delineated in [7].

The worst case analysis with respect to a given set of possible additional scenarios, say, Ω_*, can be performed similarly as in the first application provided that the first stage optimal solution $\mathbf{x}(0)$ of the original problem is unique: The lower and upper bounds for $\varphi'_Q(0^+)$ are again obtained for degenerated distributions Q on Ω_* (i.e., for extremal feasible solutions of the corresponding linear program) and the gross error sensitivity follows immediately.

4. Robust optimization model

The model introduced in Example 2
minimize

$$(5) \qquad \sum_{s=1}^{S} p_s\xi_s + \lambda \sum_{s=1}^{S} p_s \left[\xi_s - \sum_{j=1}^{S} p_j\xi_j\right]^2$$

subject to

$$(6) \qquad \begin{aligned} \mathbf{A}\mathbf{x} &= \mathbf{b} \\ \mathbf{T}_s\mathbf{x} + \mathbf{W}_s\mathbf{y}_s &= \mathbf{h}_s \\ \mathbf{c}^\top\mathbf{x} + \mathbf{q}_s^\top\mathbf{y}_s - \xi_s &= 0 \end{aligned}$$

$$\mathbf{x} \geq 0, \mathbf{y}_s \geq 0, s = 1, \dots, S$$

and its modifications have been applied in financial and other problems; see e.g. [15], [16], [18], [19]. We are not going to discuss here the pros and cons of the choice of the objective function (5) in robust optimization. The only fact important for our applications is that the model is based on a *fixed universe of scenarios* $\Omega = \{\omega_1, \dots, \omega_S\}$ and on given probabilities $p_s, \forall s$ so that the question of resistance of the results with respect to this choice of probability distribution arises once more.

We shall substitute $\mathbf{c}^\top\mathbf{x} + \mathbf{q}_s^\top\mathbf{y}_s$ for ξ_s into the objective function (5) and we shall apply the contamination technique to the problem
minimize

$$(33) \qquad \sum_{s=1}^{S} p_s \left[\mathbf{c}^\top\mathbf{x} + \mathbf{q}_s^\top\mathbf{y}_s\right] + \lambda \sum_{s=1}^{S} p_s \left[\mathbf{q}_s^\top\mathbf{y}_s - \sum_{j=1}^{S} p_j\mathbf{q}_j^\top\mathbf{y}_j\right]^2$$

subject to

$$(34) \qquad \begin{aligned} \mathbf{A}\mathbf{x} &= \mathbf{b} \\ \mathbf{T}_s\mathbf{x} + \mathbf{W}_s\mathbf{y}_s &= \mathbf{h}_s \end{aligned}$$

$$\mathbf{x} \geq 0, \mathbf{y}_s \geq 0, s = 1, \dots, S$$

with a fixed parameter value λ. We denote by $\mathcal{Z}$ the set of feasible solutions $\mathbf{z} := [\mathbf{x}, \mathbf{y}_s, \forall s]$ described by (34), we assume that the set $\mathcal{Z}_\lambda(0)$ of optimal solutions of (33), (34) is nonempty and bounded and we denote by $\varphi_\lambda(0)$ the optimal value of (33), (34). The objective function $f_\lambda(\mathbf{z}, P)$ defined by (33) is *concave* in the probability distribution P and convex with respect to all considered variables $\mathbf{z} = [\mathbf{x}, \mathbf{y}_s, \forall s]$. It can be briefly written as

$$(35) \qquad f_\lambda(\mathbf{z}, P) = \mathbf{c}^\top\mathbf{x} + E_P\{\mathbf{q}^\top\mathbf{y}\} + \lambda \mathrm{var}_P\{\mathbf{q}^\top\mathbf{y}\}$$

and its special form implies that for a unique $\mathbf{x}$-part of the optimal solution of (33), (34), there is necessarily either a unique optimal compensation $\mathbf{y}_s(0)$ for each of scenarios ω_s that enter with a positive probability p_s or that for all optimal compensations, the variance of the recourse costs $\mathbf{q}_s^\top \mathbf{y}_s, s = 1, \ldots, S$ equals zero - a trivially robust case.

The objective function that corresponds to contamination of P by another probability distribution Q is again of the form $f_\lambda(\mathbf{z}, (1-t)P + tQ) := f_{\lambda,Q}(\mathbf{z}, t)$, i.e., equals the original objective function parametrized by a scalar parameter $t \in [0, 1]$:

(36)

$$f_{\lambda,Q}(\mathbf{z}, t) = \mathbf{c}^\top \mathbf{x} + (1-t)E_P\{\mathbf{q}^\top \mathbf{y}\} + tE_Q\{\mathbf{q}^\top \mathbf{y}\} + \lambda(1-t)\mathrm{var}_P\{\mathbf{q}^\top \mathbf{y}\} + \lambda t \mathrm{var}_Q\{\mathbf{q}^\top \mathbf{y}\} + \lambda t(1-t)\left[E_P\{\mathbf{q}^\top \mathbf{y}\} - E_Q\{\mathbf{q}^\top \mathbf{y}\}\right]^2 \qquad \blacksquare$$

is a concave quadratic function of t, $f_{\lambda,Q}(\mathbf{z}, 0) = f_\lambda(\mathbf{z}, P)$, and its derivative

(37)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} f_{\lambda,Q}(\mathbf{z}, t) &= E_Q\{\mathbf{q}^\top \mathbf{y}\} - E_P\{\mathbf{q}^\top \mathbf{y}\} + \lambda \mathrm{var}_Q\{\mathbf{q}^\top \mathbf{y}\} - \lambda \mathrm{var}_P\{\mathbf{q}^\top \mathbf{y}\} \\ &\quad + \lambda(1-2t)\left[E_P\{\mathbf{q}^\top \mathbf{y}\} - E_Q\{\mathbf{q}^\top \mathbf{y}\}\right]^2 \\ &= f_\lambda(\mathbf{z}, Q) - f_\lambda(\mathbf{z}, P) + \lambda(1-2t)\left[E_P\{\mathbf{q}^\top \mathbf{y}\} - E_Q\{\mathbf{q}^\top \mathbf{y}\}\right]^2\end{aligned}$$

We shall assume that the optimal value $\varphi_{\lambda,Q}(1)$ is finite, i.e., that the problem has an optimal solution when P, identified by probabilities $p_s, s = 1, \ldots, S$, is replaced in (33) by the considered discrete contaminating distribution Q.

Application 1 - Sensitivity analysis with respect to probabilities. Let the distribution Q be carried by scenarios belonging to the given universe of scenarios $\Omega = \{\omega_1, \ldots, \omega_S\}$ with probabilities $\pi_s \geq 0, s = 1, \ldots, S, \sum_s \pi_s = 1$. According to (37), we have at any feasible point $\mathbf{z} \in \mathcal{Z}$

$$\frac{\mathrm{d}}{\mathrm{d}t} f_{\lambda,Q}(\mathbf{z}, 0^+) = f_\lambda(\mathbf{z}, Q) - f_\lambda(\mathbf{z}, P) + \lambda\left[\sum\nolimits_s (\pi_s - p_s)\mathbf{q}_s^\top \mathbf{y}_s\right]^2$$

and in the light of (10), the derivative of the optimal value function $\varphi_{\lambda,Q}(t)$ of the contaminated problem at the point $t = 0^+$ equals

$$(38) \qquad \varphi'_{\lambda,Q}(0^+) = \min_{\mathbf{z} \in \mathcal{Z}_\lambda(0)} \left[f_\lambda(\mathbf{z}, Q) + \lambda\left(\sum\nolimits_s (\pi_s - p_s)\mathbf{q}_s^\top \mathbf{y}_s\right)^2\right] - \varphi_\lambda(0)$$

where $\mathcal{Z}_\lambda(0)$ denotes the set of optimal solutions of the original problem (33), (34).

Due to the nonlinearity of the objective function $f_\lambda(\mathbf{x}, \bullet)$ with respect to the probability distribution, the marginal value (38) is equal to the difference of the objective function values computed at the initial optimal solution $\mathbf{z}(0)$ for the changed probabilities π and for the initial ones plus the minimal value of the additional quadratic term

$$\lambda\left(\sum\nolimits_s (\pi_s - p_s)\mathbf{q}_s^\top \mathbf{y}_s\right)^2$$

This term drops out if both the distribution P and Q lead to equal expected recourse costs $\sum_s p_s \mathbf{q}_s^\top \mathbf{y}_s$ and $\sum_s \pi_s \mathbf{q}_s^\top \mathbf{y}_s$ for the $\mathbf{y}$ –parts of the optimal solutions $\mathbf{z} \in \mathcal{Z}_\lambda(0)$.

The upper and lower bounds for the marginal value $\varphi'_{\lambda,Q}(0^+)$ with respect to all discrete contaminating distributions on Ω can be obtained similarly as for the two-stage SLP provided that the set of optimal solutions of (33), (34) is a singleton, say, $\mathcal{Z}_\lambda(0) = \{\mathbf{z}(0)\} = [\mathbf{x}(0), \mathbf{y}_s(0), \forall s]$. In this case, Q is represented by a vector of probabilities $\pi \in \mathcal{Q}$ defined by (23) and $\varphi'_{\lambda,Q}(0^+)$ is *linear* in π, so that the maximum and minimum values of $\varphi'_{\lambda,Q}(0^+)$ are attained at extremal points on $\mathcal{Q}$, i.e., for individual scenarios. We denote $\varphi'_{\lambda,s}(0^+)$ the marginal value for Q concentrated at scenario ω_s with probability 1. The corresponding formula for marginal value follows easily from (36):

$$\varphi'_{\lambda,s}(0^+) = \mathbf{c}^\top \mathbf{x}(0) - \varphi_\lambda(0) + \mathbf{q}_s^\top \mathbf{y}_s(0) + \lambda \left[\mathbf{q}_s^\top \mathbf{y}_s(0) - \sum\nolimits_{j=1}^S p_j \mathbf{q}_j^\top \mathbf{y}_j(0)\right]^2$$

The upper and lower bounds for $\varphi'_{\lambda,Q}(0^+)$ are given by

$$\max_{Q\in\mathcal{Q}} \varphi'_{\lambda,Q}(0^+) = \max_{s=1,\dots,S} \left\{ \mathbf{q}_s^\top \mathbf{y}_s(0) + \lambda \left[\mathbf{q}_s^\top \mathbf{y}_s(0) - \sum\nolimits_{j=1}^S p_j \mathbf{q}_j^\top \mathbf{y}_j(0)\right]^2 \right\} + \mathbf{c}^\top \mathbf{x}(0) - \varphi_\lambda(0)$$

$$\min_{Q\in\mathcal{Q}} \varphi'_{\lambda,Q}(0^+) = \min_{s=1,\dots,S} \left\{ \mathbf{q}_s^\top \mathbf{y}_s(0) + \lambda \left[\mathbf{q}_s^\top \mathbf{y}_s(0) - \sum\nolimits_{j=1}^S p_j \mathbf{q}_j^\top \mathbf{y}_j(0)\right]^2 \right\} + \mathbf{c}^\top \mathbf{x}(0) - \varphi_\lambda(0)$$

and they are attained at the locally most influential individual scenarios.

Application 2 - Analysis of the expected value solution. Similarly as in Application 2 for two-stage SLP we assume that the expectation $E\omega$ is fixed and equal to one of the considered scenarios $\omega \in \Omega$, say, $E\omega = \omega_1$ and that P is the degenerated distribution concentrated in $E\omega$. The corresponding *expected value solutions* of the robust optimization problem (33), (34) and of the two stage SLP (2), (3) are identical and can be obtained by solving the linear program (27), (28). For simplicity we assume again that the $\mathbf{x}$– part of this expected value solution is unique, equal to $\mathbf{x}(E)$; it means that the minimal costs for compensation equal $q(\mathbf{x}(E)) := (E\mathbf{q})^\top \mathbf{y}(\mathbf{x}(E))$ for all optimal second stage solutions $\mathbf{y}(\mathbf{x}(E))$ of (27), (28). Let $\mathcal{Y}^*(\mathbf{x}(E))$ denote the set of these solutions. The sets of optimal compensations of $\mathbf{x}(E)$ for the remaining scenarios $\omega_2, \dots, \omega_S$ equal the sets of the second stage feasible solutions

$$\mathcal{Y}_s(\mathbf{x}(E)) = \{\mathbf{y}_s | \mathbf{W}_s \mathbf{y}_s = \mathbf{h}_s - \mathbf{T}_s \mathbf{x}(E), \mathbf{y}_s \geq 0\}$$

Accordingly, the marginal value for Q concentrated at $\omega_1, \dots, \omega_S$ with probabilities $\pi_s \geq 0, \sum_s \pi_s = 1$ and $\sum_s \pi_s \omega_s = E\omega$ equals

(39)

$$\varphi'_{\lambda,Q}(0^+) = -q(\mathbf{x}(E)) + \min_{\mathbf{y}_s \in \mathcal{Y}_s(\mathbf{x}(E)), s=2,\dots,S} \Big\{ \pi_1 q(\mathbf{x}(E)) + \sum\nolimits_{s=2}^S \pi_s \mathbf{q}_s^\top \mathbf{y}_s + \lambda \pi_1 q^2(\mathbf{x}(E)) +$$

$$+ \lambda \sum\nolimits_{s=2}^S \pi_s \left(\mathbf{q}_s^\top \mathbf{y}_s\right)^2 - 2\lambda\pi_1 q^2(\mathbf{x}(E)) - 2\lambda q(\mathbf{x}(E)) \sum\nolimits_{s=2}^S \pi_s \mathbf{q}_s^\top \mathbf{y}_s + \lambda q^2(\mathbf{x}(E)) \Big\}$$

$$= \min_{\mathbf{y}_s \in \mathcal{Y}_s(\mathbf{x}(E)), s=2,\dots,S} \left\{ [1 - 2\lambda q(\mathbf{x}(E))] \sum\nolimits_{s=2}^S \pi_s \mathbf{q}_s^\top \mathbf{y}_s + \lambda \sum\nolimits_{s=2}^S \pi_s \left(\mathbf{q}_s^\top \mathbf{y}_s\right)^2 \right\} +$$

$$+ (\pi_1 - 1) q(\mathbf{x}(E)) + \lambda(1 - \pi_1) q^2(\mathbf{x}(E))$$

where the minimization can be obviously split into minimization of the individual terms for each of scenarios separately. Let us denote

$$r_s(\mathbf{x}(E)) = \min_{\mathbf{y}_s \in \mathcal{Y}_s(\mathbf{x}(E))} \left\{ [1 - 2\lambda q(\mathbf{x}(E))]\, \mathbf{q}_s^\top \mathbf{y}_s + \lambda \left(\mathbf{q}_s^\top \mathbf{y}_s\right)^2 \right\} \tag{40}$$

for $s = 2, \dots, S$ and for $s = 1$ put similarly

$$r_1(\mathbf{x}(E)) = [1 - 2\lambda q(\mathbf{x}(E))]\, q(\mathbf{x}(E)) + \lambda q^2(\mathbf{x}(E)) = q(\mathbf{x}(E)) - \lambda q^2(\mathbf{x}(E)) \tag{41}$$

Using (40), (41) in (39), we get the final formula for the corresponding marginal value

$$\varphi'_{\lambda,Q}(0^+) = \sum_{s=1}^{S} \pi_s r_s(\mathbf{x}(E)) - q(\mathbf{x}(E)) + \lambda \left[q(\mathbf{x}(E))\right]^2 \tag{42}$$

Once more, the sign of this marginal value helps to draw *conclusions about the position of the expected value solution*, this time for the robust optimization model (5), (6). The marginal value (42) is *linear in probabilities* π_s that identify the contaminating distribution Q and similarly as for two stage stochastic linear programs, *the worst case analysis with respect to the set* $\mathcal{Q}$ *of contaminating distributions on* Ω *identified by probabilities belonging to* (26) *reduces to the solution of a linear program*, this time of the form

$$\text{maximize} \quad \sum_s \pi_s r_s(\mathbf{x}(E)) \quad \text{on the set} \quad (26) \tag{43}$$

Application 3 - Postoptimality with respect to additional scenarios. We assume again that the problem (5), (6) has been solved for the distribution P identified by probabilities p_s of given scenarios $\omega_s, s = 1, \dots, S$. The contaminating distribution $Q = Q_*$ is a degenerated distribution that assigns probability 1 to a scenario $\omega_* \notin \Omega$ for which the corresponding single scenario problem is solvable. We can think of the original probability distribution as being carried by $\Omega \cup \{\omega_*\}$ with the original probabilities $p_1, \dots, p_S$ and with zero probability $p_* = 0$ of the additional scenario ω_*. Any optimal solution of the corresponding, formally extended problem consists of an optimal solution $[\mathbf{x}(0), \mathbf{y}_s(0), s = 1, \dots, S] \in \mathcal{Z}_\lambda(0)$ extended for an arbitrary element $\mathbf{y}_*$ of the set $\mathcal{Y}_*(\mathbf{x}(0))$ of feasible second stage solutions for the given first stage solution $\mathbf{x}(0)$ and for the additional scenario ω_*. Let $\mathcal{Z}_\lambda(0)$ be a singleton, $\mathcal{Z}_\lambda(0) = [\mathbf{x}(0), \mathbf{y}_s(0), s = 1, \dots, S]$; then the numerical evaluation of the marginal value

(44)

$$\varphi'_{\lambda,Q_*}(0^+) = \mathbf{c}^\top \mathbf{x}(0) + \min_{\mathbf{y}_* \in \mathcal{Y}_*(\mathbf{x}(0))} \left\{ \mathbf{q}_*^\top \mathbf{y}_* + \lambda \left[\mathbf{q}_*^\top \mathbf{y}_* - \sum_{s=1}^{S} p_s \mathbf{q}_s^\top \mathbf{y}_s(0) \right]^2 \right\} - \varphi_\lambda(0)$$

reduces to solution of a simple quadratic program that corresponds to the additional scenario ω_* and it is easy to obtain the bounds (12), (13) and to use them for *classification of the considered additional scenarios as to their impact on the resulting outcome.*

Notice that, once more, *the influence of the additional scenario* ω_* *does not depend solely on the difference*

$$\mathbf{c}^\top \mathbf{x}(0) + \min_{\mathbf{y}_* \in \mathcal{Y}_*(\mathbf{x}(0))} \{\mathbf{q}_*^\top \mathbf{y}_*\} - \varphi_\lambda(0)$$

between the value of the objective function (35) *evaluated for the additional scenario at the obtained optimal solution and the original optimal value* $\varphi_\lambda(0)$; the additional quadratic term relates to the *variability of the costs for compensation of the optimal first stage solution* $\mathbf{x}(0)$.

An extension of this approach to a set of out-of-sample scenarios is quite straightforward.

5. Conclusions

Postoptimality analysis via contamination technique as suggested in the preceding Sections for scenario based stochastic programs provides an easily tractable and flexible tool for exploring the effects of changes in the selected scenarios and their probabilities on the optimal value. The essence of this method implies that one does not need to worry about convexity properties with respect to random coefficients when constructing bounds for the optimal value of the perturbed problem.

The starting point for the proposed postoptimality analysis is, of course, the solution of the initial stochastic program that provides the optimal value and at least one of optimal solutions. The considered contaminated distribution P_t that models the perturbed input information is carried by the pooled sample of scenarios belonging to the original set of scenarios Ω and the out-of-sample scenarios. The described postoptimality approach provides, inter alia, bounds for the optimal value based on this pooled sample of scenarios, see (12). To compute the bounds, one has to solve the problem for the contaminating distribution Q carried by out-of-sample scenarios and to evaluate the marginal value (10).

The general formula (10) can be easily specified to the case of two-stage stochastic programs with relatively complete random recourse, see e.g., (18) or (31), and to the robust optimization model, see e.g., (38) or (42). If the initial problem has a unique optimal solution, the main computational effort needed for evaluation of the marginal value consists of evaluating the objective function at the initial optimal solution but under the contaminating distribution Q. Moreover, linearity of the marginal values with respect to the contaminating distributions reduces the worst case analysis with respect to a whole set of discrete contaminating distributions described by a given polyhedral set of probabilities to a linear programming problem.

The presence of multiple optimal solutions complicates the worst case analysis; still, for any of these optimal solutions, one gets an upper bound on the marginal value simply by omitting the min operation in the corresponding formulas based on (10).

Examples of other problems to which the contamination technique could be applied in a similar way were mentioned in Section 1. For instance, it is obvious how to get postoptimality analysis for a tracking model regarding changes of weights or inclusion of an additional scenario.

References

(1) J. R. Birge: The value of the stochastic solution in stochastic linear programs with fixed recourse. Mathematical Programming 24 (1982) 314–325.

(2) W. Bühler: Capital budgeting under qualitative data information. In: R. L. Crum and F. G. J. Derkinderen (eds.), Capital Budgeting under Conditions of Uncertainty (M. Nijhoff Publ., Boston, 1981) p. 81–117.

(3) R. S. Dembo: Scenario optimization. Annals of Oper. Research 30 (1991) 63–80

(4) M. A. H. Dempster and A. M. Ireland: A financial expert decision system. In: G. Mitra (ed.), Mathematical Models for Decision Support, NATO ASI Ser. Vol 48 (Springer, Berlin, 1988) p. 415–440.

(5) J. Dupačová: Stability in stochastic programming with recourse. Contaminated distributions. Math. Progr. Study 27 (1986) 133–144.

(6) J. Dupačová: Stability and sensitivity analysis for stochastic programming. Annals of Oper. Res. 27 (1990) 115–142.

(7) J. Dupačová: Postoptimality for multistage stochastic programs. To appear in Annals of Oper. Res. (Udine 1992).

(8) J. Dupačová: Analysis of the expected value solution and the worst case analysis under relaxed convexity assumptions. Paper presented at Minisymposium on Stochastic Programming, Berlin, Jan. 22–23, 1994.

(9) Yu. Ermoliev: Stochastic quasigradient methods and their application to systems optimization. Stochastics 9 (1983) 1 –36.

(10) E. G. Gol'shtein: Theory of Convex Programming, Translations of Mathematical Monographs 36 (American Mathematical Society, Providence RI, 1972).

(11) B. Golub et al.: A stochastic programming model for money management. RR 93–01–01, Hermes Laboratory, Wharton, Philadelphia, 1992. To appear in EJOR.

(12) J. Gondzio and A. Ruszczyński: Sensitivity method for basis inverse representation in multistage stochastic linear programming problems. JOTA 74 (1992) 221–242.

(13) F. R. Hampel: The influence curve and its role in robust estimation. J. Am. Statist. Assoc. 69 (1974) 383–397.

(14) J. L. Higle and S. Sen: Stochastic decomposition: An algorithm for two stage linear programs with recourse. Mathematics of Oper. Res. 16 (1991) 650–669.

(15) A. J. King et al.: Lake eurotrophication management: The Lake Balaton project. In: Yu. Ermoliev and R. J.-B. Wets (eds.), Numerical Techniques for Stochastic Optimization (Springer, Berlin, 1988), p. 435–444.

(16) A. J. King and D. J. Jensen: Linear-quadratic efficient frontiers for portfolio optimization. Appl. Stoch. Models and Data Anal. 8 (1992) 195–208.

(17) M. I. Kusy and W. T. Ziemba: A bank asset and liability management model. Oper. Res. 34 (1986) 356–376.

(18) J. M. Mulvey: Robust optimization on PC supercomputers. SIAM News 26 (1993) No.7.

(19) J. M. Mulvey et al.: Robust optimization of large scale systems: General modeling framework and computations. RR 91-06-04, Wharton, Univ. of Pennsylvania. To appear in Oper. Res.

(20) S. M. Robinson and R. J.-B. Wets: Stability in two-stage stochastic programming. SIAM J. Control and Optimization 25 (1987) 1409–1416.
(21) J. K. Sengupta: Robust solutions in stochastic programming. J. Opl. Res. Soc. 42 (1991) 857–870.
(22) R. J.-B. Wets: The aggregation principle in scenario analysis and stochastic optimization. In: S. W. Wallace (ed.), Algorithms and Model Formulations in Mathematical Programming (Springer, Berlin, 1989) p. 91–113.
(23) S. A. Zenios: Massively parallel computations for financial planning under uncertainty. In: J. Mesirov (ed.), Very Large Scale Computing in the 21-st Century (SIAM, Philadelphia, 1991) p. 273–294.