AUTHOR QUERIES

Journal id: RQUF_A_197233

Corresponding author: JITKA DUPAČOVÁ

Title: Stress testing for VaR and CVaR

Dear Author
Please address all the numbered queries on this page which are clearly identified on the proof for your convenience.

Thank you for your cooperation

Query number	Query
1	Please provide missing text in the x axis of figure 1
2	Please update details

Quantitative Finance, Vol. 7, No. ?, Month?? 2007, 1–11

Routledge
Taylor & Francis Group

Stress testing for VaR and CVaR

JITKA DUPAČOVÁ* and JAN POLÍVKA

Faculty of Mathematics and Physics,
Department of Probability and Mathematical Statistics,
Charles University, Prague, Czech Republic

(*Received 18 November 2005; in final form 21 August 2006*)

The practical use of the contamination technique in stress testing for risk measures Value at Risk (VaR) and Conditional Value at Risk (CVaR) and for optimization problems with these risk criteria is discussed. Whereas for CVaR its application is straightforward, the presence of the simple chance constraint in the definition of VaR requires that various distributional and structural properties are fulfilled, namely for the unperturbed problem. These requirements rule out direct applications of the contamination technique in the case of discrete distributions, which includes the empirical VaR. On the other hand, in the case of a normal distribution and parametric VaR, one may exploit stability results valid for quadratic programs.

Keywords: Stochastic programming; Risk management; Portfolio optimization; Linear programming; Fixed-income markets; Asset liability modelling; Dynamic models

1. Stress testing and contamination

Stress testing is a term used in financial practice without any generally accepted definition. It appears in the context of quantification of losses or risks that may appear under special, mostly extremal circumstances (Kupiec 2002). Such circumstances are described by certain scenarios which may come from historical experience (a crisis observed in the past)—*historical stress test*, or may be judged to be possible in the future given changes of macroeconomic, socioeconomic or political factors—*prospective stress test*, etc. The performance of the obtained optimal decision is then evaluated along these, possibly dynamic, scenarios or the model is solved with an alternative input. Stress testing approaches differ among institutions and also due to the nature of the tested problem and the way in which the stress scenarios have been selected. In this paper, we focus on the stress testing of two risk measures, VaR and CVaR, giving the 'test' a more precise meaning. This is made possible by the exploitation of parametric sensitivity results and the contamination technique.

The contamination approach was initiated in mathematical statistics as one of the tools for the analysis of the robustness of estimators with respect to deviations from the assumed probability distribution and/or its parameters. It goes back to von Mises and the concepts are briefly described, for example, in Serfling (1980). In stochastic programming, it was developed in a series of papers; see, for example, Dupačová (1986, 1996) for results applicable to two-stage stochastic linear programs. For application of contamination bounds, it is important that the stochastic program is reformulated as

$$\min_{x \in \mathcal{X}} F(x, P) := \int_{\Omega} f(x, \omega) P(\mathrm{d}\omega), \tag{1}$$

with $\mathcal{X}$ independent of P.

Via contamination, robustness analysis with respect to changes in the probability distribution P is reduced to a much simpler analysis with respect to scalar parameter λ. Assume that (1) is solved for probability distribution P. Denote $\varphi(P)$ the optimal value and $\mathcal{X}^*(P)$ the set of optimal solutions. The possible changes in the probability distribution P are modeled using contaminated distributions P_λ,

$$P_\lambda := (1 - \lambda)P + \lambda Q, \quad \lambda \in [0, 1], \tag{2}$$

with Q another *fixed* probability distribution. Limiting the analysis to a selected direction $Q - P$ only, the results are directly applicable, but they are less general than quantitative stability results with respect to arbitrary (but small) changes in P, summarized, for example, in Römisch (2003).

Corresponding author: Email:dupacova@karlin.mff.cuni.cz

Quantitative Finance
ISSN 1469–7688 print/ISSN 1469–7696 online © 2007 Taylor & Francis
http://www.tandf.co.uk/journals
DOI: 10.1080/14697680600973323

The objective function in (1) is linear in P, hence

$$F(x,\lambda) := \int_\Omega f(x,\omega)P_\lambda(\mathrm{d}\omega) = (1-\lambda)F(x,P) + \lambda F(x,Q)$$

is linear in λ. Suppose that stochastic program (1) has an optimal solution for all considered distributions P_λ, $0 \le \lambda \le 1$, of the form (2). Then the optimal value function

$$\varphi(\lambda) := \min_{x\in\mathcal{X}} F(x,\lambda)$$

is concave on $[0,1]$, which implies its continuity and the existence of directional derivatives on $(0,1)$. Continuity at the point $\lambda = 0$ is a property related to the stability results for the stochastic program in question. In general, one needs a non-empty, bounded set of optimal solutions $\mathcal{X}^*(P)$ of the initial stochastic program (1). This assumption, together with the stationarity of derivatives $\mathrm{d}F(x,\lambda)/\mathrm{d}\lambda = F(x,Q) - F(x,P)$, is used to derive the form of the directional derivative,

$$\varphi'(0^+) = \min_{x\in\mathcal{X}^*(P)} F(x,Q) - \varphi(0), \tag{3}$$

which enters the upper bound for the optimal value function $\varphi(\lambda)$:

$$\varphi(0) + \lambda\varphi'(0^+) \ge \varphi(\lambda) \ge (1-\lambda)\varphi(0) + \lambda\varphi(1), \quad \lambda \in [0,1]; \tag{4}$$

for details, see Dupačová (1986, 1996) and references therein.

If $x^*(P)$ is the *unique* optimal solution of (1), $\varphi'(0^+) = F(x^*(P),Q) - \varphi(0)$, i.e. the *local change in the optimal value function caused by a small change in P in direction $Q-P$ is the same as that of the objective function at $x^*(P)$*. If there are multiple optimal solutions, each of them leads to an upper bound $\varphi'(0^+) \le F(x(P),Q) - \varphi(0)$, $x(P) \in \mathcal{X}^*(P)$. Contamination bounds can then be written as

$$(1-\lambda)\,\varphi(P) + \lambda F(x(P),Q) \ge \varphi(P_\lambda) \ge (1-\lambda)\,\varphi(P) + \lambda\varphi(Q), \tag{5}$$

valid for an arbitrary optimal solution $x(P) \in \mathcal{X}^*(P)$ and for all $\lambda \in [0,1]$.

Contamination bounds (4) and (5) help to *quantify* the change in the optimal value due to the considered perturbations of (1). They exploit the optimal value $\varphi(Q)$ of the problem solved under the alternative probability distribution Q and the expected performance $F(x(P),Q)$ of the optimal solution $x(P)$ obtained for the original probability distribution P in situations where Q applies. Note that both of these values appear under the heading of stress testing methods.

The contaminated probability distribution P_λ may also be understood as a result of contaminating Q by P. Provided that the set of optimal solutions $x(Q)$ of the problem $\min_{x\in\mathcal{X}} F(x,Q)$ is non-empty and bounded, an alternative upper bound may be constructed in a similar way. Together with the original upper bound from (5), one may use a tighter upper bound

$$\min\Big\{(1-\lambda)\varphi(P) + \lambda F(x(P),Q), \lambda\varphi(Q) + (1-\lambda)F(x(Q),P)\Big\}, \tag{6}$$

for $\varphi(\lambda)$.

The contamination bounds are global, valid for all $\lambda \in [0,1]$. They are suitable for post-optimality analysis, out-of-sample analysis and stress testing in various disparate situations. For example, the choice of a degenerated distribution $Q = \delta_{\{\omega^*\}}$ may correspond to an additional stress or out-of-sample scenario ω^* or to increasing probability of an already considered scenario ω^*. Contamination bounds (4), (5) and (6) then provide information concerning the influence of including an additional scenario on the optimal results, etc. For stability studies with respect to small changes in the underlying probability distribution P, small values of the contamination parameter λ are typical. The choice of λ may reflect the degree of confidence in the expert's opinion, represented as the contaminating probability distribution Q, or the wish to obtain equiprobable scenarios, atoms of the contaminated distribution P_λ, and so on.

Contamination bounds were applied, *inter alia*, in Dupačová *et al.* (1998) for post-optimality analysis for multi-period two-stage bond portfolio management problems with respect to additional scenarios. In the present paper they will be exploited for the stress testing of various optimization problems related to risk measures CVaR and VaR. There are results on the stability of optimal solutions of contaminated stochastic programs and also results for the case where the set $\mathcal{X}$ depends on P. They are not ready for direct application, but possibilities will be explained in the context of VaR.

Section 2 includes definitions of CVaR and VaR and the basic formulae from Rockafellar and Uryasev (2001), which open up the possibility of applying the contamination technique to the stress testing of these risk measures with respect to changes in the probability distribution. Section 3 is devoted to the stress testing of CVaR and of its optimal value. The results are illustrated numerically. Finally, the problems encountered in the exploitation of the contamination technique to CVaR-mean return efficient solutions are explained.

Stress testing for VaR is substantially more complicated. This can be attributed to the fact that VaR is one of the optimal solutions of an auxiliary optimization problem and that its definition involves a probability constraint. Applicable contamination results can then be obtained only under additional assumptions concerning the probability distribution P. In section 4 we present stress testing for parametric VaR with respect to changes in the covariance matrix and with respect to an additional scenario. The section is concluded by an illustrative result dealing with contamination of the non-parametric VaR.

2. Basic formulae

Let $\mathcal{X} \subset R^n$ be a non-empty, closed set of feasible decisions x, and $\omega \in \Omega \subset R^m$ be a random vector with probability measure P on Ω which does not depend on x. Denote further

- $g(x, \omega)$ the random loss defined on $\mathcal{X} \times \Omega$,
- $G(x, P; v) := P\{\omega: g(x, \omega) \leq v\}$ the distribution function of the loss associated with a *fixed* decision $x \in \mathcal{X}$, and
- $\alpha \in (0, 1)$ the selected confidence level.

Value at Risk (VaR) was introduced and recommended as a generally applicable risk measure to quantify, monitor and limit financial risks, to identify losses that occur with an acceptably small probability. There exist several slightly different formal definitions of VaR that coincide for continuous probability distributions. Here, we shall also deal with VaR for discrete distributions and we shall use the definition from Rockafellar and Uryasev (2001).

The *Value at Risk* at confidence level α is defined as

$$\mathrm{VaR}_\alpha(x, P) = \min\{v \in R: G(x, P; v) \geq \alpha\}, \qquad (7)$$

and the 'upper' Value at Risk is

$$\mathrm{VaR}_\alpha^+(x, P) = \inf\{v \in R: G(x, P; v) > \alpha\}.$$

Hence, a random loss greater than VaR_α occurs with probability equal to (or less than) $1 - \alpha$. This interpretation is well understood in financial practice.

However, VaR_α does not quantify the loss, it is a qualitative risk measure, and, in general, it lacks the subadditivity property. (An exception is the elliptic distributions G (Embrechts *et al.* 2002), of which the normal distribution is a special case.) Various specific features and weak points of the recommended VaR methodology are summarized and discussed, for example, in Dempster (2002) and in chapter 10 of Rachev Mittnik (2000). To solve these problems, new risk measures have been introduced; see, for example, Acerbi and Tasche (2002). We shall exploit the results of Rockafellar and Uryasev (2001) to discuss one of them, the Conditional Value at Risk, which may be linked to integrated chance constraints (Klein Haneweld 1986), to constraints involving conditional expectations (Prékopa 1973) and to the absolute Lorenz curve at point α (Ogryczak and Ruszczuński 2002).

According to Rockafellar and Uryasev (2001), CVaR_α, the *Conditional Value at Risk* at confidence level α, is defined as the mean of the α-tail distribution of $g(x, \omega)$, which, in turn, is defined as

$$G_\alpha(x, P; v) = 0, \quad \text{for } v < VaR_\alpha(x, P),$$

$$G_\alpha(x, P; v) = \frac{G(x, P; v) - \alpha}{1 - \alpha}, \quad \text{for } v \geq VaR_\alpha(x, P). \qquad (8)$$

We shall assume below that $g(x, \omega)$ is a continuous function of x for all $\omega \in \Omega$ and $E_P|g(x, \omega)| < \infty, \forall x \in \mathcal{X}$. For $v \in R$, define

$$\Phi_\alpha(x, v, P) := v + \frac{1}{1 - \alpha} E_P(g(x, \omega) - v)^+. \qquad (9)$$

The fundamental minimization formula of Rockafellar and Uryasev (2001) helps to evaluate CVaR for general loss distributions and to analyse its stability, including stress testing.

Theorem 2.1 (Rockafellar and Uryaev 2001)**:** *As a function of v, $\Phi_\alpha(x, v, P)$ is finite and convex (hence continuous) with*

$$\min_v \Phi_\alpha(x, v, P) = CVaR_\alpha(x, P), \qquad (10)$$

and

$$\arg\min_{v\alpha}(x, v, P) = [\mathrm{VaR}_\alpha(x, P), \mathrm{VaR}_\alpha^+(x, P)], \qquad (11)$$

a non-empty compact interval (possibly one point only).

The auxiliary function $\Phi_\alpha(x, v, P)$ is evidently linear in P and convex in v. Moreover, if $g(x, \omega)$ is a *convex* function of x, $\Phi_\alpha(x, v, P)$ is convex jointly in (v, x). In addition, $\mathrm{CVaR}_\alpha(x, P)$ is continuous with respect to α (Rockafellar and Uryasev 2001).

If P is *a discrete probability distribution* concentrated on $\omega^1, \ldots, \omega^S$, with probabilities $p_s > 0$, $s = 1, \ldots, S$, and x a *fixed* element of $\mathcal{X}$, then the optimization problem (10) has the form

$$\min_v \left\{ v + \frac{1}{1 - \alpha} \sum_s p_s (g(x, \omega^s) - v)^+ \right\}, \qquad (12)$$

and can be further rewritten as

$$\min_{v, y_1, \ldots, y_S} \left\{ v + \frac{1}{1 - \alpha} \sum_s p_s y_s : y_s \geq 0, \quad y_s + v \geq g(x, \omega^s), \ \forall s \right\}.$$

There are various papers that discuss the properties of VaR and CVaR and the relations between them; see, for example, Dempster (2002) and Pflug (2001). We shall focus on contamination-based stress testing for these two risk measures.

3. Stress testing for CVaR

For a *fixed* vector x we now consider a stress test of $\mathrm{CVaR}_\alpha(x, P)$, i.e. of the optimal value of (10). Let Q be the stress probability distribution. We apply the contamination technique and proceed as explained in section 1. According to theorem 2.1, $\Phi_\alpha(x, v, P)$ is the corresponding objective function whose minimum equals $\mathrm{CVaR}_\alpha(x, P)$. Evidently, the contaminated objective function

$$\Phi_\alpha(x, v, \lambda) := \Phi_\alpha(x, v, P_\lambda)$$

is linear in λ and convex in v. Its optimal value $\mathrm{CVaR}_\alpha(x, \lambda) := \mathrm{CVaR}_\alpha(x, P_\lambda)$ is concave in λ on $[0, 1]$

and the set of optimal solutions (11) of the initial problem (10) is bounded. Hence, the derivative of $\mathrm{CVaR}_\alpha(x,\lambda)$, i.e. of the optimal value of the contaminated problem (10), at $\lambda = 0^+$ is

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathrm{CVaR}_\alpha(x,0^+) = \min_v \Phi_\alpha(x,v,Q) - \mathrm{CVaR}_\alpha(x,P), \quad (13)$$

with minimization carried over the set (11) of *optimal solutions* of (10) formulated and solved for the probability distribution P. An upper bound for the derivative is obtained when minimization over (11) is replaced by the evaluation of $\Phi_\alpha(x,v,Q)$ at an arbitrary *optimal solution* $v^*(x,P)$ of (10), for example at $v^*(x,P) = VaR_\alpha(x,P)$.

The *contamination bounds* for $\mathrm{CVaR}_\alpha(x,\lambda)$ for a fixed x follow from the concavity of $\mathrm{CVaR}_\alpha(x,\lambda)$ with respect to λ:

$$\begin{aligned}(1-\lambda)&\mathrm{CVaR}_\alpha(x,0) + \lambda\mathrm{CVaR}_\alpha(x,1)\\ &\le \mathrm{CVaR}_\alpha(x,\lambda) \le \mathrm{CVaR}_\alpha(x,0) + \lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathrm{CVaR}_\alpha(x,0^+)\\ &= (1-\lambda)\mathrm{CVaR}_\alpha(x,0) + \lambda\min_v \Phi_\alpha(x,v,Q), \qquad (14)\end{aligned}$$

for all $0 \le \lambda \le 1$. The combined upper bound (6) can be constructed in a similar way.

3.1. *Stress testing of the scenario-based form (12) of CVaR*

Consider first an application of the contamination bounds to the stress testing of the scenario-based form (12) of CVaR. Let P be a discrete probability distribution concentrated on $\omega^1,\ldots,\omega^S$ with probabilities p_s, $s = 1,\ldots,S$, x a *fixed* element of $\mathcal{X}$ and Q a discrete probability distribution carried by S' stress or out-of-sample scenarios ω^s, $s = S+1,\ldots,S+S'$, with probabilities p_s, $s = S+1,\ldots,S+S'$. Both $\mathrm{CVaR}_\alpha(x,P)$ and $\mathrm{CVaR}_\alpha(x,Q)$ can be obtained by solving the corresponding linear programs (12). Denote by $v^*(x,P)$ an optimal solution of (12) for fixed $x \in \mathcal{X}$ and for distribution P.

Bounds for CVaR_α for the contaminated probability distribution P_λ carried by the initial scenarios ω^s, $s = 1,\ldots,S$, with probabilities $(1-\lambda)p_s$, $s = 1,\ldots,S$, and by the stress scenarios ω^s, $s = S+1,\ldots,S+S'$, with probabilities λp_s, $s = S+1,\ldots,S+S'$, have the form

$$\begin{aligned}(1-\lambda)&\mathrm{CVaR}_\alpha(x,P) + \lambda\mathrm{CVaR}_\alpha(x,Q) \le CVaR_\alpha(x,P_\lambda)\\ &\le (1-\lambda)\mathrm{CVaR}_\alpha(x,P) + \lambda\Phi_\alpha(x,v^*(x,P),Q)\\ &= \Phi_\alpha(x,v^*(x,P),P_\lambda), \qquad (15)\end{aligned}$$

and are valid for all $\lambda \in [0,1]$; compare with (13) and (14).

In the special case of a *degenerate probability distribution* Q carried only by one scenario ω^*, $\mathrm{CVaR}_\alpha(x,Q) = g(x,\omega^*)$ and

$$\Phi_\alpha(x,v^*(x,P),Q) = v^*(x,P) + \frac{1}{1-\alpha}(g(x,\omega^*) - v^*(x,P))^+.$$

The difference between the upper and lower bounds in (14) is

$$\begin{aligned}&\lambda[\Phi_\alpha(x,v^*(x,P),Q) - \mathrm{CVaR}_\alpha(x,Q)]\\ &\quad= \lambda\left[v^*(x,P) + \frac{1}{1-\alpha}(g(x,\omega^*) - v^*(x,P))^+ - g(x,\omega^*)\right].\end{aligned}$$

In typical applications, the 'stress test' is reduced to evaluating the performance of the already obtained optimal solution along the new scenarios, i.e. the evaluation of $\Phi_\alpha(x,v^*(x,P),Q)$, or obtaining the optimal value such as $\mathrm{CVaR}_\alpha(x,Q)$ for Q carried by the stress scenarios. Contamination bounds (15) exploit these criteria simultaneously to *quantify* the influence of the stress scenarios, also taking into account the probability of their occurrence. As a result, they provide a genuine stress test.

3.2. *Sensitivity properties of optimal solutions*

To derive the sensitivity properties of the optimal solutions of (10) for *fixed x*, assume that the optimal solution of (10) is *unique*, $v^*(x,P)$; hence, it equals $\mathrm{VaR}_\alpha(x,P)$. This also simplifies the form of the derivative of $\mathrm{CVaR}_\alpha(x,\lambda)$ in (13) to $\Phi_\alpha(x,\mathrm{VaR}_\alpha(x,P),Q)-\mathrm{CVaR}_\alpha(x,P)$.

The general results concerning the properties of optimal solutions for contaminated distributions (see, for example, Dupačová (1986, 1987) and Shapiro (1990)) require additional properties concerning the smoothness of the objective function (9) in (10). To this end we assume that the probability distribution function $G(x,P;v)$ is continuous, with a positive, continuous density $p(x,P;v)$ on a neighbourhood of the unique optimal solution $v^*(x,P) = \mathrm{VaR}_\alpha(x,P)$ of (10).

For fixed $x \in \mathcal{X}$ we denote $\eta := g(x,\omega)$, $v := v^*(x,P)$ and use definition (9) of $\Phi_\alpha(x,v,P)$. Except for $v = \eta$, the derivative $(\mathrm{d}/\mathrm{d}v)(\eta - v)^+$ exists and

$$\frac{\mathrm{d}}{\mathrm{d}v}(\eta - v)^+ = -\frac{1}{2}\left(1 + \frac{\eta - v}{|\eta - v|}\right).$$

Thanks to the assumed properties of the distribution function $G(x,P;v)$, the expected value

$$E_P\frac{\mathrm{d}}{\mathrm{d}v}(\eta - v)^+ = -P(\eta > v) = -1 + G(x,P;v),$$

and

$$\frac{\mathrm{d}}{\mathrm{d}v}\Phi_\alpha(x,v,P) = 1 + \frac{G(x,P;v) - 1}{1-\alpha}.$$

The optimality condition $(\mathrm{d}/\mathrm{d}v)\Phi_\alpha(x,v,P) = 0$ provides, as expected,

$$\mathrm{VaR}_\alpha(x,P) = v^*(x,P) = G(x,P)^{-1}(\alpha).$$

The second-order derivative $(\mathrm{d}^2/\mathrm{d}v^2)\Phi_\alpha(x,v,P) = [p(x,P;v)/(1-\alpha)]$ is positive on a neighbourhood of $v^*(x, P)$. Direct application of the implicit function theorem to the system

$$\frac{\mathrm{d}}{\mathrm{d}v}\Phi_\alpha(x, v, P_\lambda) = 0$$

implies the existence and uniqueness of optimal solution $v^*(x, \lambda) := v^*(x, P_\lambda)$ of the contaminated problem (10) for $\lambda > 0$ sufficiently small, and the form of its derivative

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}v^*(x, P_\lambda) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\mathrm{VaR}_\alpha(x, P_\lambda) = \frac{\alpha - G(x, Q; v^*(x, P))}{p(x, P; v^*(x, P))}, \tag{16}$$

for $\lambda = 0^+$. Here, $G(x, Q; v)$ denotes the distribution loss function under probability distribution Q. Note that, except for the existence of the expected values, no further assumptions are required concerning Q. Related results for absolutely continuous probability distributions P and Q can be found, for example, in Rau-Bredow (2004).

3.3. *Optimization problems with the $\mathrm{CVaR}_\alpha(x, P)$ objective function*

For the next step, let us briefly discuss optimization problems with the $\mathrm{CVaR}_\alpha(x, P)$ objective function, which provide the optimal (with respect to the $\mathrm{CVaR}_\alpha(x, P)$ criterion) solutions

$$\text{minimize } \mathrm{CVaR}_\alpha(x, P) \text{ on a closed set } \emptyset \neq \mathcal{X} \subset \mathrm{R}^n.$$

Using (10), the problem is

$$\min_{x, v} \Phi_\alpha(x, v, P), \quad x \in \mathcal{X}. \tag{17}$$

For $\mathcal{X}$ convex, independent of P, and for loss functions $g(\bullet, \omega)$ convex for all ω, $\Phi_\alpha(x, v, P)$ is convex in (x, v) and standard stability results apply. Moreover, if P is the discrete probability distribution considered in section 3.1, $g(\bullet, \omega)$ a linear function of x, say $g(\bullet, \omega) = x^\top \omega$, and $\mathcal{X}$ convex polyhedral, we obtain the linear program

$$\min_{v, y_1, \ldots, y_S, x}\left\{v + \frac{1}{1-\alpha}\sum_s p_s y_s : y_s \geq 0,\; x^\top \omega^s - v - y_s \leq 0,\ \forall s,\ , x \in \mathcal{X}\right\}. \tag{18}$$

Let $(v_C^*(P), x_C^*(P))$ be an optimal solution of (17) and denote by $\varphi_C(P)$ the optimal value. To obtain contamination bounds for the optimal value of (17) with P contaminated by stress probability distribution Q, it is sufficient to assume a compact set $\mathcal{X}$, e.g. $\mathcal{X} = \{x \in R^n : \sum_i x_i = 1, x_i \geq 0, \forall i\}$. The bounds follow the usual pattern (compare with (15)):

$$(1-\lambda)\varphi_C(P) + \lambda\varphi_C(Q) \leq \varphi_C(P_\lambda) \leq (1-\lambda)\varphi_C(P) + \lambda\Phi_\alpha(x_C^*(P), v_C^*(P), Q). \tag{19}$$

To apply them, one has to evaluate $\Phi_\alpha(x_C^*(P), v_C^*(P), Q)$ and solve (17) with P replaced by the stress distribution Q.

3.4. *An illustrative example*

The instruments used in the portfolio management problem (18) are the total return stock and the bond indices given in table 1.

The portfolio limits were set in all cases to $x_i \leq 0.3$, hence,

$$\mathcal{X} = \left\{x \in R^n : \sum_i x_i = 1, 0 \leq x_i \leq 0.3, \forall i\right\}.$$

Assume that the probability distribution P is the distribution of losses under 'normal' conditions, whereas probability distribution Q refers to the situation when adverse conditions prevail on the world market. Both P and Q are distributions of monthly percentage losses to assets $i = 1, \ldots, 12$, which were converted into the home

Table 1. Portfolio assets (MSCI and JP Morgan indexes).

Asset	Acronym	Description
MSCI Gross Return index US, USD	1	Stock index
MSCI Gross Return index UK, USD	2	Stock index
MSCI Gross Return index Germany, USD	3	Stock index
MSCI Gross Return index Japan, USD	4	Stock index
US Government Bond index (1–3 y mat), USD	5	
US Government Bond index (7–10 y mat), USD	6	
UK Government Bond index (1–3 y mat), GPB	7	
UK Government Bond index (7–10 y mat), GPB	8	
Germany Government Bond index (1–5 y mat), EUR	9	
Germany Government Bond index (7+ y mat), EUR	10	
Japan Government Bond index (1–3 y mat), JPY	11	
Japan Government Bond index (7–10 y mat), JPY	12	

currency (EUR) using the exchange rate mid. We do not consider transaction costs.

The following approach, resembling an historical simulation, was taken to construct discrete distributions P and Q. For asset $i=1$ (US asset market returns) the percentage returns (not losses) in the home currency were computed. We took the empirical 25% quantile to be the cut-off value for all returns of asset 1. The returns below the cut-off value (and all corresponding returns of other assets on the same date) are attributed to a period of adverse conditions prevailing on the market and hence this data set serves as the input for the approximation of the distribution Q. The rest of the data sample was used for fitting the distribution P.

The two discrete probability distributions P, Q approximating the true continuous distribution of assets' percentage losses in the home currency were constructed using the method of Høyland *et al.* (2003). We prescribed that both discrete approximations P, Q were carried by 5184 equiprobable scenarios. The empirical means, variances, covariances, skewnesses and kurtoses computed separately from the two data samples enter the scenario fitting procedure for P and Q.

After solving the two CVaR minimization problems with $\alpha = 0.99$, contamination bounds (19) sharpened according to (6),

$$(1-\lambda)\varphi_C(P) + \lambda\varphi_C(Q) \le \varphi_C(P_\lambda) \le \min\{(1-\lambda)\varphi_C(P) + \lambda\Phi_\alpha(x^*_C(P), v^*_C(P), Q), \lambda\varphi_C(Q) + (1-\lambda)\Phi_\alpha(x^*_C(Q), v^*_C(Q), P)\}, \quad (20)$$

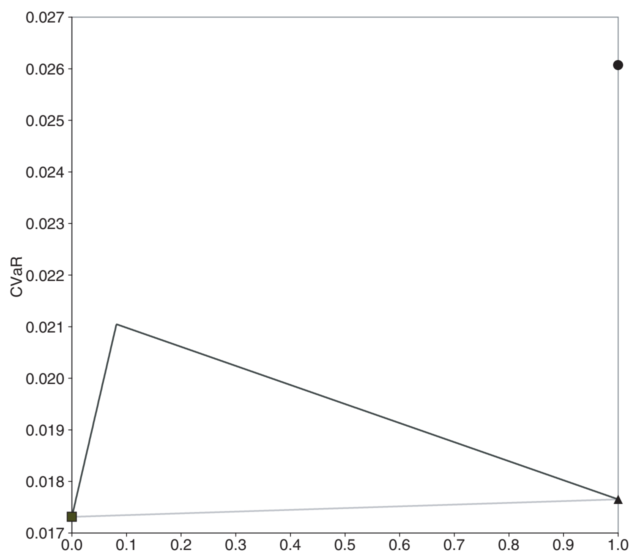
were constructed. The results of contamination are presented in figure 1 and table 2. The VaR values $v^*_C(P), v^*_C(Q)$ for distributions P, Q calculated for the optimal portfolios $x^*_C(P), x^*_C(Q)$ are obtained as a by-product.

Some observations are given below.

- The two minimal CVaR values $\varphi_C(P), \varphi_C(Q)$ (indicated in the figure by a square and a triangle, respectively) are not very different. This is the result of optimally restructuring the portfolio in the adverse market situation; see the changed composition of the optimal portfolios. The CVaR value for probability distribution Q and for the original optimal portfolio $x^*_C(P)$, i.e. without restructuring the portfolio (indicated by the isolated point in the right upper corner of the figure), is much higher.
- The value $\Phi_\alpha(x^*_C(P), v^*_C(P), Q)$ is relatively large and this determines the steep slope of the left upper bound.
- The contamination bounds in this example are not very tight (see figure 1). The maximal difference between the upper and lower bounds occurs approximately at $\lambda = 0.1$. For $\lambda = 0.5$, i.e. for the distribution carried by the pooled sample of 10 368 equiprobable scenarios, the minimal CVaR value lies in [0.0175, 0.0195]. If this precision is sufficient, one does not need to solve the problem with twice the number of scenarios—atoms of the contaminated probability distribution.

3.5. Stress testing for CVaR-mean return problems

Finally, consider stress testing for CVaR-mean return problems, i.e. for bi-criteria problems in which one aims simultaneously for a minimization of $\mathrm{CVaR}_\alpha(x, P)$ and a maximization of the expected return criterion $E_P r(x, \omega)$ on $\mathcal{X}$; see, for example, Rockafellar and Uryasev (2000), Andersson *et al.* (2001), Pflug (2001), Topaloglou *et al.* (2002), and Kaut *et al.* (2003)



1

Figure 1. Contamination bounds for the CVaR optimization problem without constraint on returns.

Table 2. Quantities used in contamination bounds (20) and non-zero components of optimal solutions $x^*_C(P)$ and $x^*_C(Q), \alpha = 0.99$.

Quantity	Value
$\varphi_C(P)$	0.01731
$\varphi_C(Q)$	0.01765
$\Phi_\alpha(x^*_C(P), v^*(P), Q)$	0.06309
$\Phi_\alpha(x^*_C(Q), v^*(Q), P)$	0.02135
$x^*_1(P)$	0.12880
$x^*_7(P)$	0.20030
$x^*_9(P)$	0.30000
$x^*_{10}(P)$	0.26470
$x^*_{11}(P)$	0.10620
$v^*_C(P)$	0.01365
$x^*_5(Q)$	0.10000
$x^*_7(Q)$	0.30000
$x^*_9(Q)$	0.30000
$x^*_{10}(Q)$	0.30000
$v^*_C(Q)$	0.01588
$CVaR(x^*_C(P), Q)$	0.02607

To obtain an efficient solution, one minimizes on $\mathcal{X}$ the parametrized objective function

$$\mathrm{CVaR}_\alpha(x, P) - \rho E_P r(x, \omega), \tag{21}$$

with parameter $\rho > 0$, or assigns a parametric bound on one of the criteria and solves, for example,

$$\min \mathrm{CVaR}_\alpha(x, P) \text{ on the set } \mathcal{X} \cap \{x : E_P r(x, \omega) \geq r\}. \tag{22}$$

The optimal solution and the corresponding values of the two criteria, CVaRα; and the expected return, depend on the chosen parameter values. To obtain the efficient frontier, (21) and (22) may be solved by parametric programming techniques with scalar parameters ρ or r, respectively. For $g(x, \omega) = x^\top \omega = -r(x, \omega)$, for polyhedral set $\mathcal{X}$ and a discrete probability distribution P, both (21) and (22) are then parametric linear programs with one scalar parameter; see, for example, Ruszczyński and Vanderbei (2003). By solving (22), the efficient frontier is obtained directly. To obtain the efficient frontier in the case of (21), values of $E_P r(x, \omega)$ and $\mathrm{CVaR}_\alpha(x, P)$ have to be computed at the optimal solution of (21) obtained for a specific value of ρ. Hence, (22) is favoured for the straightforward possibility of interpreting the trade-off between the two criteria, whereas (21) is suitable for developing sensitivity and stability results, including stress testing.

Contamination of the probability distribution P introduces an additional parameter λ into (21) and (22) and the two problems, in general, lose the readily solvable form of parametric linear programs: nonlinearity with respect to ρ and λ appears in the objective function of (21) and both the objective function and the set of feasible solutions of (22) depend on the parameters. It is still possible to obtain directional derivatives of the optimal value function for the corresponding contaminated problem. However, the optimal value function is no longer concave, hence the crucial property for the construction of contamination bounds is lost. The same applies also to problem formulations with several CVaR constraints, each with a different confidence level α, called 'risk-shaping' (Rockafellar and Uryasev 2001).

Nevertheless, contamination bounds may be obtained for the special form of the return function $r(x, \omega) = -x^\top \omega$ and for a certain class of probability distributions. Rewrite problem (22) as

$$\text{minimize } \mathrm{CVaR}_\alpha(x, P)$$

on the set

$$\mathcal{X}(P, r) = \{x \in \mathcal{X} : -x^\top E_P \omega \geq r\}. \tag{23}$$

Let $\varphi_r(P)$ denote the optimal value and $\mathcal{X}_r^*(P)$ the set of optimal solutions and assume that $\mathcal{X}_r^*(P)$ is non-empty and bounded.

Assume, in addition, that the *expected values are equal*, $E_P \omega = E_Q \omega = \bar{\omega}$. (Such an assumption is not typical for stress testing, but it is in agreement with scenario generation methods based on moment fitting (e.g. Høyland *et al.* (2003) and Høyland and Wallace (2001)), and has also been used in the stability studies of Kaut *et al.* (2003).) Then the expected return constraint is $-x^\top \bar{\omega} \geq r$, both for the initial probability distribution P and the contaminating distribution Q, as well as for all P_λ, $\lambda \in [0, 1]$, and it does not depend on λ. The optimal value function $\varphi_r(P_\lambda) = \varphi_r(\lambda)$ is concave and the contamination bounds have a form similar to (19) and (20). They are obtained for (17) with the set of feasible decisions $\mathcal{X}$ replaced by $\mathcal{X}(P, r) = \{x \in \mathcal{X} : -x^\top \bar{\omega} \geq r\}$. Moreover, there are parametric programming techniques (e.g. Guddat *et al.* (1985)) applicable to the contaminated problem (23), i.e. the minimization of $\mathrm{CVaR}_\alpha(x, P_\lambda)$ on the set $\mathcal{X}(P, r) = \{x \in \mathcal{X} : -x^\top \bar{\omega} \geq r\}$. They are discussed in Dupačová (2006) and a qualitative conclusion can be summarized as follows.

Under modest non-degeneracy assumptions, a small contamination of P does not influence the composition of CVaR-mean return efficient portfolios.

Note that, for $E_P \omega = E_Q \omega = \bar{\omega}$, problem (21) is also simplified, and the objective function is *linear* in the two parameters ρ and λ.

When the *expected loss differs* under P and Q, the optimal value $\varphi_C(P_\lambda)$ is a natural lower bound for $\varphi_r(P_\lambda)$, hence by (19),

$$\varphi_r(P_\lambda) \geq (1 - \lambda)\varphi_C(P) + \lambda \varphi_C(Q). \tag{24}$$

To construct an upper bound for $\varphi_r(P_\lambda)$ we add the additional constraint $-x^\top E_Q \omega \geq r$ to $\mathcal{X}(P, r)$. The set of feasible solutions $\mathcal{X}(P, r) \cap \mathcal{X}(Q, r) \subset \mathcal{X}(P_\lambda, r)$ is polyhedral and does not depend on λ. If $\mathcal{X}(P, r) \cap \mathcal{X}(Q, r) \neq \emptyset$ we obtain a *concave* upper bound

$$U_r(\lambda) := \min_{x \in \mathcal{X}(P,r) \cap \mathcal{X}(Q,r)} \mathrm{CVaR}_\alpha(x, P_\lambda) \geq \varphi_r(P_\lambda),$$

which may be bounded from above by the corresponding upper contamination bound. The derivative at the point $\lambda = 0^+$ is of a familiar form—$\min \Phi_\alpha(x, v, Q) - U_r(0)$ with minimization carried over the set of optimal solutions of (17) for $\mathcal{X}$ replaced by $\mathcal{X}(P, r) \cap \mathcal{X}(Q, r)$; denote one of them by $\hat{x}_r(P), \hat{v}_r(P)$:

$$(1 - \lambda)U_r(0) + \Phi_\alpha(\hat{x}_r(P), \hat{v}_r(P), Q) \geq U_r(\lambda) \geq \varphi_r(P_\lambda). \tag{25}$$

We have not tested bounds (24) and (25) numerically, but we expect that they may be quite loose.

4. Stress testing for VaR

Up to the non-uniqueness of the definitions, $\mathrm{VaR}_\alpha(x, P)$ is the same as the *α-quantile of the loss distribution* $G(x, P; v)$. One can also treat $\mathrm{VaR}_\alpha(x, P)$ as the optimal value of the stochastic program (7) with one probabilistic constraint. Such an approach enables us to exploit the existing stability results for stochastic programs of that form (Römisch 2003), which are valid under special distributional and regularity assumptions.

A normal distribution of losses is one of the manageable cases and, initially, *parametric* VaR was developed to quantify the risks associated with normally distributed losses $g(x,\omega)$, the distribution of which at a fixed point x is fully determined by its expectation $\mu(x)$ and variance $\sigma^2(x)$:

$$\text{absolute}\mathrm{VaR}_\alpha(x) = \mu(x) + \sigma(x)\cdot u_\alpha,$$
$$\text{and relative } \mathrm{VaR}_\alpha(x) = \sigma(x)\cdot u_\alpha,$$

where u_α is the α-quantile of the standard normal $\mathcal{N}(0,1)$ distribution.

For an arbitrary $\alpha > 0.5$, minimization of the relative VaR_α reduces to minimization of the standard deviation (volatility) of the portfolio losses, and minimization of the absolute VaR_α is minimization of the weighted sum of the standard deviation and the expectation.

4.1. Optimization problem with the relative $VaR_\alpha(x, P)$ objective function

Choose $\alpha > 0.5$ and assume that losses are of the form

$$g(x,\omega) = x^\top \omega,$$

$\mathcal{X}$ is a non-empty, convex polyhedral set, $0 \notin \mathcal{X}$, ω is normally distributed with mean vector μ and a positive definite variance matrix Σ.

The problem is to select portfolio composition $x \in \mathcal{X}$ such that VaR_α is minimal, i.e. to minimize the convex quadratic function $x^\top \Sigma x$ on the set $\mathcal{X}$. In this case, for all values of $\alpha > 0.5$ there is *the same, unique* optimal solution $x^*(\Sigma)$, the composition of the portfolio, which depends on the input variance matrix Σ that was obtained by an estimation procedure and is subject to an estimation error. The same optimal solution is arrived at by minimization of $\mathrm{CVaR}_\alpha(x,P)$ (Rockafellar and Uryasev 2000).

Asymptotic statistics and a detailed analysis of optimal solutions of parametric quadratic programs may help to derive asymptotic results concerning the 'estimated' optimal portfolio composition obtained for an asymptotically normal estimate $\tilde{\Sigma}$ of Σ.

Here we follow a suggestion of Kupiec (2002) and rewrite the variance matrix as $\Sigma = DCD$ with the diagonal matrix D of 'volatilities' (standard deviations of the marginal distributions) and the correlation matrix C. Changes in the covariances may then be modeled by 'stressing' the correlation matrix C by a positive semidefinite *stress correlation matrix* $\hat{C}$

$$C(\gamma) = (1-\gamma)C + \gamma\hat{C}, \tag{26}$$

with $\gamma \in [0,1]$ a parameter. This type of perturbation of the initial quadratic program allows us to apply the related stability results of Bank *et al*. (1982) to the perturbed problem,

$$\min_{x\in\mathcal{X}} x^\top DC(\gamma)Dx, \quad \gamma \in [0,1]: \tag{27}$$

- the optimal value $\varphi_V(\gamma)$ of (27) is concave and continuous in $\gamma \in [0,1]$;
- the optimal solution $x^*(\gamma)$ is a continuous vector in the range of γ where $C(\gamma)$ is positive definite;
- the directional derivative of $\varphi_V(\gamma)$

$$\varphi'_V(0^+) = x^{*\top}(0)D\hat{C}Dx^*(0) - \varphi_V(0).$$

Contamination bounds constructed as suggested in section 1,

$$(1-\gamma)x^{*\top}(0)DCDx^*(0) + \gamma x^{*\top}(1)D\hat{C}Dx^*(1) \le \min_{x\in\mathcal{X}} x^\top DCDx$$
$$\le (1-\gamma)x^{*\top}(0)DCDx^*(0) + \gamma x^{*\top}(0)D\hat{C}Dx^*(0),$$

quantify the effect of the considered change in the input data.

4.2. Stress testing of the relative VaR with respect to an additional scenario ω^*

In this case, the contaminating distribution Q is degenerate, $Q = \delta_{\{\omega^*\}}$. Rewriting (16) for the case of a normally distributed loss, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} VaR_\alpha(x, P_\lambda)|_{\lambda=0^+} = \frac{\alpha - I\{g(x,\omega^*) \le VaR_\alpha(x,P)\}}{\phi(VaR_\alpha(x,P))}. \tag{28}$$

In the above formula, x is fixed, ϕ denotes the density of the normal distribution $\mathcal{N}(\mu(x), \Sigma(x))$ of $g(x,\omega)$ and I is the indicator function.

Assume, in addition, that $g(x,\omega) = x^\top\omega$. Using the results of sections 4.1 and 3.2 for the normal distribution $P \sim \mathcal{N}(\mu,\Sigma)$ and degenerate distribution $Q = \delta_{\{\omega^*\}}$, we have the unique optimal portfolio $x^*(\Sigma)$ for P and both $\mathrm{VaR}_\alpha(x, P_\lambda)$ and its derivative with respect to λ are continuous for $\lambda \ge 0$ sufficiently small. This can be used to derive *sensitivity properties of the minimal relative VaR value*,

$$\varphi(\lambda) := \min_{x\in\mathcal{X}} \mathrm{VaR}_\alpha(x, P_\lambda),$$

in the case of $\mathcal{X} \ne \emptyset$, compact and for small $\lambda > 0$, i.e. when testing the influence of a rare stress scenario. Here, $\mathrm{VaR}_\alpha(x, P_\lambda)$ is not linear in λ. Still, using (28) and the general formula for the derivative of the optimal value of nonlinear objective functions from Danskin (1967), we obtain

$$\varphi'(0^+) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\mathrm{VaR}_\alpha(x^*(\Sigma), P_\lambda)|_{\lambda=0^+}$$
$$= \frac{\alpha - I\{g(x^*(\Sigma),\omega^*) \le VaR_\alpha(x^*(\Sigma),P)\}}{\phi(VaR_\alpha(x^*(\Sigma),P)}.$$

Then, the minimal VaR_α value for the stressed distribution P_λ is approximated by

$$\min_{x\in\mathcal{X}} \mathrm{VaR}_\alpha(x, P_\lambda) \cong \mathrm{VaR}_\alpha(x^*(\Sigma), P) + \lambda\varphi'(0^+)$$

for $\lambda > 0$ sufficiently small.

This approach may easily be extended to sensitivity analysis and stress testing of VaR with respect to an additional scenario for a broad class of probability measures P for which the probability distribution of loss $G(x, P; v)$ fulfils the assumptions of section 3.2.

4.3. *Non-parametric VaR*

For general probability distributions the evaluation of VaR_α for a fixed portfolio x is mostly based on a non-parametric approach that is distribution free and also applicable for complicated financial instruments. One exploits a finite number, S, of scenarios so that, for each fixed $x \in \mathcal{X}$, the underlying probability distribution P is replaced by a discrete distribution P_S carried by these scenarios and the probability distribution of the loss $g(x, \omega)$ is discrete with jumps at $g(x, \omega^s)\ \forall s$.

For a fixed x, let us order $g(x, \omega^s)$ as

$$g^{[1]} < \cdots < g^{[S]}, \tag{29}$$

with the probability of $g^{[s]}$ equal to $p^{[s]} > 0, \forall s$. Let s_{α, P_S} be the unique index such that

$$\sum_{s=1}^{s_{\alpha,P_S}} p^{[s]} \geq \alpha > \sum_{s=1}^{s_{\alpha,P_S}-1} p^{[s]}. \tag{30}$$

Then $\mathrm{VaR}_\alpha(x, P_S) = g^{[s_{\alpha,P_S}]}$.

The consistency of sample quantiles is valid under mild assumptions regarding the smoothness of the distribution function G, and one may even prove their asymptotic normality (Serflin 1980). For example, if there is a positive continuous density $p(x, P; v)$ of $G(x, P; v)$ on a neighbourhood of $\mathrm{VaR}_\alpha(x, P)$ and P_S denotes an associated empirical distribution, then $\mathrm{VaR}_\alpha(x, P_S)$ is asymptotically normal,

$$\mathrm{VaR}_\alpha(x, P_S) \sim \mathcal{N}\left(\mathrm{VaR}_\alpha(x, P), \frac{\alpha(1-\alpha)}{Sp^2(x, P; VaR_\alpha(x, P))}\right).$$

Estimating $\mathrm{VaR}_\alpha(x, P)$ by the *non-parametric* $\mathrm{VaR}_\alpha(x, P_S)$ calls for a large number of scenarios, especially for α close to 1; see Rachev and Mittnik (2000) for extensive numerical results. Moreover, it is evident from (30) that, even for fixed x, the inclusion of an additional scenario may cause an abrupt change in VaR_α.

Sensitivity results for VaR_α similar to (38) are obtained if the (unique) optimal solution of the CVaR_α problem (10) is differentiable (recall section 3.2). Another possibility is to derive them by a direct sensitivity analysis of the simple chance-constrained stochastic program (7). In both cases, additional assumptions concerning the probability distribution P are required, such as its continuity properties listed in section 3.2. There is more freedom as to the choice of the contaminating distribution Q. We refer to Dobiáš (2003) and Römisch for details.

4.4. *Stress testing of non-parametric VaR*

The stress testing of non-parametric VaR computed for a discrete probability distribution P carried by a finite number of scenarios ω^s, $s = 1, \ldots, S$, is more involved. To obtain an upper bound for $\mathrm{VaR}_\alpha(x, P_\lambda)$ for a fixed portfolio x, one may use the contamination-based upper bound for $\mathrm{CVaR}_\alpha(x, P_\lambda)$ in (15). Formula (30) in the definition of the empirical VaR_α implies that, for $\alpha < \sum_{s=1}^{s_{\alpha,P}} p^{[s]}$, the value of VaR_α is robust with respect to small changes in probabilities $p^{[s]}$. This indicates the possibility of covering the interval $[0, 1]$ by a finite number of non-overlapping intervals $[0, \lambda_1], (\lambda_1, \lambda_2], \ldots, (\lambda_{\bar{\imath}}, 1]$ and constructing bounds for $\mathrm{VaR}_\alpha(x, P_\lambda)$ separately for each of them.

We shall illustrate the approach for the case of one additional 'stress' scenario ω^* with

$$g^{[1]} < \cdots < g^{[s_{\omega^*}-1]} < g(x, \omega^*) < g^{[s_{\omega^*}]} < \cdots < g^{[S]}, \tag{31}$$

and with probabilities

$$(1-\lambda)p^{[1]}, \ldots, (1-\lambda)p^{[s_{\omega^*}-1]}, \lambda, (1-\lambda)p^{[s_{\omega^*}]}, \ldots, (1-\lambda)p^{[S]},$$

i.e. for degenerate probability distribution $Q = \delta_{\{\omega^*\}}$.

Suppose that the stress scenario satisfies $g^{[s_{\alpha,P}]} < g^{[s_{\omega^*}-1]}$. It is easy to see that, in the case of $\sum_{s=1}^{s_{\alpha,P}} p^{[s]} > \alpha$, we obtain $\mathrm{VaR}_\alpha(x, P_\lambda) = g^{[s_{\alpha,P}]} = VaR_\alpha(x, P)$ for sufficiently small $\lambda \geq 0$. On the other hand, if $\sum_{s=1}^{s_{\alpha,P}} p^{[s]} = \alpha$, then $\mathrm{VaR}_\alpha(x, P_\lambda) = g^{[s_{\alpha,P}+1]}$ for sufficiently small $\lambda > 0$.

The α-quantile $g^{[s_{\alpha,P_\lambda}]}$ of the contaminated distribution fulfils

$$\sum_{s=1}^{s_{\alpha,P_\lambda}} (1-\lambda)p^{[s]} \geq \alpha \quad \text{and} \quad \sum_{s=1}^{s_{\alpha,P_\lambda}-1} (1-\lambda)p^{[s]} < \alpha. \tag{32}$$

For $\lambda = 0$, these inequalities are identical to (30). They remain valid with s_{α,P_λ} replaced by the original $s_{\alpha,P}$ for

$$\lambda \leq 1 - \frac{\alpha}{\sum_{s=1}^{s_{\alpha,P}} p^{[s]}} \quad \text{and} \quad 1 - \frac{\alpha}{\sum_{s=1}^{s_{\alpha,P}-1} p^{[s]}} < \lambda.$$

The first inequality provides an upper bound λ_1 and the second is fulfilled for all $\lambda \geq 0$.

For $\lambda > \lambda_1$, $\mathrm{VaR}_\alpha(x, P_\lambda) = g^{[s_{\alpha,P}+1]}$, and by solving (32) for $s_{\alpha,P_\lambda} = s_{\alpha,P} + 1$ with respect to λ, we obtain an upper bound λ_2 of the interval on which $\mathrm{VaR}_\alpha(x, P_\lambda) = g^{[s_{\alpha,P}+1]}$ holds true. Note that $\lambda_1 = 0$ if $\sum_{s=1}^{s_{\alpha,P_S}} p^{[s]} = \alpha$ and, in this case, $\lambda_2 > 0$.

Similarly for $\lambda > \lambda_i$ with $i < s_{\omega^*} - s_{\alpha,P}$, we obtain an upper bound λ_{i+1} of the interval for which $\mathrm{VaR}_\alpha(x, P_\lambda) = g^{[s_{\alpha,P}+i]}$. This procedure stops when $i = \bar{\imath} := s_{\omega^*} - s_{\alpha,P}$. In this case, (32) is modified to

$$\sum_{s=1}^{s_{\alpha,P}+\bar{\imath}-1} (1-\lambda)p^{[s]} + \lambda \cdot 1 \geq \alpha,$$

valid for all $\lambda \geq 0$; hence, $\mathrm{VaR}_\alpha(x, P_\lambda) = g(x, \omega^*)$ for $\lambda_{\bar{\imath}} < \lambda \leq 1$.

To summarize: for contamination by one scenario as in (31), setting

$$\lambda_0 = 0,$$
$$\lambda_i = 1 - \frac{\alpha}{\sum_{s=1}^{s_{\alpha,P}+i-1} p^{[s]}}, \quad \text{for } i = 1, \ldots, s_{\omega^*} - s_{\alpha,P},$$
$$\lambda_i = 1, \quad \text{for } i > s_{\omega^*} - s_{\alpha,P},$$

we obtain the following theorem.

Theorem 4.1 *For* $g^{[s_{\alpha,P}]} < g^{[s_{\omega^*}-1]}$, $\lambda \in (\lambda_i, \lambda_{i+1}]$, $i = 0, 1, \ldots, s_{\omega^*} - s_{\alpha,P} - 1$,

(a) $\mathrm{VaR}_\alpha(x, P_\lambda) = g^{[s_{\alpha,P}+i]} \leq VaR_\alpha(x, Q)$;
(b) *if* $\sum_{s=1}^{s_{\alpha,P}} p^{[s]} > \alpha$ *or if* $i \geq 2$ *and* $\sum_{s=1}^{s_{\alpha,P}} p^{[s]} = \alpha$, *then* $\mathrm{VaR}_\alpha(x, P_{\lambda_i}) = VaR_{alpha}(x, P_\lambda) < VaR_\alpha(x, P_{\lambda_{i+1}})$; *if* $\sum_{s=1}^{s_{\alpha,P}} p^{[s]} = \alpha$, *then* $\mathrm{VaR}_\alpha(x, P_{\lambda_1}) = g^{[s_{\alpha,P}]}$ *and* $\mathrm{VaR}_\alpha(x, P_\lambda) = g^{[s_{\alpha,P}+1]}$ for $\lambda \in (\lambda_1, \lambda_2]$;
(c) $\mathrm{VaR}_\alpha(x, P_\lambda) = g(x, \omega^*) = VaR_\alpha(x, Q)$, for $\lambda > \lambda_{\bar{\imath}}$, $\bar{\imath} = s_{\omega^*} - s_{\alpha,P}$.

This procedure can be extended to stress testing with respect to another discrete probability distribution Q, carried by scenarios $\omega_1^*, \ldots, \omega_{S'}^*$ with probabilities $q^{[1]}, \ldots, q^{[S']}$ and associated losses $g(x, \omega_1^*) < \cdots < g(x, \omega_{S'}^*)$. Now, we have to determine how the support of P is related to the support of Q, i.e. that the following ordering holds:

$$\begin{aligned} g^{[1]} &< \cdots < g^{[s_{\alpha,P}]} < \cdots < g^{[s_{\omega_1^*}-1]} < g(x, \omega_1^*) \\ &< g^{[s_{\omega_1^*}]} < \cdots < g^{[s_{\omega_2^*}-1]} \\ &< g(x, \omega_2^*) < g^{[s_{\omega_2^*}]} < \cdots < g^{[s_{\omega_{S'}^*}-1]} \\ &< g(x, \omega_{S'}^*) < g^{[s_{\omega_{S'}^*}]} < \cdots < g^{[S]}. \end{aligned}$$

The covering of the interval [0, 1] depends on probabilities $q^{[s]}$, namely on the difference in their partial cumulative sums and α. For the obtained λ_i values, statements parallel to (a) and (b) of theorem 4.1 can be derived (Polívka 2005).

4.5. Minimization of $\mathbf{VaR}_\alpha(x, P)$ with respect to x

Except for the case of the normal distribution considered in sections 4.1 and 4.2, the *minimization* of $\mathrm{VaR}_\alpha(x, P)$ with respect to x is, in general, a non-convex, even discontinuous problem, which may have several local minima. It can be written as

$$\min\{v : P\{\omega : g(x, \omega) \leq v\} \geq \alpha,\ x \in \mathcal{X},\ v \in R\}. \tag{33}$$

Stability of the minimal $\mathrm{VaR}_\alpha(P)$ value $v_V^*(P)$ and of the optimal solutions $x_V^*(P)$ with respect to P holds true only under additional, restrictive assumptions (Römisch 2003). For $g(x, \omega)$ jointly continuous in x, ω and $\mathcal{H}(x, v) := \{\omega : g(x, \omega) \leq v\}$, a verifiable sufficient condition is $P(\mathcal{H}(x_V^*(P), v_V^*(P))) > \alpha$, which is fulfilled, for instance, for (non-degenerate) normal distributions, or $\alpha < \sum_{s=1}^{s_{\alpha,P}} p^{[s]}$ in (30) for the ordered sample of $g(x_V^*(P), \omega^s)$ with discrete distribution P_S (Dobiáš 2003).

To approximate VaR minimization problems, one may apply the corresponding problems with CVaR criteria, as suggested and tested numerically in Rockfellar and Uryasev (2000): the $v_C^*(P)$ part of the optimal solution of (18) is then the value of $\mathrm{VaR}_\alpha(x^*(P), P)$ for the optimal (or efficient) $\mathrm{CVaR}_\alpha(x, P)$ portfolio. Further suggestions are to approximate VaR minimization problems by a sequence of CVaR minimizations (Pflug 2001), to use a smoothed VaR objective (Gaivoronski and Pflug 2004), or to apply the worst-case VaR criterion for the family of probability distributions with given first- and second-order moments (El Ghaoui *et al.* 2003).

5. Conclusions

The application of the contamination technique to CVaR evaluation and optimization is straightforward, and the obtained results provide a genuine stress *quantification*. Stress testing via contamination for CVaR-mean return problems turns out to be more delicate.

The presence of the simple chance constraint in the definition of VaR requires that, for VaR stress testing via contamination, various distributional and structural properties are fulfilled for the unperturbed problem. These requirements rule out direct applications of the contamination technique in the case of discrete distributions, which includes the empirical VaR. Nevertheless, even in this case, it is possible to construct bounds for VaR of the contaminated distribution. In the case of a normal distribution and parametric VaR, one may exploit stability results valid for quadratic programs to stress testing of VaR minimization problems.

Using the contamination technique, we have derived computable bounds which can be extended to stress testing of other risk criteria and risk optimization problems. The presented approaches provide a deeper insight into the stress behaviour of VaR and CVaR than the common numerical evaluations based solely on backtesting and out-of-sample analysis.

Acknowledgements

This research was supported by the project 'Methods of modern mathematics and their applications' (MSM 0021620839) and by the Grant Agency of the Czech Republic (grants 201/05/061 and 402/05/1015).

References

Acerbi, C. and Tasche, D., On the coherence of expected Shortfall. *J. Bank. Finan.*, 2002, **26**, 1487–1503.

Andersson, F., Mausser, H., Rosen, D. and Uryasev, S., Credit risk optimization with Conditional Value-at-Risk criterion. *Math. Program.*, 2001, **B89**, 273–291.

Bank, B., Guddat, J., Klatte, D., Kummer, B. and Tammer, K., *Non-Linear Parametric Optimization*, 1982 (Akademie: Berlin).

Danskin, J.M., *Theory of Max–Min*, *Econometrics and Operations Research*, Vol. 5, 1967 (Springer: Berlin).

Dempster, M.A.H., editor, *Risk Management: Value at Risk and Beyond*, 2002 (Cambridge University Press: Cambridge).

Dobiáš, P., Stability in stochastic programming with applications to risk management. PhD thesis, Faculty of Mathematics and Physics, Charles University, Prague, 2003.

Dupačová, J., Stability in stochastic programming with recourse—contaminated distributions. *Math. Prog. Study*, 1986, **27**, 133–144.

Dupačová, J., Stochastic programming with incomplete information: a survey of results on postoptimization and sensitivity analysis. *Optimization*, 1987, **18**, 507–532.

Dupačová, J., Scenario based stochastic programs: resistance with respect to sample. *Ann. Oper. Res.*, 1996, **64**, 21–38.

Dupačová, J., Stress testing via contamination, in *Proceedings of the Workshop on Coping with Uncertainty*, edited by K. Marti *et al.*, 2006 (Springer: Berlin), in press.

Dupačová, J., Bertocchi, M. and Moriggia, V., Postoptimality for scenario based financial models with an application to bond portfolio management. In *World Wide Asset and Liability Modeling*, edited by W.T. Ziemba and J. Mulvey, pp. 263–285, 1998 (Cambridge University Press: Cambridge).

El Ghaoui, L., Oks, M. and Oustry, F., Worst-case value-at-risk and robust portfolio optimization: a conic programming approach. *Oper. Res.*, 2003, **51**, 543–556.

Embrechts, P., McNeil, A.J. and Straumann, D., Correlation and dependence in risk management: properties and pitfalls. In *Risk Management: Value at Risk and Beyond*, edited by M.A.H. Dempster, pp. 176–223, 2002 (Cambridge University Press: Cambridge).

Gaivoronski, A.A. and Pflug, G., Value at Risk in portfolio optimization. *J. Risk*, 2004, to appear.

Guddat, J., Guerra Vasquez, F., Tammer, K. and Wendler, K., *Multiobjective and Stochastic Optimization Based on Parametric Optimization*, 1985 (Akademie: Berlin).

Høyland, K., Kaut, M. and Wallace, S.W., A heuristic for moment-matching scenario generation. *Comput. Optim. Applic.*, 2003, **24**, 169–185.

Høyland, K. and Wallace, S.W., Generating scenario trees for multistage decision problems. *Mgmt Sci.*, 2001, **47**, 295–307.

Kaut, M., Vladimirou, H., Wallace, S.W. and Zenios, S.A., Stability analysis of a portfolio management model based on the conditional value-at-risk measure, 2003, submitted.

Klein Haneweld, W.K., *Duality in Stochastic Linear and Dynamic Programming*, LNEMS 274, 1986 (Springer: Berlin).

Kupiec, P., Stress testing in a Value at Risk framework. In *Risk Management: Value at Risk and Beyond*, edited by M.A.H. Dempster, pp. 76–99, 2002 (Cambridge University Press: Cambridge).

Ogryczak, W. and Ruszczyński, A., Dual stochastic dominance and related mean-risk models. *SIAM J. Optim.*, 2002, **13**, 60–78.

Pflug, G.Ch., Some remarks on the value-at-risk and the conditional value-at-risk. In *Probabilistic Constrained Optimization, Methodology and Applications*, edited by S. Uryasev, pp. 272–281, 2001 (Kluwer Academic: Dordrecht).

Polívka, J., ALM via stochastic programming. PhD thesis, Faculty of Mathematics and Physics, Charles University, Prague, 2005.

Prékopa, A., Contributions to the theory of stochastic programming. *Math. Program.*, 1973, **4**, 202–221.

Rachev, S. and Mittnik, S., *Stable Paretian Models in Finance*, 2000 (Wiley: Chichester).

Rau-Bredow, H., Value-at-risk, expected shortfall and marginal risk contribution. In *Risk Measures for the 21st Century*, edited by G. Szegö, pp. 61–68, 2004 (Wiley: Chichester).

Rockafellar, R.T. and Uryasev, S., Optimization of conditional value-at-risk. *J. Risk*, 2000, **2**, 21–41.

Rockafellar, R.T. and Uryasev, S., Conditional value-at-risk for general loss distributions. *J. Bank. Finan.*, 2001, **26**, 1443–1471.

Römisch, W., Stability of stochastic programming problems. In *Handbook on Stochastic Programming*, edited by A. Ruszczynski and A. Shapiro, pp. 483–554, 2003 (Elsevier: Amsterdam).

Ruszczyński, A. and Vanderbei, R.J., Frontiers of stochastically nondominated portfolios. *Econometrica*, 2003, **71**, 1287–1297.

Serfling, R.J., *Approximation Theorems of Mathematical Statistics*, 1980 (Wiley: New York).

Shapiro, A., On differential stability in stochastic programming. *Math. Program.*, 1990, **47**, 107–116.

Topaloglou, N., Vladimirou, H. and Zenios, S.A., CVaR models with selective hedging for international asset allocation. *J. Bank. Finan.*, 2002, **26**, 1535–1561.