A note on effect of errors in input parameters on mean-variance efficient portfolios

Jitka Dupačová and Tomáš Petráš[1]

Abstract. Mean-variance efficient portfolios are influenced by errors due to approximation, estimation and incomplete information. Therefore, the obtained results – recommendations for the risk and portfolio manager, should be carefully analyzed. This note presents results of a simulation study devoted to the output analysis with respect to perturbed input data – expected yields and elements of their covariance matrix. The motivation comes from results of the simulation study [2] whose conclusions about the prevailing importance of expectations turn out to be substantially influenced by the chosen value of the model parameter that quantifies the level of the risk aversion of the investor. Our simulation study complements these results comparing the influence of perturbed values of expectations, variances and covariances of yields for the whole range of the risk aversion parameter.

Keywords: Markowitz model, mean-variance efficient portfolios, perturbed parameters

JEL classification: D81, G11, C61
AMS classification: 91G10, 90C31

1 The Markowitz mean-variance model

The Markowitz model is a static, single period model which assumes a frictionless market. It applies to small rational investors whose investments cannot influence the market prices and who prefer higher yields to lower ones and smaller risks to larger ones. Let us recall the basic formulation: The composition of portfolio of I assets is given by weights of the considered assets, $x_i, i = 1, \dots, I$, $\sum_i x_i = 1$. The unit investment in the i-th asset provides the random return ρ_i over the considered fixed period. The assumed probability distribution of the vector ρ of returns of all assets is characterized by a vector of expected returns $E\rho = \mu$ and by a fixed covariance matrix $\Sigma = [\text{cov}(\rho_i, \rho_j),\, i, j = 1, \dots, I]$ whose main diagonal consists of variances of individual returns. This allows to quantify the "yield from the investment" x as the expectation $\mu(x) = \sum_i x_i\mu_i = \mu^\top x$ of its total return and the "risk of the investment" x as the variance of its total return, $\sigma^2(x) = \sum_{i,j} \text{cov}(\rho_i, \rho_j)x_i x_j = x^\top\Sigma x$. According to the assumptions, the investors aim at maximal possible yields and, at the same time, at minimal possible risks – hence, a typical decision problem with two criteria, "max" $\{\mu(x), -\sigma^2(x)\}$ or "min" $\{-\mu(x), \sigma^2(x)\}$. The mean-variance efficiency introduced by Markowitz is fully in line with general concepts of multiobjective optimization. Accordingly, mean-variance efficient portfolios can be obtained by solving various optimization problems.

In accordance with [2] we use the scalarization technique and we shall present the results for the parametric quadratic program

$$\max_{x\in\mathcal{X}}\{\mu^\top x - \lambda x^\top\Sigma x\} \tag{1}$$

where the value of parameter $\lambda \geq 0$ reflects the level of investor's risk aversion and $\mathcal{X} = \{x \in \mathbb{R}^I : x_i \geq 0\, \forall i,\, \sum_i x_i = 1\}$; in general, the approach is valid for an arbitrary nonempty bounded convex polyhedron.

Problem (1) is a convex quadratic program and there exist various solution techniques and theoretical results concerning its stability in dependence on elements of μ, Σ, and parameter λ. See e.g. Chapter 5.3 of [1] for the general theory and [3, 4, 5, 8] for applications to the Markowitz model. In the sequel, we shall assume that Σ is a positive definite matrix, briefly $\Sigma \in S^+$.

[1]Charles University in Prague, Faculty of Mathematics and Physics, Department of Probability and Mathematical Statistics, Sokolovská 83, Prague 8, 18675, Czech Republic, dupacova@karlin.mff.cuni.cz

For a *fixed* value λ, the objective function $f(x;\mu,\Sigma;\lambda)$ is linear in elements of μ,Σ, hence, for each $\lambda \geq 0$ the optimal value

$$\varphi(\mu,\Sigma;\lambda) = \max_{x\in\mathcal{X}}\{\mu^\top x - \lambda x^\top \Sigma x\}$$

is a convex function in μ, Σ, hence continuous on $\mathbb{R}^I \times S^+$. It means, inter alia, that for consistent estimates μ^ν, Σ^ν, of μ, Σ, the optimal value $\varphi(\mu^\nu, \Sigma^\nu; \lambda)$ is a consistent estimate of the true optimal value. This assertion can be complemented by the rates of convergence. Using the "delta" method, cf. [10], section 7.2.2 for a succinct explanation, for asymptotically normal estimates μ^ν, Σ^ν, asymptotic normality can be proved, and asymptotic confidence region derived.

Moreover, for each $\lambda > 0$ and for all $(\mu,\Sigma) \in \mu \times S^+$, there exists unique optimal solution $x^*(\mu,\Sigma;\lambda)$, a continuous vector function of μ,Σ; cf. Theorem 5.3.2 of [1]. However, in general, its asymptotic distribution is a mixture of normal distributions.

In this note we shall assume that problem (1) was solved for certain reference or nominal values of elements of μ,Σ and we shall study influence of perturbations in these values on the output by simulation, regardless of their origin. Of course, the results depend on the value of the model parameter λ; notice the evident differences for $\lambda = 0$, i.e. no influence of perturbations in Σ and possibly multiple optimal solutions, and $\lambda > 0$.

Notice that for multinormal $\mathcal{N}(\mu,\Sigma)$ distribution of returns the value of λ may come from an underlying problem, e.g. from maximization of expected concave utility of the total return $\sum_i \rho_i x_i$, or from minimization of its VaR or CVaR, cf. [9].

2 Simulation study

In conclusions of [11], there are *The Top 10 Points to Remeber* in applications of stochastic programming models to asset, liability and wealth management, including the Markowitz mean-variance model. Let us quote:

> The point# 1: "Means are by far the most important part of the distribution of returns, especially the direction"...

Indeed, this was also the conclusion of [2] for the Markowitz model (1):

> "...errors in means are over ten times as damaging as errors in variances, and over twenty times as damaging as errors in covariances."...

The simulation study of [2] was based on monthly returns in 1.1.1980 – 1.12.1989 of 10 randomly selected stocks from the Dow Jones Industrial Average Index for fixed $\lambda = 0.02$. The influence of changes in the nominal parameters θ_0 containing selected elements of μ_0, Σ_0 obtained from historical data on the optimal, mean-variance portfolio was quantified using values of the Cash Equivalent Loss (CEL). For Markowitz model (1) CEL is equal to the relative error ratio

$$CEL_{\bar{\theta}} := \Big|\frac{\varphi(\theta_0;\lambda) - \varphi(\bar{\theta};\lambda)}{\varphi(\theta_0;\lambda)}\Big|,$$

where $\bar{\theta}$ denotes the perturbed values of selected parameters; cf. [2]. Perturbations of components θ_{0i} of θ_0 were generated randomly according to

$$\bar{\theta}_i = \theta_{0i}(1 + k\varepsilon_i) \tag{2}$$

where ε_i are iid $N(0,1)$ random variables and the error magnitude is fixed by $k = 0.05, 0.10, 0.15, 0.20$. The average values of CEL obtained by [2] for 100 data perturbations according to (2), separately for perturbed means $\bar{\mu}$, variances $\bar{\Sigma}_{var}$ and covariances $\bar{\Sigma}_{cov}$, are contained in the first column of Table 1. The next columns of the table contain our extensions of these results for higher values of parameter λ.

Figure 1 plots the average CEL values (in %) as a function of λ for $k = 0.1$ Perturbations of means are the prevailing factor for for λ close to 0, but there is an evident change in ranking the importance of perturbations which appears approximately starting with $\lambda = 0.1$. For $\lambda \geq 0.5$ the average values of CEL remain approximately constant.

	λ=0.02	λ=0.2	λ=1	λ=2
k=0.05				
$CEL_{\bar{\mu}}$	$5.41 \cdot 10^{-3}$	$8.50 \cdot 10^{-4}$	$2.21 \cdot 10^{-5}$	$5.43 \cdot 10^{-6}$
$CEL_{\bar{\Sigma}_{var}}$	$6.75 \cdot 10^{-4}$	$3.75 \cdot 10^{-3}$	$2.22 \cdot 10^{-3}$	$2.11 \cdot 10^{-3}$
$CEL_{\bar{\Sigma}_{cov}}$	$2.02 \cdot 10^{-4}$	$1.73 \cdot 10^{-3}$	$9.74 \cdot 10^{-4}$	$9.26 \cdot 10^{-4}$
k=0.10				
CEL_{μ}	$2.37 \cdot 10^{-2}$	$3.35 \cdot 10^{-3}$	$8.53 \cdot 10^{-5}$	$2.10 \cdot 10^{-5}$
$CEL_{\bar{\Sigma}_{var}}$	$2.57 \cdot 10^{-3}$	$1.86 \cdot 10^{-2}$	$1.13 \cdot 10^{-2}$	$1.07 \cdot 10^{-2}$
$CEL_{\bar{\Sigma}_{cov}}$	$9.70 \cdot 10^{-4}$	$8.53 \cdot 10^{-3}$	$4.92 \cdot 10^{-3}$	$4.64 \cdot 10^{-3}$
k=0.15				
CEL_{μ}	$5.25 \cdot 10^{-2}$	$7.54 \cdot 10^{-3}$	$1.89 \cdot 10^{-4}$	$4.62 \cdot 10^{-4}$
$CEL_{\bar{\Sigma}_{var}}$	$5.55 \cdot 10^{-3}$	$4.70 \cdot 10^{-2}$	$2.84 \cdot 10^{-2}$	$2.70 \cdot 10^{-2}$
$CEL_{\bar{\Sigma}_{cov}}$	$2.70 \cdot 10^{-3}$	$2.16 \cdot 10^{-2}$	$1.27 \cdot 10^{-2}$	$1.21 \cdot 10^{-2}$
k=0.20				
$CEL_{\bar{\mu}}$	$8.46 \cdot 10^{-2}$	$1.35 \cdot 10^{-2}$	$3.34 \cdot 10^{-4}$	$8.09 \cdot 10^{-5}$
$CEL_{\bar{\Sigma}_{var}}$	$9.71 \cdot 10^{-3}$	$8.74 \cdot 10^{-2}$	$5.72 \cdot 10^{-2}$	$5.51 \cdot 10^{-2}$
$CEL_{\bar{\Sigma}_{cov}}$	$4.21 \cdot 10^{-3}$	$2.71 \cdot 10^{-2}$	$1.67 \cdot 10^{-2}$	$1.60 \cdot 10^{-2}$

Table 1 Average values of CEL for error model (2) using data from [2].

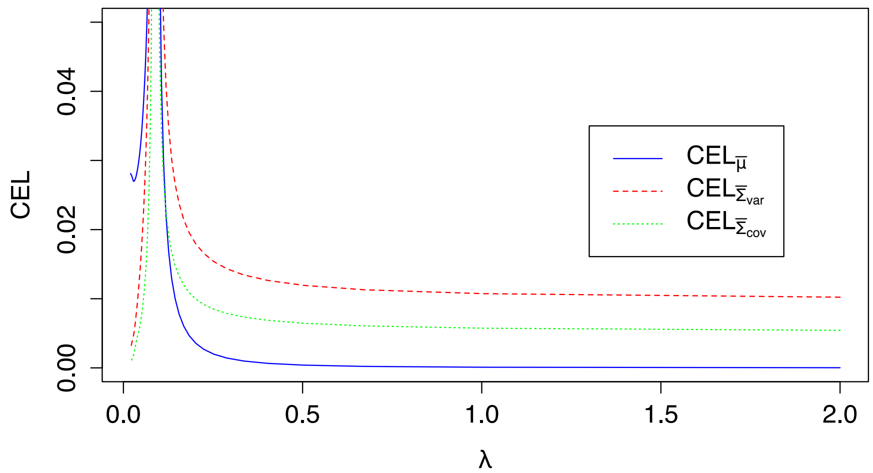

Figure 1 Dependence of CEL on risk aversion for error model (2) with $k = 0.1$.

Motivated by [6], we repeated the simulation experiment for a different construction of perturbations:

$$\bar{\theta}_i = \theta_{0i} + k\varepsilon_i\Delta_i \tag{3}$$

where ε_i are again iid random variables with $N(0,1)$ distribution and Δ_i reflects the spread of the corresponding parameter values in the sample. The results are similar, see Table 2 and Figure 2. For details see [7].

	λ=0,02	λ=0,2	λ=1	λ=2
k=0,05				
$\text{CEL}_{\bar{\mu}}$	$5.13 \cdot 10^{-3}$	$6.55 \cdot 10^{-4}$	$1.52 \cdot 10^{-5}$	$3.28 \cdot 10^{-6}$
$\text{CEL}_{\bar{\Sigma}_{var}}$	$6.56 \cdot 10^{-5}$	$5.74 \cdot 10^{-4}$	$3.99 \cdot 10^{-4}$	$3.86 \cdot 10^{-4}$
$\text{CEL}_{\bar{\Sigma}_{cov}}$	$2.42 \cdot 10^{-4}$	$2.23 \cdot 10^{-3}$	$1.21 \cdot 10^{-3}$	$1.14 \cdot 10^{-3}$
k=0,10				
$\text{CEL}_{\bar{\mu}}$	$1.91 \cdot 10^{-2}$	$2.60 \cdot 10^{-3}$	$6.40 \cdot 10^{-5}$	$1.41 \cdot 10^{-5}$
$\text{CEL}_{\bar{\Sigma}_{var}}$	$2.64 \cdot 10^{-4}$	$2.32 \cdot 10^{-3}$	$1.64 \cdot 10^{-3}$	$1.59 \cdot 10^{-3}$
$\text{CEL}_{\bar{\Sigma}_{cov}}$	$1.08 \cdot 10^{-3}$	$1.02 \cdot 10^{-2}$	$5.29 \cdot 10^{-3}$	$4.94 \cdot 10^{-3}$
k=0,15				
$\text{CEL}_{\bar{\mu}}$	$4.03 \cdot 10^{-2}$	$5.84 \cdot 10^{-3}$	$1.46 \cdot 10^{-4}$	$3.32 \cdot 10^{-5}$
$\text{CEL}_{\bar{\Sigma}_{var}}$	$5.96 \cdot 10^{-4}$	$5.59 \cdot 10^{-3}$	$3.96 \cdot 10^{-3}$	$3.84 \cdot 10^{-3}$
$\text{CEL}_{\bar{\Sigma}_{cov}}$	$2.75 \cdot 10^{-3}$	$2.02 \cdot 10^{-2}$	$1.18 \cdot 10^{-2}$	$1.12 \cdot 10^{-2}$
k=0,20				
$\text{CEL}_{\bar{\mu}}$	$6.73 \cdot 10^{-2}$	$1.03 \cdot 10^{-2}$	$2.61 \cdot 10^{-4}$	$6.04 \cdot 10^{-5}$
$\text{CEL}_{\bar{\Sigma}_{var}}$	$1.08 \cdot 10^{-3}$	$1.12 \cdot 10^{-2}$	$8.03 \cdot 10^{-3}$	$7.80 \cdot 10^{-3}$
$\text{CEL}_{\bar{\Sigma}_{cov}}$	$4.79 \cdot 10^{-3}$	$2.99 \cdot 10^{-2}$	$1.84 \cdot 10^{-2}$	$1.74 \cdot 10^{-2}$

Table 2 Average values of CEL for error model (3).

3 Conclusions

Our main conclusions can be summarized in the following two points:

- Small perturbations may cause visible relative errors in the optimal variance adjusted expected return of the portfolio.
- The main source of errors need not be the expected return, the performance depends on the model parameter λ which quantifies the level of the risk aversion of the investor.

Similar results based on simulation studies can be obtained also for maximization or minimization of other mean-risk objectives, e.g. for the mean-CVaR objective, cf. [6].

Acknowledgements

Supported by the grant No. P402/12/G097'DYME/Dynamic Models in Economics' of the Czech Science Foundation.

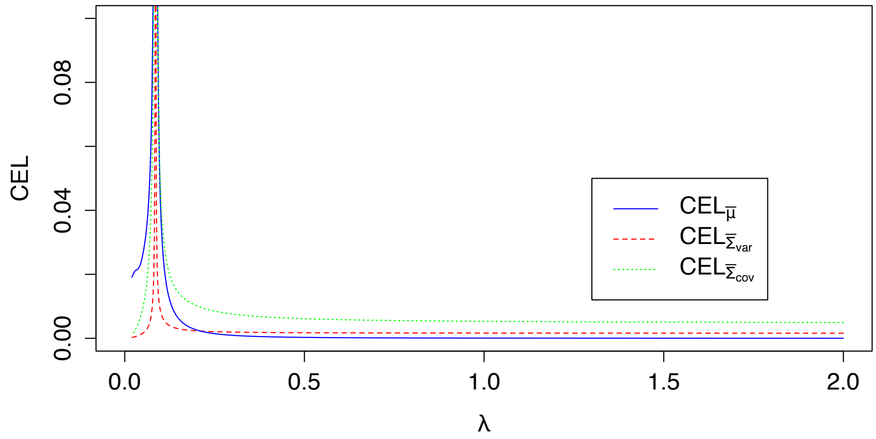

Figure 2 Dependence of CEL on risk aversion for error model (3) with $k = 0.1$.

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