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From data to model and back to data: A bond portfolio management problem

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Abstract

The bond portfolio management problem is formulated as a multiperiod stochastic program using interest rate scenarios. The scenarios are sampled from the binomial lattice from a Black–Derman–Toy model. The paper analyzes the sensitivity of the solution of the resulting large-scale mathematical program with respect to the model inputs. The numerical results are for the Italian bond market.

Keywords: Finance; Stochastic programming; Scenarios; Stability; Sensitivity and post-optimality analysis

1. The problem and the input data

The problem considered here is to preserve the value of a bond portfolio of a risk averse or risk neutral institutional investor over time. This is a problem of allocation and management of resources, not of trading. It may include additional features, e.g., presence of fixed or uncertain external inflows or outflows in the future or a required balance between assets and liabilities. There are various options concerning the choice of an appropriate model, starting with duration-based immunization models, dedicated bond portfolio management models (see Dahl et al., 1993), or goal programming type of immunization models (cf. Dembo, 1993), up to multistage stochastic programming models (see Ziemba and Mulvey, 1998), which can be used for complex assets/liabilities management problems (e.g., Cariño et al., 1994).

Why not to rely on the duration-based immunization models? A good answer is the following quotation (cf. Kahn, 1991):

> Many years ago, bonds were boring. Returns were small and steady. Fixed income risk monitoring consisted in watching duration and avoiding low qualities. But as interest-rate volatility has increased and the variety of fixed income instruments has grown, both opportunities and dangers have flourished...

Yield curves are not flat, do not move in a parallel way, the interest rates are not constant and

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investment in long maturity bonds requires an active trading strategy. These are the reasons that have led us to the exploitation of multiperiod stochastic programs with the main random element to be included – the evolution of the short interest rate over time which is regarded as the only factor that drives the prices of default free government bonds.

Given a sequence of equilibrium future short-term interest rates r_t valid for the time interval $(t, t+1]$, $t = 0, \dots, T-1$ the fair price of the jth bond at time t just after the coupon was paid equals the total cashflow $f_{j\tau}$, $\tau = t+1, \dots, T$ generated by this bond in subsequent time instances discounted to t

$$P_{jt}(\mathbf{r}) = \sum_{\tau=t+1}^{T} f_{j\tau} \prod_{h=t}^{\tau-1} (1 + r_h)^{-1}, \tag{1}$$

where T is greater than or equal to the time to maturity.

However, the time points do not coincide with the dates of coupon payments. Also the sequence of the future short-term rates r_h that determines the prices (1) is not known precisely, but prescribed ad hoc or modeled in a probabilistic way. The cashflows $f_{j\tau}$ need not be known with certainty; this is for instance the case of indexed bonds, bonds with options or default. Hence, the formula (1) should be extended for the accrued interest A_{jt} and revised to take into account the effect of options and other risks related with the jth bond. The resulting selling and purchasing prices do reflect also the transaction costs and the bid/ask spread.

Assume that the only random factor which influences the fair prices is the evolution of short-term interest rates. Their probabilistic nature will be modeled by a discrete probability distribution, say P, of T-dimensional vectors $\mathbf{r}$ of the short rates r_t, $t = 0, \dots, T-1$, where r_0 (the rate valid in the first period) is supposed to be known. The finite realizations of $\mathbf{r}$ are indexed as $\mathbf{r}^s$, $s = 1, \dots, S$ with probabilities $p_s > 0$, $s = 1, \dots, S$, $\sum_s p_s = 1$.

The recommendations for bond portfolio management depend on the chosen scenarios of interest rates, on their probabilities and on numerical values of their components r_t^s. Regarding sensitivity of the results on inclusion of additional, "out-of-sample" scenarios and on small changes of scenarios and their probabilities there are global methods and results which do not depend on the way how the scenarios were generated and which can be adapted to various stochastic programming models (cf. Dupačová, 1996, 1998; Dupačová et al., 1997a,b, 1998; Dupačová and Rømisch, 1998). We discuss them briefly in Section 2. We concentrate on the impact of data and of the estimation procedures for scenarios generated by a Black et al. (1990) model and for one possible model of bond portfolio management. Similar problems appear also for other problem settings, for different models of interest rates and they are not limited to bond portfolio management. We introduce a general methodology focused on finding and quantification of the impact of random and systematic errors, which are present in the scenario generation procedure, on the results. We illustrate the ideas numerically for the chosen stochastic programming model of bond portfolio management and for Italian bond market data from 1994 to 1997.

The Italian bond market was the fourth largest fixed-income market in the world. The government fixed-income securities represent more than 85% of this market and they include zero-coupon bonds, BOTs and CTZs, of maturities up to two years, coupon bonds without option, BTPs, with different maturities (3, 5, 10 and 30 years) which are issued two times per month through a marginal auction without minimal price, floater bonds, CCTs, there used to be puttable bonds, CTOs, etc. There are futures and options on some of BTPs and also bonds with maturities between 3 and 30 years issued by corporations. Italian bond market data were reported in risk metrics datasets.

The size of the Italian bond market provided a sound basis for application of various models of interest rates and for their calibration; its liquidity can be taken for granted when designing models which admit rebalancing strategies.

2. The model and the structure of the program

We formulate the bond portfolio management problem analogous to that in Golub et al. (1995), Zenois (1995), Dupačová and Bertocchi (1996) and Dupačová et al. (1997a,b, 1998). In comparison with earlier scenario-based stochastic programs for bond portfolio management (e.g., Hiller and Eckstein, 1993; Shapiro, 1988), the model formulation allows for the possibility of intertemporal *rebalancing* of the portfolio. Let

$j = 1, \dots, J$ be indices of the bonds and T_j the dates of their maturities; $T = \max_j T_j$.

$t = 0, \dots, T_0$ the discretization of the planning horizon;

$b_j \geqslant 0$ the initial holdings (in face value) of bond j;

b_0 the initial holding in riskless asset;

f^s_{jt} cashflow generated under scenario s from bond j at time t expressed as a fraction of its face value.

ξ^s_{jt} and ζ^s_{jt} are the selling and purchasing prices of bond j at time t for scenario s obtained from the corresponding fair prices (1) adding the accrued interest A^s_{jt} and subtracting or adding scenario independent transaction costs and spread; the initial prices ξ_{j0} and ζ_{j0} are known constants, i.e., scenario independent;

L_t is an external cashflow at time t;

x_j/y_j are the face values of bond j purchased/sold at the beginning of the planning period, i.e., at $t = 0$;

z_{j0} is the face value of bond j held in portfolio after the initial decisions $x_j,\ y_j$ have been made.

The first-stage decision variables $x_j,\ y_j,\ z_{j0}$ are nonnegative

$$y_j + z_{j0} = b_j + x_j \quad \forall j \tag{2}$$

and

$$y_0^+ + \sum_j \zeta_{j0} x_j = b_0 + \sum_j \xi_{j0} y_j, \tag{3}$$

where the nonnegative variable y_0^+ denotes the surplus.

The second-stage decisions on rebalancing the portfolio, borrowing or reinvestment of the surplus depend on individual scenarios. They have to fulfill constraints on conservation of holdings in each bond at each time period and for each of the scenarios

$$z^s_{jt} + y^s_{jt} = z^s_{j,t-1} + x^s_{jt} \quad \forall j,\ s,\ \ 1 \leqslant t \leqslant T_0, \tag{4}$$

where $x^s_{jt},\ y^s_{jt},\ z^s_{jt}$ denote the face value of bond j purchased, sold, held in the portfolio at time t, $t = 1, \dots, T_0$ under scenario s, and constraints on rebalancing the portfolio at each time period $1 \leqslant t \leqslant T_0$

$$\begin{aligned} \sum_j \xi^s_{jt} y^s_{jt} + \sum_j f^s_{jt} z^s_{j,t-1} + \left(1 - \delta_1 + r^s_{t-1}\right) y^{+s}_{t-1} + y^{-s}_t \\ = L_t + \sum_j \zeta^s_{jt} x^s_{jt} + \left(1 + \delta_2 + r^s_{t-1}\right) y^{-s}_{t-1} \\ + y^{+s}_t \quad \forall s,\ t \end{aligned} \tag{5}$$

with $y_0^{-s} = 0,\ y_0^{+s} = y_0^+,\ z^+_{j0} = z_{j0}\ \forall j$. The variables $y_t^{+s},\ y_t^{-s}$ describe the (unlimited) lending/borrowing possibilities for period t under scenario s and the spreads $\delta_1,\ \delta_2$ are model parameters to be fixed. Nonzero values of δ_1 account for the difference between the returns for bonds and for cash. Assume that $\delta_2 > 0$, i.e., there is a positive cost of borrowing.

The optimization problem is maximization of the expected utility of the final wealth at time T_0

$$\sum_s p_s U(W^s_{T_0}) \tag{6}$$

subject to constraints (2)–(5) and nonnegativity constraints on all variables, with

$$W^s_{T_0} = \sum_j \xi^s_{jT_0} z^s_{jT_0} + y^{+s}_{T_0} - \alpha y^{-s}_{T_0} \quad \forall s. \tag{7}$$

The multiplier $\alpha \geqslant 1$ is fixed according to the problem area. For instance, α can be scenario dependent and values $\alpha^s > 1$ take into account the debt service in the future. In case of preservation of portfolio value (with no liabilities considered) or for an investment project terminating at time T_0, an arbitrarily large value of α plays the role of a penalty for borrowing at the end of the accounting or planning period.

Owing to the possibility of reinvestments and of unlimited borrowing, the problem has always a feasible solution. It is a *multiperiod two-stage sto-*

chastic programming model with random relatively complete recourse and with nonlinearities in the utility function. The existence of optimal solutions is guaranteed for a large class of utility functions that are *increasing and concave* that will be assumed henceforth. Moreover, due to strict inequalities $\xi_{j0} < \zeta_{j0}\ \forall j,\ \xi^s_{jt} < \zeta^s_{jt}\ \forall j,\ t,\ s$ and $\delta_1 \geqslant 0,\ \delta_2 > 0$, the *optimal* solutions satisfy

$$y_j x_j = 0 \quad \forall j,$$
$$y^s_{jt} x^s_{jt} = 0 \quad \forall s,\ j,\ 1 \leqslant t \leqslant T_0,$$
$$y^{+s}_t y^{-s}_t = 0 \quad \forall s,\ 1 \leqslant t \leqslant T_0.$$

At optimality there is no unnecessary trading and borrowing.

We obtain a large-scale deterministic program with a concave objective function and numerous linear constraints. The size and the numerical values of the coefficients of the program result from the application and the available data: the choice of bonds, their characteristics (initial prices and future cashflows) and initial holdings, from the scheduled stream of liabilities, transaction costs and spread and from *the way how the scenarios of future interest rates are generated and sampled*. The main outcome is the optimal value of the objective function (the maximal expected utility of the final wealth) and the optimal values of the first-stage variables $x_j,\ y_j$ (and $y^+_0,\ z_{j0}$) for all j. In a dynamic setting, this decision is applied and at the end of the first period, the model is solved again for the changed input information on holdings and on scenarios of interest rates (see Kusy and Ziemba, 1986) for a detailed explanation of this idea.

Assume that the portfolio consists of *default free, liquid bonds with maturities* $T_j > T_0\ \forall j$, *all cashflows are after tax, the transaction costs and bid/ask spreads are constant quantities*. A careful tuning of the numerical values of these design parameters may contribute to a realistic performance of the model (cf. Bertocchi et al., 1996a,b or Dupačová et al., 1998), and discretization of the time horizon influences not only the tractability of the numerical procedure but also the accessibility and precision of the input data.

The following reformulation of the problem is useful for stability and post-optimality analysis. Assume that an initial trading strategy determined by scenario independent first-stage decision variables $x_j,\ y_j,\ y^+_0$ (and z_{j0}) for all j has been accepted, then the subsequent scenario-dependent decisions have to be made in an optimal way regarding the goal of the model. It means that given the values of the first-stage variables y^+_0 and $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}_0$ with components $x_j,\ y_j,\ z_{j0}\ \forall j$, the required maximal contribution of the portfolio management under the sth scenario to the value of the objective function is obtained as the value of the utility function computed for the maximal value of the wealth $W^s_{T_0}$ attainable for the sth scenario under the constraints of the model, i.e., the utility of the optimal value $W^s_{T_0}$ of the linear program

$$\text{maximize} \quad W^s_{T_0}$$

$$\text{subject to} \quad z^s_{jt} + y^s_{jt} = z^s_{j,t-1} + x^s_{jt} \quad \forall j, \quad 1 \leqslant t \leqslant T_0, \tag{8}$$

$$\sum_j \xi^s_{jt} y^s_{jt} + \sum_j f^s_{jt} z^s_{j,t-1} + \left(1 - \delta_1 + r^s_{t-1}\right) y^{+s}_{t-1} + y^{-s}_t = L_t + \sum_j \zeta^s_{jt} x^s_{jt} + \left(1 + \delta_2 + r^s_{t-1}\right) y^{-s}_{t-1} + y^{+s}_t, \quad 1 \leqslant t \leqslant T_0, \tag{9}$$

$$\xi^s_{jt} \geqslant 0, \quad y^s_{jt} \geqslant 0, \quad z^s_{j,t} \geqslant 0, \quad y^{-s}_t \geqslant 0, \quad y^{+s}_t \geqslant 0 \quad \forall j,\ 1 \leqslant t \leqslant T_0 \tag{10}$$

with

$$y^{-s}_0 = 0, \quad y^{+s}_0 = y^+_0, \quad z^s_{j0} = z_{j0} \quad \forall j$$

and with

$$W^s_{T_0} = \sum_j \xi^s_{jT_0} z^s_{jT_0} + y^{+s}_{T_0} - \alpha y^{-s}_{T_0}. \tag{11}$$

Denote the corresponding maximal value by $W_{T_0}(\mathbf{r}^s; \mathbf{x}, \mathbf{y}, \mathbf{z}_0, y^+_0)$ and rewrite the program (2)–(7) as

$$\text{maximize} \quad \sum_{s=1}^{S} p_s U\left(W_{T_0}\left(\mathbf{r}^s; \mathbf{x}, \mathbf{y}, \mathbf{z}_0, y^+_0\right)\right) \tag{12}$$

subject to nonnegativity constraints and subject to 2 and 3. Scenarios enter now only the objective function (12).

Assume a *positive market value of the initial portfolio*, $b_0 + \sum_j \xi_{j0} b_j > 0$ and in the first stage, no borrowing is allowed. This implies that *the set of the feasible first-stage solutions is nonempty and bounded.* This property holds true also in case that a *restricted borrowing* possibility in the first stage is permitted.

The remaining part of this section reviews results on the stability of the problem (2)–(7) or (2), (3) and (12), which are *independent of the way how the scenarios of interest rates were generated and selected.*

2.1. Out-of-sample scenarios

Assume that the stochastic program (2)–(7) is solved for a fixed set of scenarios $\mathbf{r}^s$, $s = 1, \dots, S$ and that the influence of including other out-of-sample scenarios should be considered. Such problem can be related to the "what–if" analysis, to various stability and sensitivity studies, to incorporating investors' views, etc. One could rewrite the program (2)–(7) for the extended set of scenarios, with additional variables and additional constraints of the type (4), (5), and (7) and solve it. Another possibility is to use the form (12), (2) and (3) whose set of feasible solutions is not influenced by inclusion of additional scenarios. The additional scenarios appear only in the objective function which is an expected value of the utility of the final wealth under a discrete probability distribution carried by a finite number of scenarios. Being an expected value, the objective function (12) is linear in the probability distribution.

Denote by P the initial probability measure carried by S interest rate scenarios indexed as $s = 1, \dots, S$ with probabilities $p_s > 0\ \forall s$, $\sum_s p_s = 1$. Let $\varphi(P)$ be the optimal value of (12) and $\mathbf{x}(P)$, $\mathbf{y}(P)$, $\mathbf{z}_0(P)$, $y_0^+(P)$ be an optimal first-stage solution. For simplicity, assume that the optimal first-stage solution is unique.

Inclusion of other out-of-sample scenarios means to consider another discrete probability distribution which is carried by the extended set of scenarios. Such distributions can be modeled as a convex mixture of two discrete probability distributions: P that is carried by the initial scenarios indexed by $s = 1, \dots, S$ with probabilities $p_s > 0$, $\sum_s p_s = 1$ and Q carried by the out-of-sample scenarios indexed by $\sigma = 1, \dots, S'$ with probabilities $\pi_\sigma > 0$, $\sum_\sigma \pi_\sigma = 1$. The weights of the two probability distributions are given by the contamination parameter λ and the contaminated distribution

$$P_\lambda = (1-\lambda)P + \lambda Q, \quad 0 \leqslant \lambda \leqslant 1$$

is carried by the *pooled sample* of $S + S'$ scenarios that occur with probabilities $(1-\lambda)p_1, \dots, (1-\lambda)p_S, \lambda\pi_1, \dots, \lambda\pi_{S'}$. For fixed probability distributions P and Q, the objective function (12) which corresponds to the contaminated distribution P_λ is a *linear* function of λ.

Small values of the contamination parameter λ are typical for various stability studies; $\lambda = 1/2$ corresponds to the analysis of the impact of doubling the sample size, by the choice of λ the degree of confidence in expert opinions (cf. Koskosides and Duarte, 1997) can be reflected, and so on.

For the fixed initial distribution P and a fixed contaminating distribution Q for which the maximal value $\varphi(Q)$ of $\sum_\sigma \pi_\sigma U(W_{T_0}(\mathbf{r}^\sigma; \mathbf{x}, \mathbf{y}, \mathbf{z}_0, y_0^+))$ is finite, the optimal value $\varphi(P_\lambda) := \varphi(\lambda)$ is a *finite convex function* on $[0, 1]$ and its derivative at $\lambda = 0^+$ equals

$$\varphi'(0^+) = \sum_\sigma \pi_\sigma U\big(W_{T_0}\big(\mathbf{r}^\sigma; \mathbf{x}(P), \mathbf{y}(P), \mathbf{z}_0(P), y_0^+(P)\big)\big) - \varphi(P). \tag{13}$$

Bounds for the optimal value $\varphi(P_\lambda)$ of the problem based on the pooled set of $S + S'$ scenarios follow by convexity arguments:

$$(1-\lambda)\varphi(P) + \lambda \sum_\sigma \pi_\sigma U\big(W_{T_0}\big(\mathbf{r}^\sigma; \mathbf{x}(P), \mathbf{y}(P), \mathbf{z}_0(P), y_0^+(P)\big)\big) \leqslant \varphi(P_\lambda) \leqslant (1-\lambda)\varphi(P) + \lambda\varphi(Q) \quad \forall 0 \leqslant \lambda \leqslant 1. \tag{14}$$

The additional numerical effort consists in solving the stochastic program

$$\text{maximize} \quad \sum_{\sigma} \pi_{\sigma} U\left(W_{T_0}\left(\mathbf{r}^{\sigma}; \mathbf{x}, \mathbf{y}, \mathbf{z}_0, y_0^+\right)\right)$$

subject to (2) and (3) and nonnegativity constraints for the distribution Q carried by S' out-of-sample scenarios to obtain $\varphi(Q)$ and in evaluation and averaging the S' function values $U(W_{T_0}(\mathbf{r}^{\sigma}; \mathbf{x}(P), \mathbf{y}(P), \mathbf{z}_0(P), y_0^+))$ for the new scenarios at the already obtained optimal first-stage solution; these are in fact the main numerical indicators which appear in various simulation studies of the portfolio performance under out-of-sample scenarios (cf. Kusy and Ziemba, 1986; McKendall et al., 1994).

The described post-optimality technique aims at the optimal value function. It is quite general and applies to a broad class of scenario-based stochastic programs (cf. Dupačová, 1996). In the context of the bond portfolio management problem, it is independent of the method which was used to generate or to select the scenarios – the atoms of the distributions P and Q. It can be extended without any problems to scenario-dependent liabilities and cashflows which allows to include the case of callable and puttable bonds and also mortgage backed securities. It was elaborated in detail and applied in Dupačová et al. (1997a,b, 1998) to data from the Italian bond market for scenarios generated from a Black et al. (1990) model. The numerical results are encouraging; they can be used to test the influence of parallel shifts of interest rate scenarios, to quantify the change in the optimal value of (12) due to inclusion of additional scenarios, etc. An extension to multistage problem setting follows from Dupačová (1995).

There are also theoretical results based on the contamination technique that deal with properties of the optimal first-stage solutions; for a survey consult Dupačová (1990). These results, however, are much more technically involved and are not yet ready for a numerical implementation.

2.2. Stability results

We now discuss the stability properties of the stochastic program (2)–(7) with respect to changes in the numerical values of its coefficients. These changes are consequences of changes in numerical values of the components r_{jt}^s of the selected S scenarios of interest rates.

It is possible to prove (see Dupačová and Bertocchi, 1997 or Dupačová, 1998), that for an arbitrary fixed scenario $\mathbf{r}^s$ and for an arbitrary feasible first-stage decision $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}_0$, y_0^+ the scenario subproblems (8)–(11) are *stable linear programs* in the sense of Robinson (1977) provided that $0 < \xi_{jt}^s < \zeta_{jt}^s \; \forall j, t$ and $\alpha > 1$, $\delta_2 > 0$. This implies that the sets of optimal solutions of the pairs of the dual scenario subproblems are nonempty and bounded, the optimal value function $W_{T_0}(\mathbf{r}^s; \mathbf{x}, \mathbf{y}, \mathbf{z}_0, y_0^+)$ is *jointly* continuous in $\mathbf{r}^s$, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}_0$, y_0^+ and for all small (but otherwise arbitrary) perturbations, the distances of optimal solutions of the pair of the dual perturbed scenario subproblems from the sets of optimal solutions of the unperturbed ones are bounded by a constant multiple of the size of perturbations. Moreover, for an arbitrary fixed scenario $\mathbf{r}^s$, the optimal value of (8)–(11), $W_{T_0}(\mathbf{r}^s; \mathbf{x}, \mathbf{y}, \mathbf{z}_0, y_0^+)$ is concave, piecewise linear in the first-stage decision variables.

The continuity properties with respect to scenarios, their probabilities and with respect to the first-stage decision variables apply also to the objective function (12) and the expectations $\sum_s p_s W_{T_0}(\mathbf{r}^s; \mathbf{x}, \mathbf{y}, \mathbf{z}_0, y_0^+)$ and $\sum_{s=1}^{S} p_s U(W_{T_0}(\mathbf{r}^s; \mathbf{x}, \mathbf{y}, \mathbf{z}_0, y_0^+))$ are concave with respect to the first-stage decision variables for an arbitrary nondecreasing concave utility function U. This means (recall that the set of the feasible first-stage decisions is nonempty and bounded) that also the optimal value function of the full problem (12), (2) and (3) is continuous with respect to the input parameters $\mathbf{r}^s$, p_s, $s = 1, \ldots, S$ (see e.g., Bank et al., 1982).

These results imply that *small errors in evaluation of scenarios of interest rates, of their probabilities and consequently of prices* ξ_{jt}^s, ζ_{jt}^s *cause only small changes to the best available scenario-based market values* $W_{T_0}(\mathbf{r}^s; \mathbf{x}, \mathbf{y}, \mathbf{z}_0, y_0^+)$ *and also to the optimal value of the overall performance function* (12). However, it is important to *quantify* at least the meaning of "small" errors in evaluation of scenarios. One way is via differentiability properties.

Directional derivatives of the optimal value functions $W_{T_0}(\mathbf{r}^s; \mathbf{x}, \mathbf{y}, \mathbf{z}_0, y_0^+)$ with respect to various changes of the scenarios can be obtained using differentiability of the coefficients and the well-known marginal value formulas (cf. Gol'shtein, 1970). This result can be extended to the *directional differentiability of the optimal value function of the full problem* (12), (2) and (3) with respect to changes in components of scenarios (see Dupačová, 1998). Differentiability of the optimal value function cannot be expected and for multi-dimensional parameters, directional derivatives are of a limited practical use in quantification of stability results.

On the other hand, under modest assumptions it is possible to prove a locally *Lipschitzian behavior* of the optimal value functions $W_{T_0}(\mathbf{r}^s; \mathbf{x}, \mathbf{y}, \mathbf{z}_0, y_0^+)$ and of the optimal value of the performance function (12) with respect to the chosen discrete probability distribution (scenarios and their probabilities). The local Lipschitzian property extends also to the sets of ε-optimal solutions of (12), (2) and (3) (see Dupačová and Rømisch, 1998 for details). The results contribute not only to the quantification of the above stability results but are also related to the problem of selection of representative scenarios and to the quantification of the influence of out-of-sample scenarios on the ε-optimal first-stage solutions. Their application is a new area of research which requires to choose a suitable and computable distance function of alternative discrete probability distributions and to get sufficiently tight bounds on the Lipschitz constants.

To summarize. It is possible to quantify the influence of inclusion of additional scenarios on the optimal value of the bond portfolio management problem. Moreover, it is possible to prove various stability results for the problem. However, (contrary to the post-optimality with respect to additional scenarios) we do not have any general numerically tractable method to quantify them. This made us to turn our attention to simulation studies. To provide well-interpretable results, these simulation studies have to be tailored to the way how the scenarios have been generated and selected.

3. Generation of scenarios

Our primary goal is to analyze the sensitivity of the optimal value of (6) to the selected scenarios of interest rates. These scenarios can be obtained by discretization of a true continuous probability distribution, from a model calibrated by market data, from historical observations and in principle, they can be also fixed ad hoc using expert forecasts (cf. Koskosides and Duarte, 1997). The best way is to exploit all sources of available information. The data from the Italian bond market considered here give a solid base for possible applications of these approaches, whereas interest rate scenarios for thin emerging markets can hardly be based on historical data or on estimation techniques discussed below.

The first step is to *choose a model of the short rates* and to calibrate it so that it fits the market data reasonably well; this can be formulated as a requirement to price precisely some of the traded financial documents, e.g., the fixed coupon government bonds. An example is the Cox et al. (1985) model applied to the daily data from Italian bond market for 1984–1990 years in Barone et al. (1991).

Similarly as in Dupačová and Bertocchi (1996) we use the Black et al. (1990) model as the basis for generation of the interest rate scenarios. The basic *assumptions of the Black–Derman–Toy model* can be summarized as follows:

- The short rate is the only factor that drives the bond prices, it can move up or down with equal probability over the next time period; the sequences of "up–down" and "down–up" moves from any fixed stage at a time point t result in the same value of interest rate at the time point $t + 2$.
- The expected returns on all securities in one period are equal, short rates are log-normally distributed with the volatility of their logarithms that depends only on time.
- The input is the yield curve and yield volatilities valid for zero-coupon government bonds at a given date; this input should be available for all maturities.

At each time point l, there are $l + 1$ possible stages and for the given horizon T there are 2^{T-1} equiprobable scenarios of short-term interest rates.

Each of them can be represented by a random binary fraction with $T-1$ 0–1 digits, say

$$\omega^s = 0 \cdot \omega_1^s \omega_2^s \ldots \omega_{T-1}^s$$

with $\omega_l^s = 0$ or $1\ \forall l,\ s$. The digit 1 at the lth position corresponds to the "up" move, the digit 0 corresponds to the "down" move of the one-period short-term interest rate in the step l. This theoretical binomial lattice has to be calibrated by the existing term structure to get the base rates r_{l0} and the volatility factors or lattice volatilities k_l for all l. The corresponding one-period short-term rates for scenario s and for the time interval $(l, l+1]$ are then

$$r_l^s = r_{l0} k_l^{i_l(s)}, \quad i_l(s) = \sum_{\tau=1}^{l} \omega_\tau^s. \tag{15}$$

That is, $i_l(s)$ equals the number of the up moves for the given scenario s which occur at time points $1, \ldots, l$ (see Rebonato, 1996 for discussions about characteristic properties of this model).

To *calibrate the Black–Derman–Toy model* means to use the yield and volatility curve related to yields to maturity of zero-coupon government bonds of all maturities corresponding to the chosen time steps of the lattice. Such bonds are rare in the market and have to be replaced by synthetic zero-coupon bonds whose yields correspond to yields of fixed coupon government bonds that do not contain any special provision such as call or put options.

Various numerical and statistical methods have been used to *fit or estimate the yield curve* from the existing market data on yields of fixed coupon government bonds at the given day. We have applied regression analysis to estimate and test the analytical form the yield curve. Instead of yields one could use the corresponding prices of these bonds as the input (see for instance Barone et al., 1991; Bliss, 1996), the discussion in Dupačová et al. (1996) or the approach recommended in Risk Metrics (1995). Regarding the assumption of homoskedasticity commonly present in regression models we decided to use yields (see the discussion in Vašíček and Fong, 1982).

Having tried different parametric nonlinear models as well as nonparametric ones, as reported in Dupačová et al. (1996) and Dupačová et al. (1997a,b), we chose to use a simple form of the yield curve applied by Bradley and Crane (1972)

$$g(t; \theta, \beta, \gamma) = \theta t^\beta \mathrm{e}^{\gamma t}. \tag{16}$$

We applied its linearized form to the logarithms of yields. For the market information consisting of yields $u_i,\ i = 1, \ldots, n$ of various fixed coupon government bonds (without option) characterized by their maturities t_i, the postulated model is

$$\lg u_i = \lg \theta + \beta \lg t_i + \gamma t_i + e_i, \quad i = 1, \ldots, n, \tag{17}$$

where the random errors $e_i,\ i = 1, \ldots, n$ are independent, normal $\mathcal{N}(0, \sigma^2)$.

The linearized model passed the nonparametric goodness-of-fit test of Eubank and Hart (1992) (see Dupačová et al., 1997a,b). There is a good reason to accept the hypothesis of approximately normal errors in (17) which is in line with the assumed log-normal process of short rates approximated by the Black–Derman–Toy binomial lattice.

The least-squares estimates $\lg \hat{\theta},\ \hat{\beta},\ \hat{\gamma}$ of parameters $\lg \theta,\ \beta,\ \gamma$ are approximately normal, with the mean values equal to the true parameter values and the covariance matrix

$$\sigma^2 \Sigma^{-1}, \quad \Sigma = \mathbf{G}^{\mathrm{T}} \mathbf{G},$$

where σ^2 is estimated by

$$s^2 = \frac{1}{n-3} \min_{\lg \theta, \beta, \gamma} \sum_{i=1}^{n} (\lg u_i - \lg \theta - \beta \lg t_i - \gamma t_i)^2. \tag{18}$$

To estimate the yields of zero-coupon bonds of all required maturities which are not directly observable, $\tilde{t} \neq t_i$, we replace the unobservable logarithm of yield by the corresponding value on the already estimated log-yield curve. Such estimates are subject to additional error.

Assume that the logarithm of the yield $\tilde{u} = u(\tilde{t})$ for maturity $\tilde{t}$

$$\lg \tilde{u} = \lg \theta^* + \beta^* \lg \tilde{t} + \gamma^* \tilde{t} + \tilde{e}$$

with $\tilde{e}$ normal, independent of e_i, $i = 1, \ldots, n$, $E\tilde{e} = 0$, $\operatorname{var}\tilde{e} = \sigma^2$ and with the true parameter values denoted by asterisks. Then $\lg \tilde{u}$ is approximately normal

$$\lg \tilde{u} - \lg \hat{\theta} - \hat{\beta} \lg \tilde{t} - \hat{\gamma}\tilde{t} \sim \mathcal{N}\Big(0, \sigma^2(1 + Q^2(\tilde{t}))\Big), \tag{19}$$

where

$$Q^2(t) = [1, \lg t, t]\Sigma^{-1}[1, \lg t, t]^{\mathrm{T}}. \tag{20}$$

The corresponding approximate $100(1-\alpha)\%$ confidence interval for the logarithm of yield $\lg \tilde{u}$ for a *fixed* maturity $\tilde{t} \neq t_i$, $i = 1, \ldots, n$ is

$$\lg g(\tilde{t}; \hat{\theta}, \hat{\beta}, \hat{\gamma}) \pm s\Big(1 + Q^2(\tilde{t})\Big)^{1/2} t_{n-p}(1 - \alpha/2)$$

and $t_{n-p}(1-\alpha/2)$ is the corresponding quantile of the t distribution with $n - p$ degrees of freedom.

The techniques for obtaining *volatilities of yields or log-yields* are less obvious. Most of the authors work with an ad hoc fixed constant volatility, say $V(t) = V$ (see e.g., the discussion in Hull and White, 1990). For a constant volatility, however, the model does not display the desirable mean reversion (see Rebonato, 1996).

The volatility curve can be estimated from the historical data (see e.g., Kahn, 1991). Risk metrics datasets provide *historical volatilities* for 14 major bond markets, including the Italian one; these volatilities are computed daily for several main maturities ranging from 1, 2, 3, 4, 5, 7, 9, 10, 15, 20 and 30 years. The proposal is to estimate the missing yields by linear interpolation and to use the volatilities and correlations of the reported yields to compute the approximate values of yield volatilities for these nonincluded maturities. There are not enough data for fitting the volatility curve by a regression model.

Another source of information is *implied volatilities* computed from quoted bond option prices (e.g., Kuberek, 1992). At a given day, this provides a set of annualized volatilities related to several different maturities. The next step is to get a volatility curve from these "observed" data. Evidently, the discussion concerning an appropriate parametric or nonparametric estimation procedure appears once more, including the plausible parametric form of the curve and the problem of a small number of available data. A suggestion is to regress the implied bond volatility on the lagged one obtained one period before (cf. Litterman et al., 1991).

In contrast to the volatility curves obtained independently on the yield curve model one could get *approximate* standard deviations of $\lg u(t)$ from the chosen parametric model of the yield curve provided that the errors in the applied regression model are normally distributed (cf. Dupačová et al., 1997a,b). For the linearized Bradley and Crane model (17) one can use directly the standard deviation which comes from (19).

For *calibration of the binomial lattice* in agreement with the (estimated) today's market term structure, both backward and forward inductions and approximate fittings have been tested (see e.g., Jamshidian, 1991; Kang and Zenios, 1992; Bjerksund and Stensland, 1996; Rebonato, 1996). The numerical results are reported in Abaffy et al., 1998.

All these steps lead to the *fitted binomial lattice* which provides different 2^{T-1} scenarios of interest rates. A smaller, manageable number of scenarios have to be *selected or sampled* from this large set (see Nielsen, 1997) for a procedure based on ideas of importance sampling and Zenios and Shtilman (1993) for a nonrandom sampling technique based on a uniform approximation of the expected utility of final wealth computed with respect to the uniform distribution over the *full* set of the 2^{T-1} scenarios of the lattice by an expected value over a subset of these scenarios.

A simplified version of the *deterministic sampling strategy* by Zenios and Shtilman (1993) can be described as follows. We fix L, $1 < L < T$ and assign the index s, $s = 1, \ldots, 2^L$ to each possible binary fraction of length L. The sample point ω^s from $(0, 1)$ is determined by one of these binary fractions and by an *arbitrary continuation* up to binary fraction of length $T - 1$.

We build S scenarios $\mathbf{r}^s$, which are identified by *scenario independent* base rates r_{t0}, $t = 1, \ldots, T-1$, volatilities k_t, $t = 1, \ldots, T-1$ and by the *scenario dependent* position on the lattice given by the exponent $i_t(s)$ equal to the number of up moves needed to reach the position on the lattice within t periods, see (15). We evaluate the prices ξ^s_{jt}, ζ^s_{jt} and cashflows f^s_{jt} along each of these scenarios and solve the problem (2)–(7). The main output is the optimal value – the maximal attainable expected utility of final wealth at time T_0 and an optimal first-stage solution, say $\mathbf{x}^*$, $\mathbf{y}^*$, $\mathbf{z}^*_0$, y_0^{+*}.

4. What–if analysis: the perturbed input

Assume that the models applied on the input side of the bond portfolio management problem have been fixed according to our past experience. In the context of the Black–Derman–Toy model of interest rates it means that the successfully tested linearized form of the Bradley and Crane (1972) yield curve (16) has been accepted to get the term structure. As one possibility, Zenios and Shtilman (1993) nonrandom sampling procedure has been used to get a modest number of scenarios out of the fitted binomial lattice. Even in this case there are numerous sources of errors that influence the input of the large-scale mathematical program (2)–(7):

- The market data of the given day are used to fit the yield curve, i.e., to estimate the coefficients in the chosen nonlinear regression model and to estimate the yields or prices of zero-coupon government bonds of all required maturities $t = 1, \dots, T$. In addition, a plausible hypothesis about volatility of these yields (i.e., about standard deviations of log-yields) is needed. The estimated prices or yields of zero-coupon government bonds of all maturities together with their volatilities are called the initial *term structure*. Evidently, both statistical and numerical errors enter the initial term structure.
- The Newton–Raphson method is used to fit the base rates and lattice volatilities of the Black–Derman–Toy model in accordance with the term structure. It requires the solution of a system of $2T$ nonlinear equations which can be done in several ways. Additional errors which stem from the chosen numerical procedure seem to be of minor importance than errors due to estimation of the yield curve and, namely, due to more or less ad hoc assessment of the volatility curve (see Abaffy et al., 1998) for numerical evidence.
- There is a certain indeterminacy as to the choice of representative scenarios. In case of the Zenios and Shtilman nonrandom sampling strategy, this includes the choice of the number $S = 2^L$ of scenarios and also the choice of the $L + 1$ and further components of the binary fractions which identify the selected scenarios.

The final task is a solution of the large mathematical program (2)–(7) whose coefficients are burdened by errors of various kinds. The question is the sensitivity of the optimal first-stage decision (the first-period trading strategy) and of the optimal value of the objective function on the abovementioned errors.

The form of the fitted interest rates allows us to separate the influence of the input data and of methods used for the lattice calibration from the impact of the chosen sampling procedure. Hence, we can concentrate now solely on an *analysis of errors in the estimated term structure and their effect on the results*. We recall only that the contamination technique explained briefly in Section 2.1 is a suitable method for bounding errors due to inclusion of additional scenarios and/or due to changes of parameter L in the Zenios and Shtilman (1993) sampling procedure.

4.1. Simulation studies

The results summarized in the context of estimating the yield curve by parametric regression (cf. (16)–(20)), provide a basis for simulation of log-yields at individual points t which are needed for fitting the binomial lattice provided that the *volatility curve is not subject to any perturbations*:

(i) At each point t of the discretization of the time horizon generate the random error e by sampling from the normal distribution $\mathcal{N}(0, \sigma^2(1 + Q^2(t)))$; the corresponding simulated log-yield at the given time instant t is $\lg u = \lg g(t; \hat{\theta}, \hat{\beta}, \hat{\gamma}) + e$. Let $\mathbf{e}$ be the vector of the independent normally distributed components e obtained in the described way.

(ii) For each vector of log-yields obtained according to (i) get the vector of simulated yields

u, fit the lattice and evaluate the interest rates r_t^s, prices P_{jt}^s, ξ_{jt}^s, ζ_{jt}^s and cashflows f_{jt}^s.

By a repeated solution of the scenario-based programs (2)–(7) for various sets of coefficients obtained by the simulation procedure (i) and (ii) one gets repeated "observations" of the optimal value and of the optimal initial trading strategy which allows to construct empirical distribution of the maximal expected utility of the final wealth, a useful information for subsequent, sample-based statistical inference, and to classify the considered bonds. See Section 5 for selected numerical results.

Using (i) and (ii), it is also possible to design a simpler procedure which aims only at properties of scenario subproblems.

The bond portfolio model is solved with prices and interest rates which come from the binomial lattice fitted in agreement with the estimated log-yield curve (17) and with a fixed volatility curve. The first-stage optimal solution is *kept fixed* in the subsequent steps and used together with simulated coefficients computed according to (i) and (ii) as an input for solution scenario subproblems (8)–(11). Repeated simulation runs according to (i) and (ii) and solution of the scenario subproblems provide empirical distributions of the optimal values of scenario subproblems.

Finally, it is possible to adapt the approaches suggested by Mak et al. (1997) to test the quality of the optimal first-stage solution based on the estimated yield curve through designing asymptotic confidence bounds on the optimal value. This simulation experiment proved to be rather demanding even in a parallel environment and its description is beyond the scope of this paper. We refer to Bertocchi et al. (1998) for the first numerical experience, related to the considered application within the Italian bond market.

The specific form of the linearized Bradley and Crane yield curve (17) is not essential for simulation experiments delineated above. They can be applied whenever there is a sound basis for assuming *random errors in the model input*; the examples are other regression models and/or other assumed distribution of errors and also random sampling procedures for selection of scenarios.

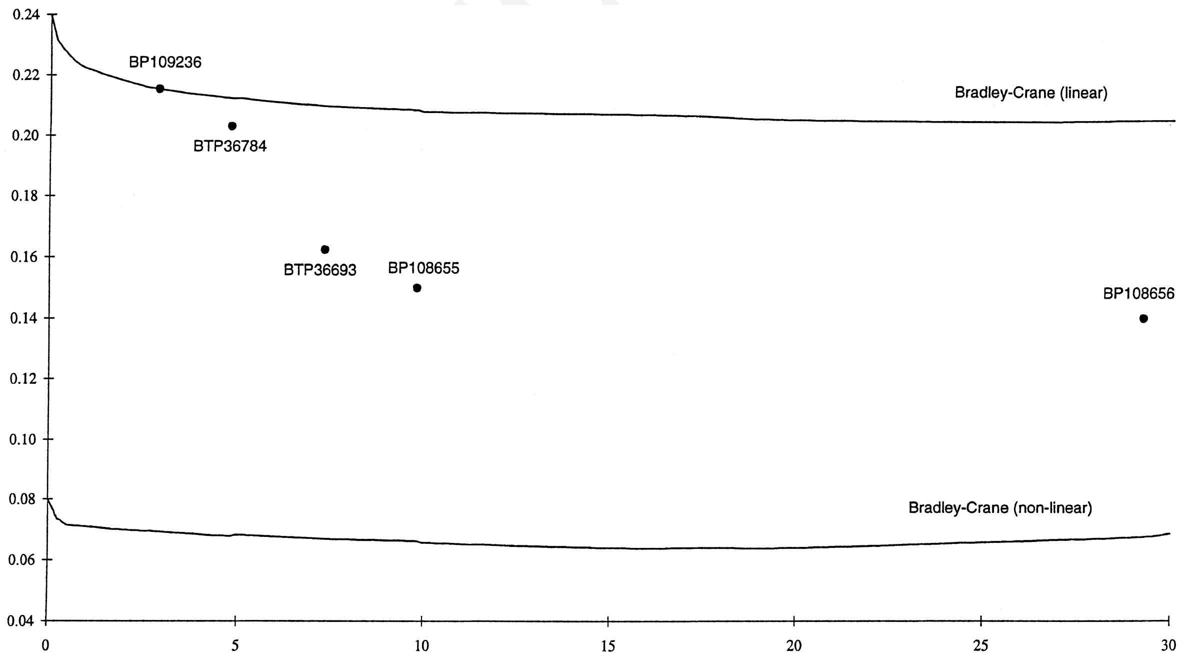


Fig. 1. Volatility curves and implied volatilities.

Table 1
Lattice parameters for various constant volatilities

Time	Volatility = 0.15		Volatility = 0.16		Volatility = 0.20	
	r_{l0}	k_l	r_{l0}	k_l	r_{l0}	k_l
0	0.036643	1.000000	0.036643	1.000000	0.036643	1.000000
1	0.031953	1.236311	0.031705	1.253919	0.030718	1.326896
2	0.028652	1.233381	0.028216	1.250765	0.026512	1.322819
3	0.025822	1.232824	0.025237	1.250190	0.022994	1.322232
4	0.023327	1.232797	0.022625	1.250205	0.019986	1.322514
5	0.021101	1.233010	0.020310	1.250493	0.017388	1.323258
6	0.019101	1.233384	0.018244	1.250972	0.015131	1.324361
7	0.017297	1.233894	0.016392	1.251616	0.013161	1.325797
8	0.015663	1.234531	0.014726	1.252418	0.011436	1.327568
9	0.014181	1.235296	0.013224	1.253377	0.009922	1.329687
10	0.012833	1.236190	0.011867	1.254498	0.008592	1.332174
11	0.011605	1.237218	0.010639	1.255788	0.007420	1.335053
12	0.010485	1.238386	0.009527	1.257256	0.006388	1.338355
13	0.009463	1.239701	0.008518	1.258910	0.005479	1.342115
14	0.008528	1.241171	0.007603	1.260764	0.004678	1.346375
15	0.007674	1.242804	0.006772	1.262829	0.003974	1.351179
16	0.006893	1.244612	0.006017	1.265119	0.003355	1.356584
17	0.006178	1.246604	0.005331	1.267650	0.002812	1.362651
18	0.005524	1.248793	0.004709	1.270440	0.002338	1.369454
19	0.004925	1.251192	0.004145	1.273509	0.001926	1.377078

4.2. Alternative volatility curves

At the present state of knowledge, the simulation technique delineated above is not suitable for sensitivity with respect to the volatility data. The results of different approaches provide quite disparate volatility curves; compare the volatility curves plotted in Fig. 1 which come from the linearized and nonlinear Bradley–Crane model used to estimate the yield curve for data of 17 April 1997 with values of implied volatilities related to the same day.

At the same time, the numerical evidence (cf. Abaffy et al., 1998), shows that the *assumed values of the log-yield volatilities influence essentially the rates along the fitted binomial lattice*, see Table 1 for results based on market data of 24 June 1996 and on constant volatilities and Table 2 and Fig. 2 for lattice parameters that have been computed for the volatility curve obtained from (19) and from exponential smoothing of the implied volatilities for 17 April 1997. Moreover, the mapping from the log-yield volatility curve values to the lattice parameters is approximately linear; see the last column of Table 2 which corresponds to the volatility curve equal to the average of the two initial volatility curves.

All computations were done in double precision with 15 significant decimal digits, the step of time discretization was fixed to six months. The numerical results reported in Tables 1 and 2 show that there is a great sensitivity of the base rates and lattice volatilities to the input volatility of log-yields. It emphasizes both the need for deep studies of volatility aspects and the necessity to analyze the progression of errors in input volatilities through the fitting procedure up to the final results.

Assume that the sampling strategy and the applied values of yields are kept fixed and *two alternative volatility curves* are taken into account. Let $\mathbf{V}_1, \mathbf{V}_2$ be the two corresponding vectors of their function values computed for all points t of the considered discretization of the time horizon T. Each of these vectors provides S scenarios of interest rates, say, $\mathbf{r}^s(\mathbf{V}_1)$, $\mathbf{r}^s(\mathbf{V}_2)$, $s = 1, \ldots, S$; the two scenario bets are different. The stochastic program (2)–(7) can be solved for each volatility curve separately; denote the optimal values $\varphi(\mathbf{V}_1)$, $\varphi(\mathbf{V}_2)$ and the optimal first-stage solution

Table 2
Lattice parameters for various volatility curves

Time	Volatility of $\lg y$		Implied volatility		Averaged volatility	
	r_{l0}	k_l	r_{l0}	k_l	r_{l0}	k_l
0	0.028754	1.000000	0.028754	1.000000	0.028754	1.000000
1	0.024023	1.365638	0.024387	1.330074	0.024205	1.347740
2	0.020588	1.353971	0.021176	1.320724	0.020880	1.337243
3	0.017732	1.349114	0.018532	1.314332	0.018129	1.331608
4	0.015313	1.346372	0.016318	1.308736	0.015809	1.327408
5	0.013245	1.344693	0.014444	1.303537	0.013833	1.323963
6	0.011465	1.343746	0.012845	1.298726	0.012138	1.321016
7	0.009927	1.343344	0.011471	1.294151	0.010675	1.318466
8	0.008593	1.343440	0.010283	1.289817	0.009405	1.316285
9	0.007433	1.343959	0.009252	1.285704	0.008300	1.314389
10	0.006421	1.344915	0.008351	1.281795	0.007331	1.312808
11	0.005538	1.346266	0.007561	1.278077	0.006481	1.311510
12	0.004766	1.348054	0.006864	1.274538	0.005733	1.310441
13	0.003498	1.350286	0.006249	1.271168	0.005071	1.309706
14	0.002980	1.352931	0.005702	1.267956	0.004486	1.309162
15	0.002527	1.356094	0.005215	1.264894	0.003965	1.308937
16	0.002132	1.359770	0.004779	1.261972	0.003502	1.308984
17	0.001786	1.363915	0.004389	1.259183	0.003097	1.309206
18	0.001487	1.368718	0.004038	1.256519	0.002722	1.309810
19	0.004931	1.374090	0.003721	1.253973	0.002395	1.310579

for the initial choice of the volatility curve as $\mathbf{x}^*(\mathbf{V}_1),\ \mathbf{y}^*(\mathbf{V}_1),\ \mathbf{z}_0^*(\mathbf{V}_1),\ y_0^{+*}(\mathbf{V}_1)$. The influence of the passage from the already applied volatility curve $\mathbf{V}_1$ to the alternative volatility curve $\mathbf{V}_2$ can be studied by the *contamination technique*; we refer to Section 2.1 and to earlier theoretical papers (e.g., Dupačová, 1996), and financial applications (see Dupačová et al., 1997a,b, 1998) for a detailed explanation of the method. The directional derivative of the optimal value function computed for scenarios $\mathbf{r}^s(\mathbf{V}_1),\ s = 1,\dots,S$ in the direction of $\mathbf{r}^s(\mathbf{V}_2) - \mathbf{r}^s(\mathbf{V}_1)$ equals

$$\varphi'(0^+) = \sum_s p_s U\big(W_{T_0}\big(\mathbf{r}^s(\mathbf{V}_2); \mathbf{x}^*(\mathbf{V}_1), \mathbf{y}^*(\mathbf{V}_1), \mathbf{z}_0^*(\mathbf{V}_1), y_0^{+*}(\mathbf{V}_1)\big)\big) - \varphi(\mathbf{V}_1), \tag{21}$$

if the optimal solutions are unique; in the opposite case (21) is a lower bound for the directional derivative.

The change of input in a specified direction can be also interpreted as a passage to stochastic program (2)–(7) based on a convex mixture of scenarios coming from the two scenario beds with coefficients $1-\lambda$ and λ, or equivalently, as a passage to stochastic program (2), (3) and (12) based on the pooled sample of $2S$ scenarios $\mathbf{r}^s(\mathbf{V}_1),\ \mathbf{r}^s(\mathbf{V}_2),\ s = 1,\dots,S$ with probabilities $(1-\lambda)p_s,\ \lambda p_s,\ \lambda \in (0,1)$. The objective function (12) depends linearly on the parameter λ, its optimal value $\varphi(\lambda)$ is convex in λ and differentiable at $\lambda = 0^+$. Hence the bounds

$$(1-\lambda)\varphi(\mathbf{V}_1) + \lambda\varphi(\mathbf{V}_2) \geqslant \varphi(\lambda) \geqslant \varphi(\mathbf{V}_1) + \lambda\varphi'(0^+) \quad \forall \lambda \in [0,1],$$

which quantify the sensitivity of the optimal value on changes in the initial volatility curve. For sensitivity purposes, one chooses small values of λ whereas post-optimality with respect to scenarios $\mathbf{r}^s(\mathbf{V}_2),\ s = 1,\dots,S$ based on another volatility curve might correspond to $\lambda = 0.5$.

4.3. Refinement of scenarios

Given a scenario s, the fair prices P^s_{jt} depend on random and nonrandom errors of the input obtained from market data and on other errors due to implementation and essence of the applied nu-

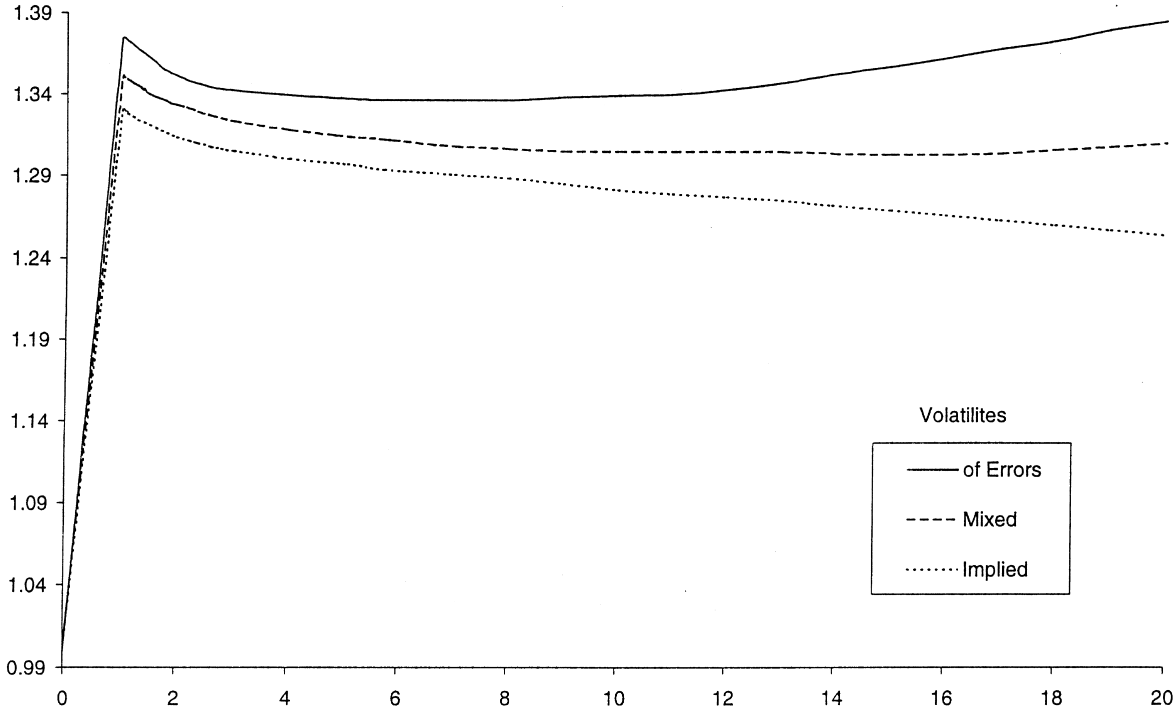


Fig. 2. Lattice volatilities.

merical methods. We deal with the errors in the input, hence, we assume that $P^s_{jt} = P^s_{jt}(\mathbf{e}, \mathbf{V})$ depend on the random errors $\mathbf{e}$ in log-yields and on the volatilities. By solving the model for one choice of these parameters we omit other possibilities. To hedge against a more rich set of possible evolutions of the interest rates we can design scenarios of the input parameters $\mathbf{e}$ coming from normal distribution and $\mathbf{V}$ belonging for each t to an interval by a suitable discretization. There are many sophisticated ways to do it; we use a discretization that has Q scenarios $(\mathbf{e}^q, \mathbf{V}^q)$, $q = 1, \ldots, Q$ of inputs for fitting the binomial lattice. Hence there are Q couples of base rates and lattice volatilities. In conjunction with suitably assigned weights or probabilities $\pi_q > 0$, $\sum_q \pi_q = 1$ of these scenarios and with the presumably *fixed* sampling procedure we have thus arrived at QS scenarios $\mathbf{r}^{qs}$ of the considered evolution of interest rates with components $r^{qs}_t = r^q_{t0} k_t^{q,i_t(s)}$ (compare with (15)) for $t = 1, \ldots, T-1$ and with probabilities $\pi_q p_s$. For each of them, scenario subproblem of the type (8)–(11) provides the maximal value $W_{T_0}(\mathbf{r}^{qs}; \mathbf{x}, \mathbf{y}, \mathbf{z}_0, y_0^+)$ and the full program (2)–(7) reads

$$\text{maximize} \quad \sum_{q=1}^{Q} \sum_{s=1}^{S} \pi_q p_s U(W_{T_0}(\mathbf{r}^{qs}; \mathbf{x}, \mathbf{y}, \mathbf{z}_0, y_0^+))$$

subject to nonnegativity constraints and subject to (2) and (3).

The main stumbling block of this procedure is to obtain an adequate and numerically manageable scenario representation of the underlying stochastic program.

5. Selected numerical results

This section provides selected numerical results related with the simulation experiment (i) and (ii) described in Section 4.1.

To simulate the behavior of a value preserving portfolio of fixed income securities on the Italian bond market we use the model described by (2)–(7) with the linear utility function and within the time horizon of one year ($T_0 = 12$).

The sample of bonds and all the information come from a local bank. Similarly as in Bertocchi et al. (1996b) and Dupačová et al. (1997a,b, 1998), the portfolio is composed of cash (500 mil. Liras) and of typical government bonds, paying semi-annual coupons and covering two year forward till 29 year maturities (the so-called BTPs) as well as puttable bonds (CTOs), paying semi-annual coupons with the maturity of 8 years and a possible exercise of the option in the fourth year or with the maturity of 6 years and an exercise at the third year (see Table 3). The quantities (Qt) of bonds included in the initial portfolio are expressed in lots of million items so that the nominal value of the portfolio is 10,500 mil. Liras. The portfolio and the initial term structure are related to 1 September 1994, the coupon yields and the redemption prices are after tax. The market value of the portfolio was 10,466 mil. Liras.

To estimate the term structure of interest rates we used the linearized model (17) of Bradley and Crane (1972) applied to the yields obtained by the market quotation of the BTPs on the relevant day. Volatilities of log-yields were set equal to the standard deviations of the normal distribution in (19) (see Dupačová et al., 1997a,b) for a detailed discussion.

In this application, liabilities are not considered, liquidity can be obtained from the interbank market at a rate greater than that one at which surplus can be always reinvested. The additive transaction costs are fixed at ± 0.01, $\delta_1 = 0.0005$ and $\delta_2 = 0.0016$.

Various beds of scenarios were considered. All of them were based on the data from the Italian bond market and were sampled from the fitted Black–Derman–Toy binomial lattice. Among others, they included scenario beds selected according to the simplified version of the nonrandom sampling strategy by Zenios and Shtilman (1993) with $L = 3$, 4, 5, 6, as described in Dupačová and Bertocchi (1996), Bertocchi et al. (1996a,b) and Bertocchi et al. (1998).

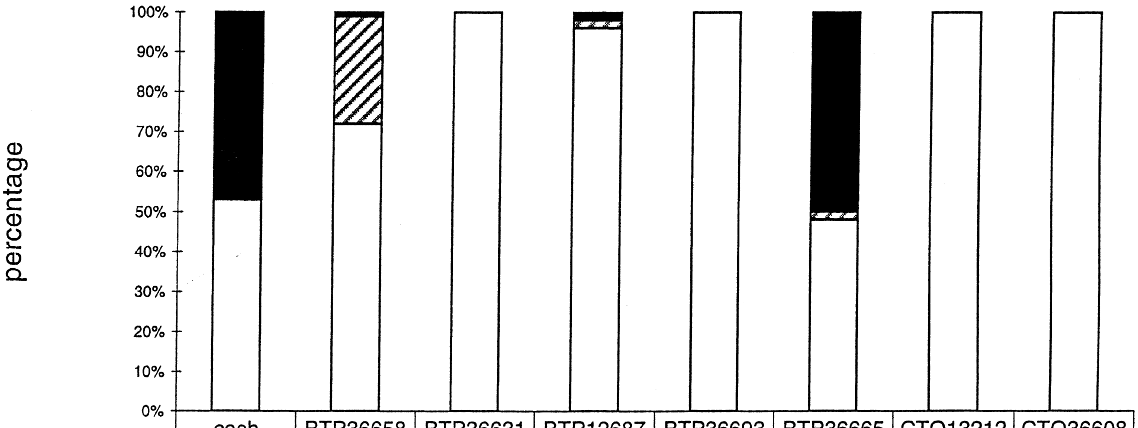
The parameters of the binomial lattice were computed using the forward Bjerksund and Stensland (1996) procedure with a monthly discretization along 5 years. After 5 years the interest rates have been kept fixed. This produces scenarios that can be coded by 2^{60} binary fractions uniformly distributed in $[0, 1]$.

One characteristic which is common to all experiments is that all perturbations to the initial yield curve are maintained in a small range. More precisely, the error vector $\mathbf{e}$ is constructed to satisfy the property that its components belong to a normal distribution with zero mean and standard deviation equal to $h10^{-2}\sigma[1 + Q^2(t)]^{1/2}$, where h is in the range $(0, 1]$. This was necessary to guarantee that the numerical method used to fit the Black–Derman–Toy lattice gave reasonable interest rates when applied to the perturbed yield curves. This observation quantifies partly the meaning of "small perturbations" of the data.

We shall report only one of the many simulation experiments related with the first part of the simulation study suggested in Section 4.1. The number of simulations of vectors $\mathbf{e}$ has been fixed to 100 and a particular bed of eight scenarios was used to represent the random evolution of interest

Table 3
Initial portfolio composition

Bonds	Qt	Coupon	Payment dates	Exercise	Redemp.	Maturity
BTP36658	10	3.9375	1 April and 1 October		100.187	1 October 1996
BTP36631	20	5.0312	1 March and 1 September		99.531	1 March 1998
BTP12687	15	5.2500	1 January and 1 July		99.231	1 January 02
BTP36693	10	3.7187	1 August and 1 February		99.387	1 August 04
BTP36665	5	3.9375	1 May and 1 November		99.218	1 November 23
CTO13212	20	5.2500	20 January and 20 July	20 January 1995	100.000	20 January 1998
CTO36608	20	5.2500	19 May and 19 November	19 May 1995	99.950	19 May 1998



	cash	BTP36658	BTP36631	BTP12687	BTP36693	BTP36665	CTO13212	CTO36608
■ Buy	47	1	0	2	0	50	0	0
▨ Do Nothing	0	27	0	2	0	2	0	0
□ Sell	53	72	100	96	100	48	100	100

Fig. 3. Investment strategies.

rates. The choice of scenarios aimed at a representative coverage of the lattice up to the planning horizon T_0 and then to proceed with alternating up–down movements. For results of the complete simulation study and for extensions (see Bertocchi et al., 1998). The obtained unperturbed optimal value of the stochastic program was 11,499, the optimal first-stage solution kept BTP12687 and sold all other bonds to use the obtained cash plus the initial one for purchasing the long bond BTP36665. This is a consequence of a comparably low initial market price of this long bond.

The simulation experiment provides a survey on how the considered bonds are distributed with respect to the strategies of selling, holding and buying for slightly perturbed yield curves (see Fig. 3). Similarly as for the unperturbed input, the long bond is dominant for the most of the perturbed cases.

The average value of the optimal values φ^k, $k = 1, \ldots, 100$, for the perturbed data was 13,041 with the standard deviation of 2479, median 11,591, minimal value 11,462 and maximal value 22,850. The average of expected values of the Buy and Hold strategy, computed again for the same 100 simulations of the particular choice of eight scenarios, was 11,389 with the standard deviation of 265, median 11,389, minimal value 10,867 and maximal value 12,230. Comparing the two descriptive statistics results, it is clear that the empirical distribution of the stochastic programming optimal values is shifted to the higher function values, and is nonsymmetrical and provides possibilities of rather large values. More details can be found in Bertocchi et al. (1998).

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