European Journal of Operational Research 52 (1991) 28–44
North-Holland

Theory and Methodology

Stochastic programming in water management: A case study and a comparison of solution techniques *

Jitka Dupačová
Department of Mathematical Statistics, Charles University, Sokolovská 83, 18600 Prague, Czechoslovakia

Alexei Gaivoronski
V. Glushkov Institute of Cybernetics, Kiev 207, SU-252207, USSR

Zdeněk Kos
Faculty of Engineering, Technical University, Thakurova 7, 16629 Prague, Czechoslovakia

Tamás Szántai
Department of Operations Research, Eötvös Loránd University, Muzeum Krt. 6-8, H-1088 Budapest, Hungary

Abstract: The present stage of developments in stochastic programming gives already a good base for real-life applications. The possibility of using alternative models is studied on a small-size but meaningful example connected with water management of a real-life water resource system in Eastern Czechoslovakia. Both of the considered conceptually different stochastic programming models take into account intercorrelations within a group of random parameters and provide comparable optimal decisions. At the same time, these models are used for comparison of existing numerical procedures for stochastic programming, namely, approximation schemes that result in large-size linear programs, stochastic quasigradient methods and special techniques for handling joint chance constraints.

Keywords: Water management, probabilistic programming, stochastic quasigradient methods, computational analysis, stochastic modeling

* This paper is a result of the joint research activity of the authors connected with their stay at the International Institute for Applied Systems Analysis (IIASA), A-2361 Laxenburg in 1985–86. The first version of the paper appeared as a Working Paper of this institute (see [4]).

Received March 1989; revised August 1989

1. Introduction

Stochastic programming deals with situations where some random parameters appear in a formulation of a mathematical program. This is a relatively frequent case if the mathematical program stems from a real-life problem and the examples of random parameters are (random) demands, (random) inflows, (random) yields, etc. Decision models of stochastic programming have been designed to treat the cases when a decision has to be chosen before a realization of random parameters can be observed.

There is a rich choice of decision models of stochastic programming (see e.g. [9,23]). One can consider different objective functions and model the same problem using chance constraints or penalizing the constraints violation or a combination of the two. Very often there is no convincing reason to prefer one approach over another. The model choice is often influenced by the decision maker's individual attitude and knowledge, by the data structure and by the software available. To use the penalty approach means to assign costs to discrepancies, which may be a very difficult problem in practice. That is why decision makers often prefer to assign probability thresholds and to use chance constraints. However, chance constraints do not distinguish the magnitudes of the discrepancies and the joint chance constraints seem to be the most complicated choice from the point of view both of theory and of software. We feel that, especially in costly projects, the decisions should not be made relying on an ad hoc chosen or favorite models and we suggest to use multimodeling.

The progress in the field of numerical solution techniques for stochastic programming problems, see [7], has already resulted in special packages such as the ADO/SDS collection of Stochastic Programming Codes available at the International Institute for Applied Systems Analysis (IIASA), Laxenburg, Austria (see [5], [7, Part III]), and made it possible to solve successfully nontrivial stochastic programs. We shall use [7] as the source of reference whenever possible.

We have chosen a real-life application of stochastic programming to water resources and management; see Section 2 for the description of the problem and Section 6 for the results. The state-of-the-art of mathematical models including stochastic programming that were developed for reservoir management is contained in the review [24]. Among others, the use of individual chance constraints has a long tradition in this problem area; see e.g. [17]. The joint chance constraints were treated in a series of papers by Prékopa et al., see e.g. the collection [15]. The use of the stochastic programming penalty or recourse models is not that frequent, see e.g. [2]. In Czechoslovakia, where the analyzed system is located, applications of stochastic programming models with chance constraints were described in [3] and [10] in conjunction with the linear decision rule or with the direct control.

In this paper we take into account intercorrelations between the successive values of water requirements for irrigation purposes; this cannot be achieved via individual chance constraints. It is the theory of log-concave measures developed by Prékopa [14] which gives the theoretical background for handling the joint chance constraints. Their use, however, brings along limitations as to the dimension of the corresponding vector of random parameters and to the type of its distribution. Therefore we have also formulated an alternative model of penalty type (see Section 3). Both types of models are compared on the considered aggregated problem whose size is sufficient for the early screening stage of the decision process and, at the same time, gives a realistic possibility to multimodeling and to subsequent comparison of different numerical procedures (see Sections 4 and 5). Among other solution techniques, the stochastic quasigradient method is introduced and its basic features are explained and illustrated.

2. Description of the water resources management problem

The problem considered here is a part of a multiobjective large-scale system that arises primarily from decision making on the governmental level. In this paper, we suppose that the subsystems having their own independent objective have been coordinated by means of resource allocation so as to maximize the performance of the total system. Out of many possible goals, the water supply for industry and irrigation, flood control and recreation purposes are considered in this study. The most important decision variable is the storage capacity of reservoirs. In the corresponding mathematical model we use a single objective function that is assumed to be an additive sum of individual objective commensurable functions. The issues that are analyzed in this paper can be related to the multipurpose water resources sys-

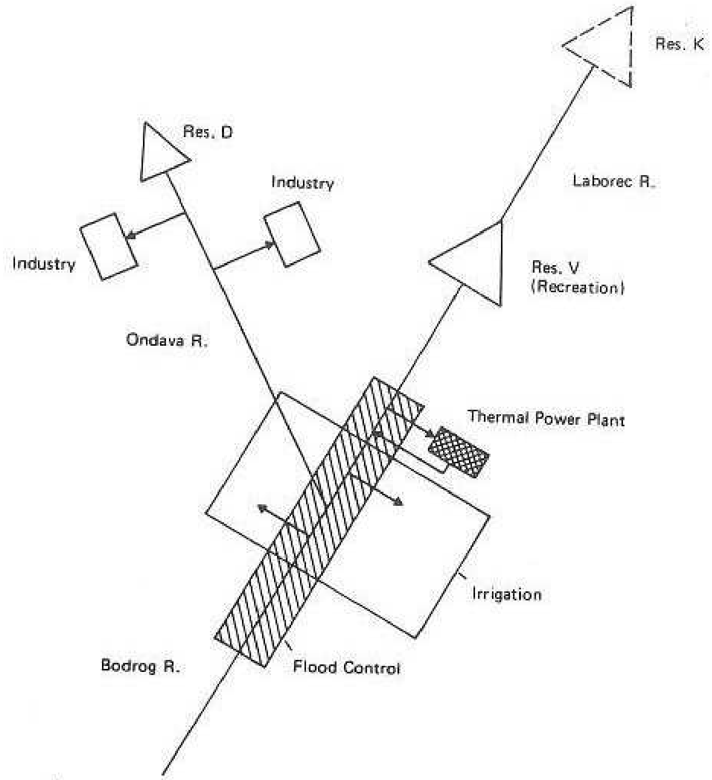

Figure 1. Schematic representation of the water resources system of the Bodrog river

tems with the objective to design preliminarily the system with minimum costs (screening). These costs are determined either by the capacity of the designed reservoirs (the chance-constrained model) or by the overall costs that include also the losses due to failure of some reservoir functions (the penalty models).

As an example of this analysis, the water resource system of the Bodrog River in Eastern Czechoslovakia was chosen.

The water resources system consists of three reservoirs, V, D, K, two of which are in operation (D, V) and the third one is to be built or not (K), see Figure 1. The main purposes of the water resources system are the irrigation water supply, industrial uses — mainly water withdrawal for the thermal power station, flood control (better flood alleviation), environmental conservation and recreation. An accelerated development of the recreation and growing irrigation requirements poses the main questions for the decision makers:

– Can the existing water resources system still meet all the requirements? And if so, with which reliability?

– Is the construction of reservoir K necessary and when will it be necessary?

The analysis of this problem was divided into two steps. The first one comprises the screening modeling and is discussed in this paper. Optimization models, however, cannot reflect all the details of the water resources system operation. Therefore the results of the stochastic optimization model are supposed to be verified using the stochastic simulation model with the input generated by the methods of stochastic hydrology [11]. For the stochastic programming screening model, which is the subject of the present paper, an aggregated model was used and the monthly flows and the irrigation water requirements were aggregated into four periods:

(1) November till April of the following year,
(2) May and June,
(3) July and August,
(4) September and October.

The first period starts at the beginning of the hydrological year and comprises the winter and the spring periods filling the reservoirs. The second and fourth periods include irrigation and industrial demands, the third period includes in addition the recreation demands. The requirements for the minimum pool due to environmental control and enhancement and flood control pertain to all the periods.

Following [19] we use as criterion the net present value of the project. To simplify the economic considerations we assume that the optimal water allocation has been performed in advance so that restrictions caused by limited supply of water can be taken into account by minimization of the project costs and losses due to reduced level in water supply or water service. As the decisive factor in determination of the project value is the reservoir capacity the objective function is briefly described as the cost of reservoir of the considered capacity.

3. The mathematical models

Three types of models are presented: (i) minimizing reservoir capacity subject to joint and individual chance constraints and simple deterministic bounds, (ii) minimizing the overall expected costs subject to simple deterministic bounds and (iii) minimizing the expected cost that equals the cost of reservoir increased with the expected penalty for unsaturated needs of irrigation water subject to individual chance constraints and simple deterministic bounds.

3.1. The chance-constrained model

Using *chance constraints*, the result identifies the capacity x_0 of reservoir V that should meet the needs with a prescribed reliability. The task is formulated as the cost of reservoir V minimization. As the cost is an increasing function of reservoir capacity x_0, we can evidently minimize the capacity x_0 instead of minimizing the cost.

The first type of constraints is used for sustaining the total release x_i in excess to the water supply need $\beta_i + d_i$ in periods 2, 3 and 4 (vegetation periods). The needs consist of the fixed demand d_i (minimum flow and industrial water needs) and random demand β_i (irrigation water requirements). As to the first period, the fixed demand d_1 (caused mainly by the needs of the thermal power station) should be met with such a high probability that the deterministic constraint $x_1 \geqslant d_1$ was used. Taking into account the intercorrelations of random demands β_i, $i = 2, 3, 4$, the constraint for the vegetation period was formulated as the *joint* chance constraint

$$P\{x_i \geqslant \beta_i + d_i,\ i = 2, 3, 4\} \geqslant \alpha, \tag{1}$$

where α is the required joint probability level.

The second type of constraints involves the minimum storage and freeboard constraints in their classical form of individual chance constraints:

$$P\{s_i \geqslant m_i\} \geqslant \alpha_i, \quad i = 1, \ldots, 4, \tag{2}$$

$$P\{s_i + v_i \leqslant x_0\} \geqslant \gamma_i, \quad i = 1, \ldots, 4, \tag{3}$$

where s_i is the reservoir storage, m_i is the prescribed minimum storage, v_i is the prescribed minimum freeboard volume, and α_i, γ_i are the chosen probability levels in the i-th period.

The resulting optimization problem has the form

minimize x_0

subject to

$$\begin{aligned} &P\{x_i \geqslant \beta_i + d_i,\ i = 2, 3, 4\} \geqslant \alpha, \\ &P\{s_i \geqslant m_i\} \geqslant \alpha_i, \quad i = 1, \ldots, 4, \\ &P\{s_i + v_i \leqslant x_0\} \geqslant \gamma_i, \quad i = 1, \ldots, 4, \end{aligned} \tag{4}$$

and subject to additional constraints

$$\begin{aligned} &d_1 \leqslant x_1 \leqslant u_1, \\ &l_0 \leqslant x_0 \leqslant u_0, \\ &0 \leqslant x_i \leqslant u_i, \quad i = 2, 3, 4, \end{aligned} \tag{5}$$

which stem mostly from natural hydrological and morphological situation.

We shall use the direct (zero-order) decision rule so that the decision variables will be x_0 — the reservoir capacity — and the planned releases x_i, $i = 1, 2, 3, 4$. (Notice that the use of the chance constraints does not exclude the possibility that these planned releases cannot be implemented, e.g., due to an actual shortage of water, and the problem of actual releases from an already existing reservoir is subsequently solved via a less aggregated reservoir operation model.) Neglecting losses due to evaporation, the reservoir storages s_i can be expressed via the water inflows and releases in the relevant periods. Let r_j denote the water inflow in the j-th period and let ζ_i denote the cumulated water inflow,

$$\zeta_i = \sum_{j=1}^{i} r_j, \quad i = 1, \ldots, 4.$$

Denote further by s_0 the initial reservoir storage at the beginning of the hydrological year. As a rule, we can set $s_0 = m_4$, i.e., the reservoir storage is supposed to be at its minimum after the vegetation period. (This assumption is realistic for relatively small reservoirs without carry-over.) Repeated use of the continuity equation gives

$$s_i = m_4 + \zeta_i - \sum_{j=1}^{i} x_i, \quad i = 1, \ldots, 4. \tag{6}$$

Substituting into (2) and (3) and using the corresponding 100% quantiles $z_k(p)$ of the distribution of the random variables ζ_k, $k = 1, 2, 3, 4$, we can rewrite the individual chance constraints (2) and (3) in the form

$$\sum_{i=1}^{k} x_i \leqslant z_k(1 - \alpha_k) + m_4 - m_k, \quad k = 1, \ldots, 4, \tag{7}$$

$$\sum_{i=0}^{k} x_i \geqslant z_k(\gamma_k) + m_4 + v_k, \quad k = 1, \ldots, 4. \tag{8}$$

Unfortunately, neither this simple device nor the well-known linear decision rule (cf. [17]) apply to the joint chance constraint (1).

For solving the resulting optimization problem,

$$\text{minimize} \quad x_0 \tag{9a}$$

subject to

$$P\{x_i \geqslant \beta_i + d_i,\ i = 2, 3, 4\} \geqslant \alpha \tag{9b}$$

and subject to linear constraints (5), (7), (8), special techniques have been developed, see Chapter 5 of [7] and Section 4.

3.2. The penalty models

Alternatively, stochastic programming decision models can be built solely on the evaluation and minimization of the overall expected costs, which contain not only the cost of the reservoir of the capacity x_0 but also the losses connected with the fact that the needs were not fulfilled and/or the requirements on the minimum reservoir storage and on the flood control storage were not met. This type of models is called mostly *stochastic program with penalties* and it includes also the *stochastic programs with recourse* as a special case.

Suppose first that the constraints on water supply, minimum water storage and flood control storage do not contain random variables, i.e., we have (besides of the upper and lower bounds (5))

$$x_i \geqslant \beta_i + d_i, \quad i = 2, 3, 4, \tag{10a}$$

$$\sum_{i=1}^{k} x_i \leqslant \zeta_k + m_4 - m_k, \quad k = 1, \dots, 4, \tag{10b}$$

$$\sum_{i=0}^{k} x_i \geqslant \zeta_k + m_4 + v_k, \quad k = 1, \dots, 4. \tag{10c}$$

In case of random β_i, $i = 2, 3, 4$, and ζ_k, $k = 1, \dots, 4$, the chosen decision $x_0, x_1, \dots, x_4$ need not fulfill constraints (10) for the actual (observed) values of β_i, ζ_k. If this is the case, costs evaluating losses on crops (not being irrigated on a sufficiently high level), on the decrease of recreation (due to the lower reservoir pool) and the economic losses due to flood, etc. can be attached to the discrepancies.

Let $c(x_0)$ denote the cost of the reservoir of the capacity x_0, let the penalty functions be of the type

$$\phi(y) = 0 \quad \text{if } y \leqslant 0$$
$$\geqslant 0 \text{ and nondecreasing if} \quad y > 0. \tag{11}$$

Denote by ϕ_i^1, ϕ_k^2, ϕ_k^3 the penalty functions corresponding to the considered three types of constraints in (10), $i = 2, 3, 4$, $k = 1, \dots, 4$. We try to find a decision for which the total expected cost will be minimal subject to inequality constraints (5):

minimize

$$c(x_0) + E\left\{ \sum_{i=2}^{4} \phi_i^1(\beta_i + d_i - x_i) + \sum_{k=1}^{4} \phi_k^2\left(\sum_{i=1}^{k} x_i - \zeta_k - m_4 + m_k \right) + \sum_{k=1}^{4} \phi_k^3\left(\zeta_k + m_4 + v_k - \sum_{i=0}^{k} x_i \right) \right\} \tag{12}$$

subject to the constraints (5).

The expectation in (12) is taken with respect to random variables β_i and ζ_k. The choice of the penalty functions should be based on a deep economic and environmental analysis of the underlying problem. On the other hand, for a screening study, it seems satisfactory to restrict the choice to piece-wise linear or piece-wise quadratic penalty functions (see e.g. [18]).

In the simplest case (*simple recourse model*) all penalty functions ϕ are of the form

$$\phi(y) = qy^+, \quad \text{where } q \geqslant 0 \text{ and } y^+ = \max(0, y),$$

where the coefficients q have to be given by the decision maker. As a result, we have to

minimize

$$c(x_0) + E\left\{ \sum_{i=2}^{4} q_i^{(1)}(\beta_i + d_i - x_i)^+ + \sum_{k=1}^{4} q_k^{(2)}\left(\sum_{i=1}^{k} x_i + m_k - m_4 - \zeta_k \right)^+ + \sum_{k=1}^{4} q_k^{(3)}\left(\zeta_k + m_4 + v_k - \sum_{i=0}^{k} x_i \right)^+ \right\} \tag{13}$$

subject to the constraints (5).

The solution method can be based on approximation of the marginal distributions of β_i, ζ_k by discrete ones and, in case of $c(x_0)$ linear or convex piece-wise linear, the resulting program can be solved by simplex method with upper-bounded variables (see Chapter 14 of [7]).

The use of the stochastic program (12) with separable penalties does not require the knowledge of the joint distribution of the random needs for irrigation water. It means that due to the assumed separability of the penalty functions (i.e., due to the fact that shortage in irrigation water is penalized in each of the three vegetation periods separately and the total penalty is taken as the sum over the three periods), no intercorrelations are considered. Alternatively, we can attach a penalty cost to the situation, when the total needs for water irrigation in the vegetation period as a whole were not met. In that case, we can take e.g.

$$\tilde{\phi}(y) = \phi\left(\max_i y_i\right),$$

with ϕ of the form (11), and minimize

$$c(x_0) + E\Big\{\phi\Big(\max_{i=2,3,4}(\beta_i + d_i - x_i)^+\Big) + \sum_{k=1}^{4} \phi_k^2\left(\sum_{i=1}^{k} x_i - \zeta_k - m_4 + m_k\right) + \sum_{k=1}^{4} \phi_k^3\left(\zeta_k + m_4 + v_k - \sum_{i=0}^{k} x_i\right)\Big\} \tag{14}$$

subject to the constraints (5), which fits in better with our aim to reflect the interdependencies of the needs in the individual vegetation periods.

3.3. The mixed model

Finally, it is possible to *combine the chance constraints and the penalization*: one can define the set of admissible decisions e.g. by means of individual chance constraints (2), (3) and inequalities (5) and, at the same time penalize the occurrence of the discrepancies between water supply needs and releases by corresponding penalty terms in the objective function. Instead of the joint chance constraint (1) on the water supply needs, one penalty term of the form

$$\tilde{\phi}(\beta_i + d_i - x_i,\ i = 2, 3, 4) = c\Big\{\max_{i=2,3,4}(\beta_i + d_i - x_i)^+\Big\} = c\max\Big\{0, \max_{i=2,3,4}(\beta_i + d_i - x_i)\Big\}$$

is used. The resulting problem

$$\text{minimize} \quad x_0 + cE\Big\{\max_{i=2,3,4}(\beta_i + d_i - x_i)^+\Big\} \tag{15}$$

subject to the constraints (5), (7), (8), can be solved by the stochastic quasigradient method ([6] and Chapter 6 of [7]) which is also applicable to the problems (12)–(14) or by techniques designed for the solution of the complete recourse problem via discrete approximation of the joint distribution of β_2, β_3, β_4; see Section 4. For the optimal solution of (15), the values of the joint probability

$$p(x) = P\{x_i \geqslant \beta_i + d_i,\ i = 2, 3, 4\} \tag{16}$$

can be computed ex post.

Observe that in (15), the cost $c(x_0)$ of the reservoir of capacity x_0 is supposed to be linear in the interval $l_0 \leqslant x_0 \leqslant u_0$. The penalty coefficient c should have a meaningful economic interpretation. Two situations can be considered in this respect. Firstly, it may be possible to evaluate the unit losses due to the water shortage relative to the cost per unit capacity of the reservoir and in this case penalty c represents these losses. In the second situation, the decision maker may be mostly interested in maintaining the desired level of reliability in (1). Then we can consider the expectation model (15) as a tool for getting a solution of the original model (9) if such a solution exists and, consequently, to proceed with solving (15) for increasing values of c until the joint chance constraint (1) is fulfilled. Of course, the problems (9) and (15) are not equivalent, but it can be proved under reasonable assumptions that the joint probability

$$p(x) = P(x_i \geqslant \beta_i + d_i,\ i = 2, 3, 4)$$

evaluated at the unconstrained minimizers of

$$x_0 + cE\Big\{\max_{i=2,3,4}(\beta_i + d_i - x_i)\Big\}$$

tends to 1 as $c \to \infty$. In the presence of the constraints (5), (7), (8) we have to experiment with values of c to obtain the penalty level c^* which guarantees (1). This penalty can be interpreted as the *relative price for maintaining the desired level of reliability*.

In their final form, both the penalty models (12)–(14) and the mixed model (15) can be included into the class of *expectation models*: their objective function has the form

$$F(x) = E_\omega f(x, \omega)$$

where $f(x, \omega)$ is a function that depends on decision variables x and on random parameters ω and the set X of admissible decisions is defined via deterministic constraints. Moreover, the set X considered in our models is convex polyhedral, say

$$X = \{x:\ Ax \geqslant b\}$$

and the considered function $f(\cdot, \omega)$ is convex piece-wise linear for each fixed value of ω.

4. Solution techniques used to solve the problem

In this section we shall describe briefly the solution techniques used for numerical experiments with models (9) and (15) described in the previous section. We shall preferably choose among those techniques that are supported by the programs from the SDS/ADO collection of Stochastic Programming Codes available at IIASA.

4.1. Model with joint chance constraint

For solving problem (9) with the joint chance constraint, nonlinear programming techniques can be used. The choice among them depends on the properties of the set of feasible solutions. For log-concave probability measures, which is the case of multidimensional normal, gamma, uniform, and Dirichlet distributions of β, the set described by

$$p(x) = P\{x_i \geqslant \beta_i + d_i,\ i = 2, 3, 4\} \geqslant \alpha$$

is convex and among others, the method of feasible directions, supporting hyperplane method and penalty methods supplemented by an efficient routine for computing the values of the function $p(x)$ and its derivatives can be applied. (For a survey see Chapter 5 of [7].)

For multinormal distribution, the supporting hyperplane method was implemented by Szántai as PCSP code (see Chapter 10 of [7]). The program solves problems of stochastic programming with joint probability constraints under assumption of multinormal distribution of random right-hand sides (β_i, $i = 2, 3, 4$, in our case). It is based on Veinott's supporting hyperplane algorithm (see [22]). The individual upper bounds on variables are handled separately and the parameters of the multinormal distribution are used to get a starting feasible interior solution. For constructing the necessary linear and stochastic data files, one can turn to Chapter 9 of [7].

Alternatively, it is possible to apply the nonlinear version of the MINOS system [13]. For this purpose one has to use a separate subroutine named CALCON which calculates the value of the probabilistic constraint and its gradient. This subroutine for multinormal distributions is contained in the PCSP code. It was coded on the base of an improved simulation technique [20].

For the special multivariate gamma distribution with prescribed expectations, variances and the correlation matrix designed in [16], the corresponding subroutine CALCON can be found in [20,21].

4.2. Approximization of the expectation model by a large-scale linear program

For solving expectation models such as (15), one possibility is to approximate the initial continuous distribution of the random vector by a sequence of discrete distributions. Various ways of doing this are described in [1].

In our model (15), the penalty $\tilde{\phi}$ can be given implicitly via the so-called *second stage program*: for fixed values of x_i, β_i, d_i, $i = 2, 3, 4$, the penalty term $\tilde{\phi}(\beta_i + d_i - x_i,\ i = 2, 3, 4)$ equals to the optimal value of the objective function in the following linear program

minimize $\quad cy$

subject to

$$x_i + y \geqslant \beta_i + d_i, \quad i = 2, 3, 4,$$

$$y \geqslant 0,$$

so that problem (15) can be considered as the complete recourse problem and solved accordingly.

For a discrete distribution of the random vector β that is concentrated in N points β^j, $j = 1, \ldots, N$, with probabilities $p_j > 0$, $j = 1, \ldots, N$, $\sum_{j=1}^{N} p_j = 1$, the problem (14) can be written in the following way:

$$\text{minimize} \quad x_0 + c \sum_{j=1}^{N} p_j y_j$$

subject to the constraints

$$x_i + y_j \geqslant \beta_{i-1}^j + d_i, \quad i = 2, 3, 4, \quad j = 1, \ldots, N, \tag{17}$$

and subject to constraints (5), (7), (8), which is a large linear program of a special structure. If the random vector β has a continuous distribution, it is necessary to approximate this distribution by a discrete one and build the linear program (17) for the approximate discrete distribution. Two discretization schemes were used; we shall call them the *intelligent approximation* and the *random approximation*.

For the *intelligent approximation*, the discrete distribution concentrated in N points with equal probabilities $p_j = 1/N$, $j = 1, \ldots, N$, is constructed in such a way that the original values of moments are saved. If the initial distribution was multinormal, then the discrete distribution is symmetric around its expectation and has the same expectation and the same variance matrix as the initial one. For details see [4].

In the less sophisticated *random approximation*, the discrete distribution is concentrated in N points with equal probabilities $1/N$ and these points are generated as independent observations of the random vactor β (according to its initial distribution).

The resulting linear program (17) can be of a very large dimension if an accurate approximation is needed and/or if the random vector is of larger dimensionality than that considered here. In similar situations, adaptive approximation schemes (see Chapter 2 of [7]) can improve the performance of approximation techniques substantially.

4.3. *The stochastic quasigradient method*

We shall describe more in detail the stochastic quasigradient (SQG) method for solving the expectation models as this method, in spite of its simplicity and effectiveness, has not yet been spread among the operations research community. The theoretical background can be found in [6] or in Chapter 6 of [7], the basic reference for its implementation is Chapter 16 of [7]. For extensions consult Chapters 17–20 of [7] and references there.

The basic scheme resembles the gradient projection method: The optimal solution of the expectation model,

$$\text{minimize} \quad F(x) := E_\beta f(x, \beta)$$
$$\text{on the set} \quad X = \{x \colon Ax \geqslant b\}, \tag{18}$$

is approached iteratively starting from an initial point x^0 by applying the following iterative procedure:

$$x^{s+1} = \pi_X(x^s - \rho_s \xi^s) \tag{19}$$

where ρ_s is the stepsize, ξ^s the step direction (stochastic quasigradient), π_X the projection operator on the set X:

$$\| \pi_X(x) - x \| = \min_{z \in X} | z - x |, \quad \pi_X(x) \in X.$$

The projection in the SQG is performed using QPSOL quadratic programming package [8].

The step direction ξ^s should be, roughly speaking, in average close to the gradient of the objective function $F(x) = E_\beta f(x, \beta)$ at point x^s, although individual ξ^s may be far from actual values of the gradient. This explains the term quasigradient. The desired property of the quasigradient is expressed with the help of conditional expectations:

$$E(\xi^s | x^0, \ldots, x^s) = \nabla_x F(x^s) + \alpha_s, \tag{20}$$

where α_s is some vanishing term. Each particular strategy of choosing a sequence of stepsizes ρ_s and step directions ξ^s leads to a particular algorithm and many such stategies are implemented in the program SQG, some of them are fairly sophisticated. It is also possible to change strategies interactively during the optimization process.

Here we shall describe briefly some of the solution strategies, namely those implemented in the numerical experiment.

The following methods of choosing the *step direction*, i.e. the quasigradient ξ^s, can be implemented:

– gradients or subgradients of the random function $f(x, \beta)$ with respect to x, say

$$\xi^s = f_x(x^s, \beta^s) \in \partial f(x^s, \beta^s), \tag{21}$$

– finite differences

$$\xi^s = \sum_{i=1}^{n} \frac{f(x^s + \delta_s e_i, \beta^{s,i}) - (f(x^s, \beta^{s,0})}{\delta_s} e_i, \tag{22}$$

where e_i are unit vectors,

– random search techniques

$$\xi^s = \sum_{i=1}^{M} \frac{f(x^s + \delta_s\eta_i, \beta^{s,i}) - f(x^s, \beta^{s,0})}{\delta_s}\eta_i, \quad (23)$$

where η_i are unit random vectors uniformly distributed on the sphere,

– aggregation (moving average)

$$\xi^{s+1} = (1-\alpha_s)\xi^s + \alpha_s v^s, \quad (24)$$

where v^s is chosen according to any of the methods (21)–(23).

It is also possible to form a step direction as an average of several directions chosen according to (21)–(24), normalize direction, etc.

As to the stepsize ρ_s, the necessary condition for convergence with probability 1 is

$$\rho_s \to 0, \quad \sum_{s=0}^{\infty} \rho_s = \infty, \quad \sum_{s=0}^{\infty} \rho_s^2 < \infty. \quad (25)$$

(See [12].) In principle, it is possible to select the stepsize sequence in advance but this leads to poor performances. Therefore two strategies, *interactive* and *automatic*, were adopted for implementation.

In the *interactive* option, the value of the stepsize is changed by the user who obtains information about process behavior from the computer monitor. The user keeps stepsize constant if process exhibits 'regular' behavior and decreases it if behavior becomes 'irregular' or 'oscillatory'; In order to use this possibility effectively the user has to get certain measure of familiarity with the problem. In experiments with interactive mode the step direction ξ^s was computed according to (23). It means, more in detail, that the following subroutines were used:

– Generate l random vectors $\eta^{s1}, \ldots, \eta^{sl}$, whose components η_j^{si} are uniformly and independently distributed on an interval $[-\delta_s, \delta_s]$.

– Generate l independent random vectors $\beta^{s1}, \ldots, \beta^{sl}$, with the given multivariate distribution.

– Compute ξ^s:

$$\xi^s = \sum_{i=1}^{l} \frac{f(x^s + \eta^{si}, \beta^{si}) - f(x^s, \beta^{si})}{\| \eta^{si} \|}\eta^{si}. \quad (26)$$

In order to get the exact solution, it is necessary to take $\delta_s \to 0$; however, in many situations a sufficiently good approximation can be obtained even with a fixed δ.

The inconvenience of the interactive option is that it requires too much experience from the user. Therefore the *automatic* option was developed in which the computer simulates the behavior of an experienced user.

The stepsize ρ_s is computed automatically according to the following rule:

– On each iteration, one observation $f(x^s, \beta^s)$ is made and these observations are used to compute an estimate $\hat{F}(s)$ of the current value of the objective function and of the current path length $L(s)$:

$$\hat{F}(s) = \sum_{i=1}^{s} f(x^i, \beta^i)/s,$$

$$L(s) = \sum_{i=1}^{s} \| x^{i+1} - x^i \|. \quad (27)$$

– The initial stepsize ρ_1 is chosen sufficiently large and each M iterations the condition for reducing stepsize is checked using the ratio

$$\Phi(x) = \frac{\hat{F}(s-K) - \hat{F}(s)}{L(s) - L(s-K)} \quad (28)$$

as the algorithm performance measure. If $\Phi(x)$ is decreasing, it means either the algorithm path becomes large due to oscillations, or the function difference becomes small due to arrival in the vicinity of optimum, or both. This justifies the stepsize decrease.

As a result

$$\begin{aligned} \rho_{s+1} &= D\rho_s \quad \text{if } \Phi(s) \leqslant k, \\ \rho_{s+1} &= \rho_s \quad \text{otherwise}. \end{aligned} \quad (29)$$

In connection with the automatic option, the quasigradient was chosen according to (29): For the objective function $F(x) = E_\beta f(x, \beta)$ in our problem (15) we have

$$f(x, \beta) = x_0 + c \max\left\{0, \max_{i=2,3,4} (\beta_i + d_i - x_i)\right\},$$

and its subgradient $f_x(x, \beta)$ is easy to compute:

$$f_x(x, \beta) = \{1, 0, -ct_1, -ct_2, -ct_3\},$$

where

$$t_j = 1 \quad \text{if } \beta_{j+1} + d_{j+1} - x_{j+1} = \max\left\{0, \max_{i=2,3,4} \{ \beta_i + d_i - x_i \}\right\},$$

$$t_j = 0 \quad \text{otherwise.}$$

Accordingly, $\xi^s = f_x(x^s, \beta^s)$ in the SQG scheme (19).

Notice that for the described implementation of the SQG method, only a generator of random vectors β with the given multivariate distribution is needed and that the form of the procedure does not depend on the assumed type of this distribution. Moreover, neither the values $F(x)$ of the objective function nor of its gradient are computed, which results in an essential decrease of the computational effort. The performance measures such as (28) are constructed using the estimate $\hat{F}$ (see (27)) of the current value of the objective function F. The procedure is not monotonous and the estimates $\hat{F}$ show clearly an oscillatory behavior with a tendency of decreasing in the average (see the numerical results contained in Tables 5–7 of the next section). The performance of the SQG method depends on the rules for choosing the stepsizes and there is strong evidence that the interactive mode cannot be fully avoided. The iterations terminate when the stepsize becomes smaller than some value specified by the user. This stopping rule is quite reliable if coupled with repeated runs. For more details on this subject, see Chapter 17 of [7].

5. Comparison of numerical results

To this purpose, we solved problems (9) and (15) of Section 3 for specific input values of the model parameters and for the assumed distribution of random variables using the numerical techniques introduced in Section 4. The resulting particular problem with chance constraints of type (9) reads:

$$\text{minimize} \quad x_0 \tag{30a}$$

subject to

$$P\{x_i \geqslant \beta_i + d_i,\ i = 2, 3, 4\} \geqslant \alpha, \tag{30b}$$

and subject to linear inequalities

$$\begin{aligned}
x_1 + x_2 &\leqslant 156.4,\\
x_1 + x_2 + x_3 &\leqslant 201.9,\\
x_1 + x_2 + x_3 + x_4 &\leqslant 225.3,\\
x_0 + x_1 &\geqslant 512.9,\\
x_0 + x_1 + x_2 &\geqslant 595.9,\\
x_0 + x_1 + x_2 + x_3 &\geqslant 654.2,\\
x_0 + x_1 + x_2 + x_3 + x_4 &\geqslant 720.2,
\end{aligned} \tag{30c}$$

$$\begin{aligned}
100.0 \leqslant x_0 &\leqslant 500.0,\\
38.1 \leqslant x_1 &\leqslant 102.3,\\
0 \leqslant x_2 &\leqslant 252.0,\\
0 \leqslant x_3 &\leqslant 252.0,\\
0 \leqslant x_4 &\leqslant 252.0,
\end{aligned} \tag{30d}$$

where $d_i = 12.7$, $i = 2, 3, 4$. The random vector $\beta = (\beta_2, \beta_3, \beta_4)$ is distributed normally with expectations:

$$E(\beta) = (20.2, 27.37, 10.65),$$

standard deviations:

$$\sigma(\beta) = (8.61, 10.65, 6.00)$$

Table 1
Results of the calculations by the PCSP code

x_0	x_1	x_2	x_3	x_4	$\hat{F}(s)$	Prob. level	CPU time	No. of cutting planes
494.88	38.23	106.39	51.95	28.73	605.04	0.75		
494.88	49.98	58.30	81.97	35.05	500.89	0.973	25.52	3
494.88	42.93	55.82	63.61	62.94	501.20	0.983	36.38	5
494.88	41.49	59.40	63.16	61.25	501.53	0.984	32.37	4
495.38	43.28	58.23	86.26	37.53	497.38	0.989		
494.88	41.31	69.23	74.60	40.17	495.67	0.997	41.87	5
494.88	39.78	66.92	76.14	42.46	495.17	0.998	34.45	5

In column 6 of this table are the estimated values of the objective function (27) computed using the same 10000 observations of multivariate normal vectors as in the relevant column of Table 4. Column 7 gives CPU time of the VAX 11/780.

Table 2
Results of the calculations by using the nonlinear MINOS, multivariate normal distribution

x_0	x_1	x_2	x_3	x_4	Prob. lev.	CPU time	No. of major iterations
494.89	51.22	63.07	77.35	33.69	0.973	46.46	11
494.89	50.00	63.07	77.40	34.92	0.983	56.22	14
494.89	49.91	63.07	77.35	35.01	0.984	58.06	14
494.89	46.17	63.07	77.40	38.75	0.997	84.48	23
494.89	43.41	63.04	77.35	41.50	0.999	112.58	31

and correlation matrix:

$$\begin{pmatrix} 1.0 & 0.360 & 0.125 \\ 0.360 & 1.0 & 0.571 \\ 0.125 & 0.571 & 1.0 \end{pmatrix} = R(\beta). \tag{31}$$

Table 1 contains the optimal solution obtained by the PCSP code for different values of α, whereas Table 2 contains the corresponding results obtained by the nonlinear version of the MINOS system. These computational results show that the direct application of the MINOS system for the solution of the optimization problem (9) is less successful than the use of the PCSP code. We obtained that without giving a good initial setting on the values of variables the MINOS system failed to find a feasible solution. The presented computational results correspond to initial settings $x_2 = 63.035$, $x_3 = 77.345$, $x_4 = 30.0$ for which the joint probability is equal to 0.866.

For comparison, parallel results were obtained for the special multivariate gamma distribution [16] with the given mean values $E\beta_i$, standard deviations $\sigma(\beta_i)$, $i = 2, 3, 4$, and the correlation matrix $R(\beta)$, see (31). The numerical results are given in Table 3. The CPU time in this table reflects the fact, that in the case of multivariate gamma distribution it was necessary first to fit the multivariate gamma probability distribution to the empirical data, and then to generate the corresponding random vectors needed for calculation the values of the distribution function and of its gradient.

The particular problem of the type (15) which was used for comparisons is formulated below:

minimize

$$F(x) = E_\beta f(x, \beta) = x_0 + cE_\beta\left\{\max\left\{0, \max_{i=2,3,4} \{\beta_i + d_i - x_i\}\right\}\right\} \tag{32}$$

subject to constraints (30c,d),

with the same values $d_i = 12.7$, $i = 2, 3, 4$, with the same multivariate normal distribution of the random vector β given by the expectations, standard deviations and correlation matrix (31), and with the penalty coefficient c equal to 100. The convenient feature of this problem is that we can easily obtain a very good lower bound for the solution by minimizing x_0 subject to the constraints stated above. This gives $F(x^*) \geqslant 494.9$ where x^* is the optimal solution of (32), (30c,d).

The results are summarized in Table 4. The left column of this table contains an abbreviation of

Table 3
Results of calculations by using the nonlinear MINOS system and the multivariate gamma distribution

x_0	x_1	x_2	x_3	x_4	Prob. lev.	CPU time	No. of major iterations
494.89	46.78	63.07	77.38	38.07	0.973	845.57	12
494.89	44.00	63.07	77.38	40.85	0.983	923.55	14
494.89	43.48	63.07	77.38	41.37	0.984	1020.17	15
494.89	38.10	60.60	84.33	42.27	0.997	1886.77	30
494.89	38.10	59.89	78.59	48.72	0.999	2502.17	40

Table 4
The summary of experiments with the model of expectation type

Experiment		x_0	x_1	x_2	x_3	x_4	Function value	Prob. value	CPU time
MINOS	125_A1	494.88	38.1	88.02	61.22	39.96	505.20	0.9728	55.46
MINOS	125–A2	494.88	38.1	118.35	43.34	25.49	833.04	0.4878	27.18
MINOS	343_A1	494.88	38.1	84.92	63.06	39.23	501.40	0.9832	488.92
MINOS	343_A2	494.88	38.11	114.92	42.42	29.84	820.39	0.5578	71.72
MINOS	1331_A1	494.88	64.4	54.75	65.34	40.80	499.85	0.9844	264.76
MINOS	1331_A2	494.88	38.1	92.09	61.81	33.30	513.26	0.9391	385.96
SQG int	100	494.88	39.24	71.48	74.03	40.53	495.60	0.9978	
SQG int	24	494.88	38.43	67.80	78.52	40.53	495.42	0.9981	
SQG aut	1000	494.88	38.10	63.39	77.38	46.42	495.15	0.9994	51.52
SQG aut	1000	494.88	40.65	57.32	74.81	52.49	495.73	0.9972	49.38

the particular numerical experiment, each row describing the results of one experiment. The first six rows correspond to experiments with the large-scale linear program (17) solved by MINOS 4.0, see [13]. The numbers in the first column in these rows indicate the number of approximating points, A1 means the intelligent approximation scheme and A2 means the random one. The column

Table 5
Results of the calculations with the interactive mode of SQG

Step number	Stepsize	x_0	x_1	x_2	x_3	x_4	$\hat{F}(s)$
The value of step G in random search was set to 10.0							
2	10.0	497.96	54.92	56.57	57.12	56.67	522.66
4	10.0	495.45	54.01	56.13	57.75	56.82	517.10
6	10.0	500.00	45.75	110.69	28.86	34.86	1702.02
8	10.0	494.88	96.35	60.08	2.81	66.06	4222.28
10	10.0	500.00	89.51	63.60	1.03	71.14	4405.28
11	2.0	500.00	102.31	51.58	3.52	62.74	4156.32
12	2.0	497.28	95.58	0.0	106.28	21.02	3796.11
14	2.0	500.00	91.40	37.49	18.36	42.92	754.44
16	2.0	500.00	38.10	54.77	108.99	19.94	948.33
18	2.0	494.88	39.74	64.96	84.68	35.90	498.72
20	2.0	494.88	38.10	67.45	78.98	40.75	492.37
22	2.0	494.88	38.62	67.74	78.42	40.50	495.42
24	2.0	494.88	38.43	67.80	78.52	40.53	495.42
26	0.20	497.90	39.04	66.94	75.44	43.49	498.29
28	0.20	497.76	39.00	66.93	75.44	43.47	498.13
30	0.20	497.60	39.04	66.95	75.47	43.49	497.49
At this point the step in random search was changed to 5.0							
31	0.2	497.56	39.06	66.98	75.47	43.47	497.87
35	0.2	497.03	39.24	67.18	75.47	43.39	497.35
40	0.2	496.69	39.24	67.20	75.42	43.42	497.04
50	0.2	496.04	39.25	67.20	75.35	43.42	496.35
60	0.2	495.40	39.22	67.25	75.32	43.35	495.80
70	0.2	495.52	39.08	71.34	73.79	40.44	496.19
80	0.2	495.05	39.18	71.46	74.00	40.51	495.66
90	0.2	494.88	39.16	71.50	74.03	40.59	495.59
100	0.2	494.88	39.24	71.48	74.03	40.53	495.60

Table 6
Results of the computations with the automatic option of SQG

Step number	Stepsize	x_0	x_1	x_2	x_3	x_4	$\hat{F}(s)$
20	5.00	494.88	56.32	56.32	56.32	56.32	524.38
40	5.00	494.88	40.65	57.32	61.28	66.03	504.56
60	2.50	494.88	40.65	57.32	61.28	66.03	504.56
80	1.25	494.88	40.65	57.32	61.28	66.03	504.56
100	0.62	498.43	38.88	55.55	94.30	36.55	502.24
140	0.31	494.88	40.31	57.67	86.94	40.36	495.70
200	0.31	494.88	40.31	59.67	86.94	40.36	792.70
220	0.31	497.96	39.11	55.78	72.86	57.53	499.34
260	0.31	494.88	40.65	57.32	86.91	40.36	495.48
400	0.31	494.88	40.65	57.32	86.94	40.36	495.78
460	0.15	496.71	39.74	56.41	87.86	41.28	497.64
480	0.15	495.15	40.52	57.19	87.08	40.50	496.00
560	0.15	494.88	38.10	63.39	77.38	46.42	495.15
1000	0.15	494.88	38.10	63.39	77.38	46.42	495.15

'Function value' contains the estimate (27) of the objective function (32) at the approximation to the optimal solution (x_0, x_1, x_2, x_3, x_4), which was made using a sample of 10 000 points generated by a subroutine for multivariate normal distribution from the IMSL library; the same set of samples was used for all estimates of the function value. The column 'Prob. value' contains the values of probability constraint (25), this allows comparison of the expectation model with the probability constrained model.

The last four rows of Table 4 correspond to experiments with stochastic quasigradient techniques, described in detail in Section 4.3. The first two of these rows correspond to the interactive stepsize run, presented in detail in Table 5, after 100 and 24 iterations. Each iteration in this run requires 40 observations of random function $f(x, \beta)$ from (32). The last two rows include results of two runs with the automatic stepsize selection described in Tables 6 and 7 after 1000 iterations; each iteration required one observation of the subgradient $f_x(x, \beta)$. The last column presents solution times in seconds of CPU of VAX-11/780. For the interactive option of SQG these times are not included because they do not measure the time of user's response. As a measure of method effectiveness these times should be taken

Table 7
Results of the computations with the automatic option of SQG (continued)

Step number	Stepsize	x_0	x_1	x_2	x_3	x_4	$\hat{F}(s)$
20	5.00	494.88	56.32	56.32	56.32	56.32	524.38
40	2.50	494.88	48.99	48.99	61.28	66.03	513.80
60	2.50	496.00	40.10	56.77	61.28	67.14	505.75
80	1.25	497.00	39.60	56.27	61.28	68.14	506.89
100	1.25	494.88	40.65	57.32	61.28	66.03	504.56
200	1.25	494.88	40.65	57.32	61.28	66.03	504.56
220	1.25	499.50	38.35	55.02	108.49	23.43	736.52
240	0.62	494.88	40.65	57.39	74.81	52.49	495.73
420	0.62	494.88	40.65	57.32	74.81	52.49	495.73
440	0.62	496.87	39.66	56.33	75.80	53.49	497.94
460	0.31	494.88	40.65	57.32	74.81	52.49	495.73
940	0.15	494.88	40.65	57.32	74.81	52.49	495.73
1000	0.15	494.88	40.65	57.32	74.81	52.49	495.73

with some reservation, for instance for MINOS they do not represent effort spent on preparation of the huge MPS file.

For the two presented runs with the automatic stepsize selection, two different sequences of the vector β were generated, the same starting points (1000, 100, 100, 100, 100) were taken and the parameters of the performance measure (28) and of the corresponding stepsize reduction (29) were fixed to $D = 0.5$, $K = 20$, and $k = 0.1$. The first run displayed smoother behavior of the process, the second one shows jumps. When the current point approaches optimum, the event $\{t_j = 1\}$ becomes less and less likely. Therefore the method spends much of the iterations standing at the same point (e.g. iterations 260–400 in Table 3). The last column again was obtained afterwards using the same 10000 observations of β as in the previous tests. It shows that the algorithm reaches quite good the vicinity of the solution after 220 iterations. The jumps are due to too large stepsizes.

The overall comparison shows that SQG performed better on this problem then the general linear programming tool. Approximation techniques started to provide comparable results only when the number of approximating points exceeded 1000, i.e., when the number of rows in the deterministic equivalent linear programming problem (17) exceeded 4000. The more sophisticated, intelligent approximation scheme provided much better results than the random one, which was based on a random selection of approximation points.

We complete the discussion by presenting values of the empirical expectation and correlation matrix computed using the same 10000 random numbers which were used for the objective function values estimates (compare with (31)). It gives an idea how accurate the estimates are.

Empirical expectations:

$$\hat{E}(\beta) = (20.29, 27.39, 10.68),$$

empirical correlation matrix:

$$\hat{R}(\beta) = \begin{pmatrix} 1.0 & 0.354 & 0.108 \\ 0.354 & 1.0 & 0.573 \\ 0.108 & 0.573 & 1.0 \end{pmatrix}.$$

Experiments suggest that both the expectation model and the model with probability constraint give comparable results in terms of reliability level and prices for maintaining the desired level of reliability. The methods contained in the ADO/SDS library of stochastic programming codes performed better on this particular problem than the direct use of standard LP or NLP packages. To solve the case study, model (15) was selected as it opens straightforward possibilities of extension to alternative types of the probability distribution of β. For obtaining the numerical results we have chosen the SQG method on account of its easy implementation and of its good performance in light of the above comparisons.

6. The case study

In the case study of the water resources system in the Bodrog River basin, the model (15) was used with added constraints (see discussion):

$$d_i \leqslant x_i, \quad i = 2, 3, 4. \tag{33}$$

The input values of the 3-dimensional multinormal distribution of β_i were those given by (31). The parameters of the marginal normal distributions of cumulated monthly inflows ζ_i, the corresponding values of quantiles for $\alpha_i = 0.9$ and $\gamma_i = 0.4$, $k = 1, 2, 3, 4$, and the values d_i, $i = 1, 2, 3, 4$, (in Mm^3) are given in Table 8.

The minimum and maximum reservoir capacity x_0 was fixed as $l_0 = 100$ Mm^3 and $u_0 = 334$ Mm^3. (In the numerical study in the previous section, a less realistic value $u_0 = 500$ Mm^3 together with $\gamma_i = 0.75$, $i = 1, 2, 3, 4$, was used.) The upper bounds for variables were set as $u_i = 252$, $i = 1, 2, 3, 4$, in Mm^3, which is the volume of a long-term flood. These constraints were not found to be effective and therefore they were not analyzed.

Due to recreation purposes, the acceptable minimum storage in the third period is $m_3 = 194$ Mm^3. However, the comparison of the third inequality of (7), the third inequality of (8) together with $x_0 \leqslant 334$ gives an upper bound of 189,4 for

Table 8
The input data for cumulated inflows

Period	$E(\zeta_i)$	$\sigma(\zeta_i)$	$z_i(1-0.9)$	$z_i(0.4)$	d_i
1	303.47	122.28	146.8	272.5	38.1
2	375.94	133.43	205.0	342.1	12.7
3	432.61	140.27	252.9	397.1	12.7
4	486.26	158.64	283.0	446.1	12.7

the sum $m_3 + v_3$, so that the parameter value $m_3 = 194$ would lead to contradictory constraints. That is why the minimum storage value m_3 has been put up to a maximum of 137 Mm^3 (see alternative C) and the storage values m_k, $k =$ 1, 2, 3, 4, have been kept fixed equal to 57 Mm^2 over all periods in alternatives A and B. The reliability of maintaining the summer reservoir pool has been evaluated ex post.

6.1. Choice of reliability values

The very important parameters of the model are the required probabilities α, γ_i and α_i. The value α is the required joint probability of the water supply. Tests with the model have shown that it is necessary to add the deterministic constraint (33) in order to secure the required values of constant industrial water demands. Using the deterministic constraint (33), a relatively low value of α, e.g. $\alpha = 0.85$, may be acceptable.

The values α_i refer to the relatively strong environmental and technical requirements for maintaining the minimum reservoir pool. Therefore $\alpha_i = 0.9$, $i = 1, 2, 3, 4$, was chosen.

The choice of the values γ_i was rather difficult. They refer to the important constraints imposed on the reservoir V operation that arise from flood control requirements that stipulate that a certain space — flood control storage — be held empty. This requirement cannot be easily expressed in the model due to its aggregated character. The probability that the freeboard storage is empty means also that there is no spill during this period. Therefore the flood control problems are often treated in a separate model and the required probabilities γ are adapted to the resulting values of this separate model. Accordingly, the value $\gamma_i =$ 0.4, $i = 1, 2, 3, 4$, was chosen in this section.

6.2. Results

The results of the selected alternatives for the design and operation of the reservoir V (using different input parameters m_k and v_k) are contained in Table 9. The probabilities γ_k and α_k were fixed in advance. The value α of the joint probability $p(x)$ was computed at the resulting point ex post and compared with the chosen probability level 0.85. The probability of maintaining the sufficient recreation pool of at least $s_3 = 194$ Mm^3 was computed according to a simple scheme: the identity (6) was applied to evaluate the necessary cumulated inflow, say

$$z_3 = s_3 - m_4 + x_1 + x_2 + x_3$$

for the desired recreation storage $s_3 = 194$, for the

Table 9
Case study summary

Alternative:	A	B	C	D	E
Model/Code:	(17)/STO	(17)/STO	(17)/STO	(17)/STO	(11)/PCSP
Parameters:	$m_k = 57\ \forall k$	$m_k = 131\ \forall k$	$m_k = 57$, $k = 1, 2, 4$, $m_3 = 137$	$m_k = 57\ \forall k$	$m_k = 57\ \forall k$
	$v_k = 70\ \forall k$	$v_k = 10\ \forall k$	$v_k = 70$, $k = 1, 2, 4$, $v_3 = 10$	$v_k = 70\ \forall k$	$v_k = 70\ \forall k$
	$u_0 = 334$	$u_0 = 334$	$u_0 = 334$	$u_0 = 500$	$u_0 = 500$
Optimal solution					
x_0	291.6	304.1	334.0	494.9	494.9
x_1	107.9	109.4	67.55	40.7	39.8
x_2	69.6	69.6	67.55	57.3	66.9
x_3	69.8	65.1	37.8	74.8	76.1
x_4	35.7	38.9	110.10	52.5	42.5
Reliabilities					
α (releases)	0.979	0.987	0.412	0.99	0.99
$\hat{\alpha}_3$ (recreation)	0.633	0.814	0.809	0.81	0.79
γ (freeboard)	0.4	0.4	0.4	0.75	0.75
Goals (met)					
min storage	YES (0.9)	YES (0.9)	YES (0.9)	YES (0.9)	YES (0.9)
recreation pool	NO	YES (0.8)	YES (0.8)	YES (0.8)	YES (0.79)
freeboard volume	YES	NO	YES	YES	YES
water supply	YES	YES	NO	YES	YES

optimal releases x_i and for $m_4 = 57$. The parameters of the given marginal distribution of ζ_3 were used to get the corresponding probability

$$\hat{\alpha}_3 = P\{\zeta_3 \geqslant z_3\}.$$

For a more detailed discussion, see [4].

None of the alternatives A, B, C satisfies the contradictory requirements for reservoir V operation. (The seemingly good result of alternative B is not acceptable due to the conjunction of a very low freeboard volume v_k and of a low value of the probability γ.) Whereas the optimal solutions of alternatives A, B are comparable, their dissimilarity with the optimal solution of the rather extremal alternative C is remarkable. The last two columns of Table 9 (alternatives D and E) correspond to the nonrealistic upper bound of the reservoir capacity that was used in the numerical experiments of Section 5.

6.3. Conclusions

The possibility of treating intercorrelations (in our case, intercorrelations of random needs) within the framework of stochastic programming models of water management was explored via two conceptually different models that have proved a good agreement of the technological results. For the analysis of the different design alternatives, the more easily implementable expectation type model was used and the method of multimodeling proved to be of use in planning the water resources system development. The analysis of the design alternatives shows the contradictory character of the main goals of the water resources system — water supply for industry and irrigation, flood control, environmental conservation and recreation. As the optimum alternatives do not meet all these goals, the water resources system has to be enlarged by the reservoir K. As a screening aggregate model was used and the multidimensional distribution and marginal distributions were approximated by the multinormal and normal distributions respectively, the optimum design and operation variables derived later by this model are rough approximations only. However, the more precise values that were derived by a stochastic simulation model [11] do not differ to such degree that the main result (i.e. the necessity to plan a new reservoir) need to be altered.

As mentioned in Section 6.1, the model reliability parameters have to be carefully chosen, otherwise the model could be too conservative requiring high values of reservoir capacity. In real-life planning, less aggregated models with monthly time periods will be used and the value of resulting released volumes will better approximate the rule curves of reservoirs.

References

[1] Birge, J., and Wets, R.J-B., "Designing approximation schemes for stochastic optimization problems, in particular for stochastic programs with recourse", *Mathematical Programming Study* 27 (1986) 54–102.

[2] Dupačová, J., "Water resources system modelling using stochastic programming with recourse", in: P. Kall and A. Prékopa (eds.), *Recent Results in Stochastic Programming* (Proc. Oberwolfach 1979), Lecture Notes in Economics and Math. Systems 179, Springer, Berlin, 1980, 121–133.

[3] Dupačová, J., and Kos, Z., "Chance — constrained and simulation models for water resources systems", *Ekonomicko — matematicky' obzor* 15 (1979) 178–191.

[4] Dupačová, J., Gaivoronski, A., Kos, Z., and Szántai, T., "Stochastic programming in water resources system planning: A case study and a comparison of solution techniques", WP-86-40, IIASA, Laxenburg, 1986.

[5] Edwards, J. (ed.), "Documentation for the ADO/SDS collection of stochastic programming codes", WP-85-02, IIASA, Laxenburg, 1985.

[6] Ermoliev, Yu., *Methods of Stochastic Programming* (in Russian), Nauka, Moscow, 1976.

[7] Ermoliev, Yu., and Wets, R. (eds.), *Numerical techniques for stochastic optimization problems*, Springer, Berlin, 1988.

[8] Gill, P.E., Murray, W., Saunders, M.A., and Wright, H.M., "User's guide for SOL/QPSOL: A FORTRAN package for quadratic programming", SOL 83-7, Systems Optimization Laboratory, Stanford University, 1983.

[9] Kall, P., "Stochastic programming", *European Journal of Operations Research* 10 (1982) 125–130.

[10] Kos, Z., "Operation of water resource systems in Czechoslovakia", in: S. Kaczmarek and J. Kindler (eds.), *The Operation of Multiple Reservoir Systems*, CP-82-S4, IIASA, Laxenburg, 1979, 181–204.

[11] Kos, Z., "Model of water management in the basin: Water engineering, irrigation and drainage in the basin of the Bodrog river" (in Czech), RR 120.86, Technical University, Prague, 1988.

[12] Kushner, H., and Clark, D., *Stochastic Approximation Methods for Constrained and Unconstrained Systems*, Springer, New York, 1978.

[13] Murtagh, B.A., and Saunders, M.A., "MINOS 5.0 User's Guide", Report No. SOL 83-20, Systems Optimization Laboratory, Stanford University, 1983.

[14] Prékopa, A., "Logarithmic concave measures with application to stochastic programming", *Acta Scientiarum Mathematicarum* 32 (1971) 301–316.

[15] Prékopa, A. (ed.), "Studies in applied stochastic programming", Hungarian Academy of Sciences, MTA SzTAKI, Budapest, 1978.

[16] Prékopa, A., and Szántai, T., "A new multivariate gamma distribution and its fitting to empirical stream flow data", *Water Resource Research* 14 (1978) 19–24.

[17] ReVelle, Ch., Joeres, E., and Kirby, W., "The linear decision rule in reservoir management and design", *Water Resources Research* 5 (1969) 767–777.

[18] Rockafellar, R.T., and Wets, R.J.-B., "A Lagrangian finite generation technique for solving linear quadratic problems in stochastic programming", *Mathematical Programming Study* 28 (1986) 63–93.

[19] Sassone, P.G., and Straffer, W.A., *Cost-benefit analysis: A Handbook*, Academic Press, New York, 1978.

[20] Szántai, T., "Evaluation of a special multivariate gamma distribution function", *Mathematical Programming Study* 27 (1986) 1–16.

[21] Szántai, T., "Calculation of the multivariate probability distribution function values and their gradient vectors", WP-87-82, IIASA, Laxenburg, 1987.

[22] Veinott, A.F., "Supporting hyperplane method", *Operations Research* 15 (1967) 147–152.

[23] Wets, R.J-B., "Stochastic programming", in: *Handbooks in Operations Research and Management Science*, Vol. 1, Elsevier Sci. Publ. (North-Holland), Amsterdam, 573–629.

[24] Yeh, W.W-G., "Reservoir management and operations models: A state-of-the-art review", *Water Resources Research* 21 (1985) 1797–1818.