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Testing the structure of multistage stochastic programs

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Abstract A fixed topology of stages and/or a fixed branching scheme are common assumptions for applications and numerical solution of scenario based multistage stochastic programs. Using contamination technique to test this structure, we extend the results of Dupačová (Contamination for multistage stochastic programs. In: Hušková M, Janžura M (eds) Prague stochastics. Matfyzpress, Praha, pp 91–101, 2006a) to stochastic programs with multistage polyhedral risk objectives. The ideas are exemplified by bond portfolio management problems and complemented by illustrative numerical results.

Keywords Contamination · Additional stages · Out-of-sample scenarios · Polyhedral risk objectives · Bond portfolio management

1 Multiperiod and multistage stochastic programs

When formulating multistage stochastic programs it is common to assume that the horizon and the sequence of times at which decisions will be made have been already

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fixed. An important requirement is that the decisions must be nonanticipative, i.e. in any stage of the decision process the decisions are allowed to depend only on the past observations and decisions.

As the discrete approximations of the data process may be available at much finer timestep than the intervals between the decision points, see e.g. Dempster et al. (2000), Dupačová et al. (2003), Eichhorn and Römisch (2006), the crucial task is then to relate the time instants and stages.

In the general T-*stage stochastic program* we think of a stochastic data process $\omega = (\omega_1, \dots, \omega_{T-1})$ and a decision process $\boldsymbol{x} = (\boldsymbol{x}_1, \dots \boldsymbol{x}_T)$. The components $\omega_1, \dots, \omega_{T-1}$ of ω and the decisions $\boldsymbol{x}_2, \dots, \boldsymbol{x}_T$ are assumed to be random vectors, not necessarily of the same dimension, defined on some probability space $(Z, \mathscr{F}, \mu)$, while $\boldsymbol{x}_1$ is a nonrandom vector-valued variable and ω_0 denotes a fixed initial input value.

The decision process is *nonanticipative* which means that decisions taken at any stage of the process do neither depend on future *realizations* of stochastic data nor on future decisions, whereas the past information as well as the knowledge of the probability distribution of the data process are exploited. This can be expressed as follows: Let $\mathscr{F}_0$ be the trivial σ-field $\{\emptyset, \Omega\}$ and $\mathscr{F}_{t-1} \subseteq \mathscr{F}$ be the σ-field generated by the part $\omega^{t-1,\bullet} := (\omega_1, \dots, \omega_{t-1})$ of the stochastic data process ω that precedes stage t. The dependence of the t-th stage decision $\boldsymbol{x}_t$ only on the past means that $\boldsymbol{x}_t$ is $\mathscr{F}_{t-1}$-measurable. We denote $\boldsymbol{x}^{t-1,\bullet} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_{t-1})$ the sequence of decisions at stages $1, \dots, t-1$, P the distribution function of ω, P_t denotes the marginal probability distribution of ω_t, and $P_t(\cdot|\omega^{t-1,\bullet})$, $t = 2, \dots, T-1$, its conditional probability distribution. E_P is the expectation operator under P.

The first-stage decision vector $\boldsymbol{x}_1$ consists of all decisions that have to be selected before further information is revealed whereas the second-stage decisions are allowed to adapt to this information, etc. In each of the stages, the decisions are limited by constraints that may depend on the previous decisions and observations. *Stages do not necessarily refer to time periods, they correspond to steps in the decision process.*

An example is the *nested form* of the multistage stochastic linear program (MSLP) which resembles the backward recursion of stochastic dynamic programming with an additive overall cost function:

$$\min_{\boldsymbol{x}_1 \in \mathscr{X}_1} [\boldsymbol{c}_1^\top \boldsymbol{x}_1 + E_P\{\varphi_1(\boldsymbol{x}_1, \omega_1)\}] \text{ with } \mathscr{X}_1 := \{\boldsymbol{x}_1 \mid \boldsymbol{A}_1 \boldsymbol{x}_1 = \boldsymbol{b}_1,\ \boldsymbol{l}_1 \le \boldsymbol{x}_1 \le \boldsymbol{u}_1\} \tag{1}$$

where the cost-to-go function $\varphi_{t-1}(\cdot,\cdot)$, $t = 2, \dots, T$, is defined recursively as

$$\varphi_{t-1}(\boldsymbol{x}^{t-1,\bullet}, \omega^{t-1,\bullet}) = \min_{\boldsymbol{x}_t} [\boldsymbol{c}_t(\omega^{t-1,\bullet})^\top \boldsymbol{x}_t + E_{P_t(\cdot|\omega^{t-1,\bullet})}\{\varphi_t(\boldsymbol{x}^{t-1,\bullet}, \boldsymbol{x}_t, \omega^{t-1,\bullet}, \omega_t)\}] \tag{2}$$

subject to constraints

$$\boldsymbol{B}_t(\omega^{t-1,\bullet})\, \boldsymbol{x}_{t-1} + \boldsymbol{A}_t(\omega^{t-1,\bullet})\, \boldsymbol{x}_t = \boldsymbol{b}_t(\omega^{t-1,\bullet}), \quad \boldsymbol{l}_t(\omega^{t-1,\bullet}) \le \boldsymbol{x}_t \le \boldsymbol{u}_t(\omega^{t-1\bullet}) \text{ a.s.}$$

and φ_T is explicitly given, e.g. $\varphi_T \equiv 0$.

Matrices $\boldsymbol{A}_t$ are of a fixed (m_t, n_t) type and the remaining vectors and matrices are of consistent dimensions. For the first stage, known values of all elements of $\boldsymbol{b}_1, \boldsymbol{c}_1, \boldsymbol{A}_1, \boldsymbol{l}_1, \boldsymbol{u}_1$ are assumed and the main decision variable is $\boldsymbol{x}_1$ that corresponds to the first stage. Constraints involving random elements hold almost surely. For simplicity we will assume that *all infima are attained* and that *all expectations exist.* See recent books Kall and Mayer (2005) or Ruszczyński and Shapiro (2003) for more general cases. Notice that the first-stage problem (1) has the form of the expectation-type stochastic program with the set of feasible decisions *independent* of P.

For applications one mostly approximates the true probability distribution P of ω by a discrete probability distribution carried by a finite number of atoms, say, $\omega^1, \dots, \omega^K$. Accordingly, the supports of marginal and conditional probability distributions P_t, $P_t(\cdot|\omega^{t-1,\bullet})\, \forall t$ are finite sets. For disjoint sets of indices $\mathscr{K}_t$, $t = 2, \dots, T$, let us list as $\tilde{\omega}_{k_t}$, $k_t \in \mathscr{K}_t$ all possible realizations of $\omega^{t-1,\bullet}$ and denote by the same subscripts the corresponding values of the t-th stage coefficients. The total number of scenarios K equals the number of elements of $\mathscr{K}_T$. Each scenario $\omega^k = \{\omega_1^k, \dots, \omega_{T-1}^k\}$ thus generates a sequence of coefficients $\{\boldsymbol{c}_{k_2}, \dots, \boldsymbol{c}_{k_T}\}$, $\{\boldsymbol{A}_{k_2}, \dots, \boldsymbol{A}_{k_T}\}$, $\{\boldsymbol{B}_{k_2}, \dots, \boldsymbol{B}_{k_T}\}$, $\{\boldsymbol{b}_{k_2}, \dots, \boldsymbol{b}_{k_T}\}$, $\{\boldsymbol{l}_{k_2}, \dots, \boldsymbol{l}_{k_T}\}$, $\{\boldsymbol{u}_{k_2}, \dots, \boldsymbol{u}_{k_T}\}$. The data are organized in the form of the scenario tree: Its nodes are determined by all considered realizations $\tilde{\omega}_{k_t}$, $k_t \in \mathscr{K}_t, t = 2, \dots, T$, and by the root indexed as $k_1 = 1$; each realization $\tilde{\omega}_{k_{t+1}}$ of $\omega^{t\bullet}$, $t = 1, \dots, T-1$, has a unique ancestor $\tilde{\omega}_{k_t}$ (a realization of $\omega^{t-1,\bullet}$), we denote it by subscript $a(k_{t+1})$, and a finite number of descendants—realizations of $\omega^{t+1,\bullet}$.

This allows to rewrite the T-stage stochastic linear program (1)–(2) in the following *arborescent form*:

$$\min \left[\boldsymbol{c}_1^\top \boldsymbol{x}_1 + \sum_{k_2 \in \mathscr{K}_2} p_{k_2} \boldsymbol{c}_{k_2}^\top \boldsymbol{x}_{k_2} + \sum_{k_3 \in \mathscr{K}_3} p_{k_3} \boldsymbol{c}_{k_3}^\top \boldsymbol{x}_{k_3} + \cdots + \sum_{k_T \in \mathscr{K}_T} p_{k_T} \boldsymbol{c}_{k_T}^\top \boldsymbol{x}_{k_T} \right] \tag{3}$$

subject to

$$\begin{array}{llllll}
\boldsymbol{A}_1 \boldsymbol{x}_1 & & & & = \boldsymbol{b}_1 & \\
\boldsymbol{B}_{k_2} \boldsymbol{x}_1 + & \boldsymbol{A}_{k_2} \boldsymbol{x}_{k_2} & & & = \boldsymbol{b}_{k_2}, & k_2 \in \mathscr{K}_2 \\
& \boldsymbol{B}_{k_3} \boldsymbol{x}_{a(k_3)} & + \boldsymbol{A}_{k_3} \boldsymbol{x}_{k_3} & & = \boldsymbol{b}_{k_3}, & k_3 \in \mathscr{K}_3 \\
& \quad \ddots & \quad \ddots & & \vdots & \\
& & \boldsymbol{B}_{k_T} \boldsymbol{x}_{a(k_T)} & + \boldsymbol{A}_{k_T} \boldsymbol{x}_{k_T} & = \boldsymbol{b}_{k_T}, & k_T \in \mathscr{K}_T
\end{array}$$

$$\boldsymbol{l}_1 \le \boldsymbol{x}_1 \le \boldsymbol{u}_1,\ \boldsymbol{l}_{k_t} \le \boldsymbol{x}_{k_t} \le \boldsymbol{u}_{k_t},\ k_t \in \mathscr{K}_t,\ t = 2, \dots, T. \tag{4}$$

The *path probabilities* $p_{k_t} > 0\, \forall k_t$, $\sum_{k_t \in \mathscr{K}_t} p_{k_t} = 1$, $t = 2, \dots T$, of partial sequences of coefficients are probabilities of realizations of $\omega^{t-1,\bullet}\, \forall t$. They may be obtained by stepwise multiplication of the marginal probabilities p_{k_2} by the *conditional arc (transition) probabilities*, say, $\pi_{k_{\tau-1}k_\tau}$, $\tau = 3, \dots, t$. Probabilities p^k of individual scenarios ω^k, $k = 1, \dots, K$, are equal to the corresponding path probabilities p_{k_T}.

Nonanticipativity constraints are included in an implicit way. Notice, that (3)–(4) may correspond also to a T-period two-stage stochastic program based on the same scenarios: Except for the root, there is only one descendant $d(k_t)$ of each of t-th stage nodes, that is, the transition probabilities $\pi_{k_t,d(k_t)} = 1\, \forall k_t \in \mathscr{K}_t, t = 2, \ldots, T-1$. Scenarios are identified by sequences $\{k_2, \ldots, k_T\}$ such that $k_t \in \mathscr{K}_t,\ k_{t+1} = d(k_t)\,\forall t$ and the objective function (3) may be simplified to

$$\boldsymbol{c}_1^\top \boldsymbol{x}_1 + \sum_{k_T \in \mathscr{K}_T} p_{k_T} \left[\boldsymbol{c}_{k_2}^\top \boldsymbol{x}_{k_2} + \boldsymbol{c}_{k_3}^\top \boldsymbol{x}_{k_3} + \cdots + \boldsymbol{c}_{k_T}^\top \boldsymbol{x}_{k_T} \right]. \tag{5}$$

Problem (4), (5) is called the *two-stage relaxation* of MSLP (3)–(4) and it corresponds to $\mathscr{F}_t = \mathscr{F}_1\,\forall t$.

With explicit inclusion of nonanticipativity constraints, the scenario-based multiperiod or multistage stochastic programs with linear constraints can be again written as a large-scale deterministic program: Given scenario ω^k denote by $\boldsymbol{c}(\omega^k)$ the vector composed of all corresponding coefficients, say, $\boldsymbol{c}_1, \boldsymbol{c}_{k_t}, t = 2, \ldots, T$, in the objective function, by $\boldsymbol{A}(\omega^k)$ the matrix of all coefficients of system of constraints (4) for scenario ω^k, and, similarly, by $\boldsymbol{b}(\omega^k),\ \boldsymbol{l}(\omega^k),\ \boldsymbol{u}(\omega^k)$ the vectors composed of right-hand sides in (4) and bounds of the box constraints for scenario ω^k. The *scenario-splitted* form of the T-stage stochastic linear program is

$$\min_{\mathscr{X} \cap \mathscr{C}} \left\{ \sum_{k=1}^{K} p^k \boldsymbol{c}(\omega^k)^\top \boldsymbol{x}^k \mid \boldsymbol{A}(\omega^k)\boldsymbol{x}(\omega^k) = b(\omega^k),\ \boldsymbol{l}(\omega^k) \le \boldsymbol{x}(\omega^k) \le \boldsymbol{u}(\omega^k)\,\forall k \right\}. \tag{6}$$

Set $\mathscr{X}$ is defined by deterministic constraints on $\boldsymbol{x}_t(\omega^k)\,\forall t, k$, $\mathscr{C}$ by the nonanticipativity conditions, and $\boldsymbol{x}(\omega^k)$ is the corresponding decision vector composed of stage related subvectors $\boldsymbol{x}_t(\omega^k)\forall t$. For two-stage stochastic programs the nonanticipativity constraints boil down to the requirement that the first-stage decisions must be scenario independent, i.e. $\boldsymbol{x}_1(\omega^k) = \boldsymbol{x}_1(\omega^{k'})\,\forall k, k'$. Similar constraints guarantee that the t-th stage decisions based on the same history are equal. Such constraints can be expressed as $\boldsymbol{x} = \boldsymbol{U}\boldsymbol{x}$ where $\boldsymbol{x}$ contains carefully grouped components of all decision vectors $\boldsymbol{x}(\omega^k)$ and $\boldsymbol{U}$ is a 0-1 matrix.

Besides the formulation of goals and constraints and identification of the driving random process, building a scenario-based multiperiod or multistage stochastic program requires specification of the horizon, stages and generation of the input in the form of a scenario fan or a scenario tree. The choice of stages, of the branching scheme, of scenarios and their probabilities influence the optimal first-stage decision and the overall optimal value. To use multiperiod two-stage model or to assign one stage to each of possible discretization points are two extreme cases. Requirements of various applications may lead to different topologies of the decision points: With a fixed time discretization of the data process $\omega = (\omega_t,\ t = 1, \ldots, T-1)$ the stages may be allocated to selected time points, say $\tau_1 < \cdots < \tau_D < T$. The decisions are made at $\tau_d,\ d = 1, \ldots, D$, using the past information $\omega^{\tau_d - 1, \bullet}$ and the probabilistic specification. Similarly as the second stage decisions for multiperiod two-stage stochastic programs, all decisions at time points t *between* stages $\tau_d,\ \tau_{d+1}$ are made at $t = \tau_d$ using the past information up to τ_d. The formulation exploits then a fixed suitable coarser structure (filtration) $\{\mathscr{F}_d,\ d = 1, \ldots, D\}$, $\mathscr{F}_d \subseteq \mathscr{F}$ defined by the data available at time τ_d which corresponds to stage d. The whole procedure has been developed in Dempster et al. (2000) for a specific application, see also Kall and Mayer (2005) for the corresponding scenario tree construction.

We shall assume in the sequel that the *horizon has been fixed*; for numerical experiments and for a discussion of various choices related with the nature of the decision problem see e.g. Frauendorfer and Haarbrücker (2000) and Bertocchi et al. (2006b), Dupačová et al. (2002), respectively. Also selection of stages follows sometimes from the problem formulation (e.g. the dates of maturity of bonds, cf. Frauendorfer and Marohn (2004), yearly rebalancing of the fund Dempster et al. (2006) or expiration dates of options) but more frequently, stages are fixed ad hoc, by application of heuristic rules and/or experience and regarding software and computer facilities.

Hence, for an already chosen horizon, the crucial step is to relate the time instants and stages. It is a problem specific task and we will discuss it mainly for financial applications. There are some general recommendations: Accept unequal lengths of time periods between subsequent stages, starting with a short first period. Together with repeated rolling the model over time, this may replace well the full dynamics of the decision process even for problems with a few stages. Another suggestion Grinold (1986) is to break the problem with a long (possibly infinite) horizon into three phases: To use the scenario tree structure for $1 \le t \le T_1$, to design just one descendant from each node for $T_1 + 1 \le t \le T_2$ (i.e. the horse-tail structure) and to aggregate the rest of the process into one additional stationary stage.

A detailed analysis of the origin and of the initial structure of the solved problem may be exploited to aggregate the stages, may help to prune the tree or to extend it for other out-of-sample scenarios or branches.

To generate the required input means to approximate the probability distribution of the random factors bearing in mind the chosen type of the model and to get the scenario-dependent coefficients according to the assigned tree structure. As a rule, the selected procedure should take into account the chosen type of model, the level of the existing information, software and hardware possibilities. Finally, a validation of results is necessary: an approximate stochastic program is solved instead of the underlying "true" decision problem and compromises between the size of the resulting problem and the desired precision of the results cannot be avoided.

It has been observed that various theoretical results valid for two-stage stochastic programs do not carry over to the multistage case (e.g. Dupačová (2004), Heitsch et al. (2006), Shapiro (2003, 2008)). At the same time, input generation (e.g. generation of a scenario tree instead of a fan of scenarios) and the numerical solution of multistage programs is substantially more complicated. Hence, a natural question is how many stages and what topology of stages should be used, why to use multistage stochastic programs at all and how much we loose when simplifying them to their multiperiod two-stage variant by relaxation of nonanticipativity constraints. As the set of feasible decisions gets enlarged, the optimal value of the two-stage relaxation (4), (5) *based on identical data* provides a lower bound of the optimal value of the original multistage problem.

Depending on the context, analysis of the impact of including additional stages on the results may be classified as quantitiative stability, postoptimality, stress testing or output analysis. There exist several numerical studies comparing different topologies of stages and branching structures mostly for various financial applications of stochastic programming, e.g. Bertocchi et al. (2006b), Dempster and Thompson (2002), Dempster et al. (2006), Nielsen and Poulsen (2004), Nielsen and Zenios (1996). A detailed analysis of special financial problems supplemented by numerical experiments may add to the heuristics, cf. Blomwall and Shapiro (2007).

We shall approach these problems via the contamination technique. Basic ideas will be briefly explained in section 2 with emphasis on the results which may be used to testing the topology of stages for multistage stochastic linear programs. The approach is then applied to stress testing the structure of multistage stochastic programs with polyhedral convex risk objectives cf. Eichhorn and Römisch (2005) and of a multistage bond portfolio management problem Bertocchi et al. (2006a,b).

2 Quantitative analysis based on contamination

2.1 Contamination technique

Contamination approach was initiated in mathematical statistics as one of the tools for analysis of robustness of estimators with respect to deviations from the assumed probability distribution and/or its parameters. It goes back to von Mises and the concepts are briefly described e.g. in Serfling (1980). In stochastic programming, it was developed in a series of papers up to results applicable to two-stage stochastic linear programs, e.g. Dupačová (1986, 1996), and to the first ideas dealing with the multistage case Dupačová (1995, 2006a).

Contamination technique is a quantitative stability method of modeling and quantifying the impact of perturbations of the underlying probability distribution on the results of the considered stochastic programming problem. It is suitable, e.g. for the examination of the influence of including additional scenarios on the optimal value function.

Let $\mathcal{X} \subset \mathbb{R}^n$ be nonempty, closed and $P \in \mathcal{P}$, the set of all Borel probability distributions on $\Omega \subset \mathbb{R}^m$. We shall deal with stochastic programs which can be formulated as

$$\min_{\boldsymbol{x} \in \mathcal{X}} F(\boldsymbol{x}, P) \tag{7}$$

with $\mathcal{X}$ independent of P and the objective function F *concave in* $P \in \mathcal{P}$. The concavity assumption is fulfilled for various common types of stochastic programs including those with risk objectives or robust optimization problems of Mulvey et al. (1995). Expectation type objective functions

$$F(\boldsymbol{x}, P) = E_P f(\boldsymbol{x}, \omega) = \int_{\Omega} f(\boldsymbol{x}, \omega) P(d\omega)$$

which are *linear* in P are the most frequent special case.

Via contamination, robustness analysis with respect to changes in P gets reduced to a much simpler analysis with respect to a scalar parameter λ : Assume that (7) was solved for an already constructed scenario tree corresponding to the discrete probability distribution P. Possible changes in probability distribution P are modeled using *contaminated distributions* P_λ,

$$P_\lambda := (1-\lambda)P + \lambda Q,\ \lambda \in [0, 1]$$

with $Q \in \mathscr{P}$ another *fixed* probability distribution. Limiting the analysis to a selected direction only, the results are directly applicable but they are less general than quantitative stability results with respect to arbitrary (but small) changes in P summarized e.g. by Römisch in Chapter 8 of Ruszczyński and Shapiro (2003).

Let $\varphi(P)$ and $\mathscr{X}^*(P)$ be the optimal value function and the optimal set mapping of (7) and $\varphi_{PQ}(\lambda)$ and $\mathscr{X}^*_{PQ}(\lambda)$ be the optimal value function and the optimal set mapping of (7) when using contaminated distribution P_λ instead of P. For $\mathscr{X}$ independent of P concavity of $F(\boldsymbol{x}, P)$ in P ensures that $\varphi_{PQ}(\lambda)$ is concave as well. The Gâteaux derivative

$$\varphi'(P; Q-P) := \lim_{t\to 0^+} \frac{\varphi(P_t) - \varphi(P)}{t}$$

of $\varphi(P)$ in the direction of $Q-P$ can be exploited to construct bounds on the optimal value function $\varphi_{PQ}(\lambda)$ using the values of φ_{PQ} only for $\lambda = 0$ or $\lambda = 1$, i.e. the optimal values for probability distributions P and Q. The following inequality is a simple consequence of concavity of φ_{PQ} :

$$\varphi_{PQ}(0) + \lambda\varphi'_{PQ}(0^+) \geq \varphi_{PQ}(\lambda) \geq (1-\lambda)\varphi_{PQ}(0) + \lambda\varphi_{PQ}(1), \quad \lambda \in [0, 1]. \quad (8)$$

It provides bounds on the *relative change* of the optimal value function of the considered stochastic program due to contamination:

$$\varphi_{PQ}(1) - \varphi_{PQ}(0) \leq \frac{\varphi_{PQ}(\lambda) - \varphi_{PQ}(0)}{\lambda} \leq \varphi'_{PQ}(0^+), \quad \lambda \in [0, 1].$$

Any use of these bounds is of course conditioned on the existence of Gâteaux derivative and our ability to compute it. If the objective function $F(\boldsymbol{x}, P)$ is linear in P,

$$F(\boldsymbol{x}, P_\lambda) := \int_\Omega f(\boldsymbol{x}, \omega) P_\lambda(d\omega) = (1-\lambda)F(\boldsymbol{x}, P) + \lambda F(\boldsymbol{x}, Q)$$

is linear in λ and its derivative with respect to λ equals $\frac{dF(\boldsymbol{x},\lambda)}{d\lambda} = F(\boldsymbol{x}, Q) - F(\boldsymbol{x}, P)$. The following result follows by application of Theorem 1, Chapter 3 of Danskin (1967):

Lemma 1 *Let $\mathscr{X}$ be compact and $F(\boldsymbol{x}, P)$ linear in P. Let $P, Q \in \mathscr{P}$ be such that $F(\bullet, P)$, $F(\bullet, Q)$ are finite continuous functions. Then the Gâteaux derivative of the optimal value function φ at P in the direction $Q-P$ exists and can be computed as*

$$\varphi'(P; Q - P) = \min_{\boldsymbol{x} \in \mathscr{X}^*(P)} F(\boldsymbol{x}, Q) - \varphi(P). \tag{9}$$

If $\boldsymbol{x}^*(P)$ is the *unique* optimal solution, $\varphi'(0^+) = F(\boldsymbol{x}^*(P), Q) - \varphi(0)$, i.e. the *local change of the optimal value function caused by a small change of P in direction $Q - P$ is the same as that of the objective function at $\boldsymbol{x}^*(P)$*. If there are multiple optimal solutions, each of them leads to an upper bound

$$\varphi'(P; Q - P) \leq F(\boldsymbol{x}(P), Q) - \varphi(P),\ \boldsymbol{x}(P) \in \mathscr{X}^*(P). \tag{10}$$

Contamination bounds can be then relaxed to

$$(1 - \lambda)\varphi(P) + \lambda F(\boldsymbol{x}(P), Q) \geq \varphi(P_\lambda) \geq (1 - \lambda)\varphi(P) + \lambda\varphi(Q) \tag{11}$$

valid for an arbitrary $\boldsymbol{x}(P) \in \mathscr{X}^*(P)$ and $\lambda \in [0, 1]$. If $\boldsymbol{x}(P)$ is an ε-optimal solution of (7) for probability distribution Q then the difference of the upper and lower bound in (11) is less than or equal to $\lambda\varepsilon$.

Similarly, one may think of P_λ as a result of contamination of Q by P and derive (right) upper bounds based on $\varphi'(Q; P - Q)$. In this context, (10) will be called a left upper bound.

Concavity of the optimal value function $\varphi_{PQ}(\lambda)$ is important for constructing the above global bounds which hold true for all $\lambda \in [0, 1]$. It cannot be obtained, in general, when the set $\mathscr{X}$ depends on the probability distribution P. In such cases and under additional assumptions, only local stability results can be proved. On the other hand, Lemma 1 can be generalized in various ways:

- If F is convex in $\boldsymbol{x}$ and continuous for all probability distributions belonging to a neighborhood of P, the same result and formula (8) follow from Theorem 16 of Gol'štejn (1972). Using a similar approach, it can be proved even for local minimization; see Teorem 8 of Dupačová (1990).
- For F concave in P and convex in $\boldsymbol{x}$ additional assumptions are needed to get persistence and stability of the parametric program $\min_{\boldsymbol{x} \in \mathscr{X}} F(\boldsymbol{x}, P_\lambda)$, to prove the existence and the form of the Gâteaux derivative, now

$$\varphi'(P; Q - P) = \min_{\boldsymbol{x} \in \mathscr{X}^*(P)} \frac{d}{d\lambda} F(\boldsymbol{x}, P_\lambda)|_{\lambda = 0^+} \tag{12}$$

  cf. Dupačová (1990, 1996, 2006b).

Contamination bounds (8), (11) help to *quantify* the change in the optimal value due to the considered perturbations of (7). They were applied in Dupačová (2006b), Dupačová and Polívka (2007) and Bertocchi et al. (2006a), Dupačová et al. (1998), to stress test of CVaR and of multiperiod two-stage bond portfolio management problems, respectively. An application to two-stage stochastic linear programs with polyhedral risk objectives is derived in Dupačová (2008).

It is also possible to prove the existence of Gâteaux derivative of the optimal value function when the set of feasible decisions depends on P and is given by explicit constraints. For problems convex in $\boldsymbol{x}$ this follows from Theorem 17 of Gol'štejn (1972), otherwise one has to rely on various results on stability of parametric programs; see e.g. Dupačová (1986, 1990), Shapiro (1990). Using classical stability results, Gâteaux differentiability of *optimal solutions* of (7) can be proved if the optimal solution of (7) is unique and if the multifunction $\mathscr{X}^*$ is in fact a function on a neighborhood of P, e.g. Dupačová (1986, 1990), Shapiro (1990). There exist also differential stability results for the optimal solution multifunction; again, they are based on properties of parametric programs obtained for contaminated probability distributions and are valid under specific assumptions concerning the probability distribution P and the structure of the stochastic program; an example is Dentcheva and Römisch (2000).

2.2 Contamination for multistage stochastic linear programs

Also multistage stochastic programs can be formulated as (7), with $\mathscr{X}$ the fixed set of feasible first-stage decisions, recall (1). Still, a note of warning is needed: In (7), the random objective $f(\cdot, \cdot)$ is a given function whereas the random objective $\varphi_1(\cdot, \cdot)$ in (1) changes when the topology of stages, i.e. the filtration, gets changed. This indicates that for a *fixed topology of stages* contamination with respect to additional scenarios goes its usual way. Indeed, the corresponding contamination bounds were derived in Dupačová (1995) for multistage stochastic linear programs with respect to additional out-of-sample scenarios which increase the branching number of selected nodes of the scenario tree but do not change the topology of stages; see also Dupačová (2006a). The results were applied to multistage problems with a *fixed* topology of stages in Bertocchi et al. (2006a), Dupačová and Polívka (2004). On the other hand, applications to MSLP with a varying topology of stages are not straightforward. Let us introduce first a motivating example.

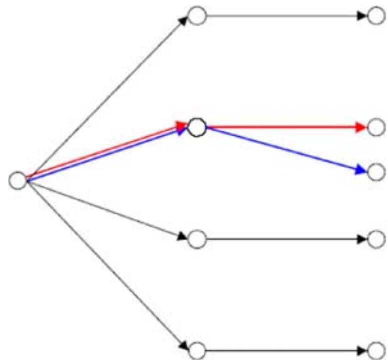
Example 1 Let us describe briefly the stochastic dedicated bond portfolio selection problem modeled as two-stage multiperiod stochastic linear program, see e.g. Dupačová et al. (2002), Shapiro (1988). The goal is to minimize the cost of portfolio so that the portfolio's cash flows cover future liabilities. The problem is solved over T time periods for a portfolio consisting of N bonds and for probability distribution P carried by a fan of selected scenarios of interest rates, $T-1$ dimensional vectors $\omega^k = \{\omega_t^k, t = 1, \ldots, T-1\}$, $k = 1, \ldots, K$, with probabilities p^k; ω_0 is a known initial interest rate valid for the first period.

We assume that the vector of bonds' acquisition prices $\boldsymbol{c} = (c_1, \ldots, c_N)^\top$ and the T-vectors $\boldsymbol{f}_n$, $n = 1, \ldots, N$, $\boldsymbol{l}$ of the cash flows and liabilities are known. The sought composition of portfolio $\boldsymbol{x} = (x_1, \ldots, x_N)^\top$ has to be scenario independent whereas the surpluses y_t^{+k} and short-term shortfalls y_t^{-k} in time periods $t = 1, \ldots, T$ may adapt to individual scenarios and are constructed at once for all subsequent periods $t = 2, \ldots, T$. In addition, there is a penalty $\sum_k p_k \boldsymbol{q}^{k\top} \boldsymbol{y}^{-k}$ for borrowing included into the objective function and a spread δ between the rate for borrowing and lending.

The resulting problem is

$$\text{minimize } \boldsymbol{c}^\top \boldsymbol{x} + y_0^+ + \sum_k p_k \boldsymbol{q}^{k\top} \boldsymbol{y}^{-k}$$

Fig. 1 Contaminated scenario tree



subject to

$$\sum_{n=1}^{N} f_{tn}x_n+(1+\omega_{t-1}^k)y_{t-1}^{+k}-y_t^{+k}-(1+\omega_{t-1}^k+\delta)y_{t-1}^{-k}+y_t^{-k}=l_t,\quad \forall k, t=1,\dots,T$$

with $y_0^{+k} = y_0^+$, $y_0^{-k} = y_T^{-k} = 0$, $\omega_0^k = \omega_0\ \forall k$ and nonnegativity of all vectors $\boldsymbol{x}$, $\boldsymbol{y}^{+k}$, $\boldsymbol{y}^{-k}$, $k = 1,\dots,K$.

The alternative probability distribution Q corresponds to a possible call option at $t = t_1 > 1$ for certain bond, under some of scenarios. It provides an optimal first stage decision if the call option is exercised. The interest rate scenarios are identical with those for P but the related cash flows of the bond with call option differ—the full nominal value plus coupon and premium get paid at t_1 and zero cash flows follow in subsequent periods. Of course, the first stage investment decisions for Q will be different.

Contaminated probability distribution P_λ takes into account both possibilities and the contamination parameter λ reflects the belief that the call option will not be exercised. To get a 3-stage SLP one keeps the time horizon T and the time discretization and includes an additional decision point at t_1, see Figure 1. This means, inter alia, that the system of linear constraints written for the pooled set of scenarios corresponding to P_λ must be extended for the nonanticipativity condition:

Decisions $\boldsymbol{x}$ and y_t^{+k}, y_t^{-k} at $t < t_1$ cannot count upon the outcome of the option at time t_1.

Consider now a general scenario-based multistage stochastic program with stages allocated at certain time points $\{\tau_1,\dots,\tau_D\} \subset \{t_1,\dots,T\}$ which may involve an additional stage (not additional time discretization point!) at $t = t_1 \notin \{\tau_1,\dots,\tau_D\}$. To include an additional stage means to reflect in the arborescent form of MSLP (3)–(4) for contaminated probability distribution P_λ additional nonanticipativity conditions: For $t < t_1$, all coefficients and decision variables for P and Q are equal. The corresponding subsystem of constraints in (4), briefly

$$\boldsymbol{B}_{k_t}\boldsymbol{x}_{a(k_t)} + \boldsymbol{A}_{k_t}\boldsymbol{x}_{k_t} = \boldsymbol{b}_{k_t},\quad \boldsymbol{l}_{k_t} \le \boldsymbol{x}_{k_t} \le \boldsymbol{u}_{k_t},\quad k_t \in \mathscr{K}_t,\ t \le t_1 \tag{13}$$

(with the first term $\boldsymbol{B}_{k_1}\boldsymbol{x}_{a(k_1)}$ missing for $k_1 = 1$) will be called *common constraints*.

At $t = t_1$ realization of an additional random factor is observed. Assume that there are two possible outcomes which correspond to two alternative probability distributions P, Q. The initial constraints for scenarios ω^k, atoms of P,

$$\boldsymbol{B}_{k_t}\boldsymbol{x}_{a(k_t)} + \boldsymbol{A}_{k_t}\boldsymbol{x}_{k_t} = \boldsymbol{b}_{k_t}, \quad \boldsymbol{l}_{k_t} \le \boldsymbol{x}_{k_t} \le \boldsymbol{u}_{k_t}, \quad k_t \in \mathscr{K}_t,\ t > t_1, \tag{14}$$

are kept and will be called P-system. Another Q-system of constraints for scenarios ω^h, atoms of Q,

$$\boldsymbol{B}_{h_t}\boldsymbol{x}_{a(h_t)} + \boldsymbol{A}_{h_t}\boldsymbol{x}_{h_t} = \boldsymbol{b}_{h_t}, \quad \boldsymbol{l}_{h_t} \le \boldsymbol{x}_{h_t} \le \boldsymbol{u}_{h_t}, \quad h_t \in \mathscr{H}_t,\ t > t_1 \tag{15}$$

will be attached. The ancestors $a(h_{t_1+1})$ and the decision variables $\boldsymbol{x}_{a(h_{t_1+1})}$ in the Q-system come from the common constraints for corresponding indices k_{t_1} and variables $\boldsymbol{x}_{a(h_{t_1+1})} \sim \boldsymbol{x}_{k_{t_1}}$. Thus using the *pooled set of scenarios* from P and Q we get a *fixed* set $\mathscr{X}$ of solutions which fulfil the system of linear constraints (13)–(15). The contaminated stochastic program is a *linear parametric program* with parameter λ only in the objective function:

$$\begin{aligned} F(\boldsymbol{x}, \lambda) := \boldsymbol{c}_1^\top \boldsymbol{x}_1 + \sum_{t=2}^{t_1} \sum_{k_t \in \mathscr{K}_t} p_{k_t} \boldsymbol{c}_{k_t}^\top \boldsymbol{x}_{k_t} + (1-\lambda) \sum_{t=t_1+1}^{T} \sum_{k_t \in \mathscr{K}_t} p_{k_t} \boldsymbol{c}_{k_t}^\top \boldsymbol{x}_{k_t} \\ + \lambda \sum_{t=t_1+1}^{T} \sum_{h_t \in \mathscr{H}_t} q_{h_t} \boldsymbol{c}_{h_t}^\top \boldsymbol{x}_{h_t} \end{aligned} \tag{16}$$

minimized with respect to (13)–(15). In the last term of (16), q_{h_t} are the path probabilities in the scenario tree for probability distribution Q.

The optimal value of (16) with respect to (13)–(15) is denoted $\varphi_{PQ}(\lambda)$ and $\mathscr{X}^*_{PQ}(\lambda)$ is the set of optimal solutions of (13)–(16). The symbols $\varphi(P)$, $\mathscr{X}^*(P)$, $\varphi(Q)$, $\mathscr{X}^*(Q)$ are kept for optimal values and sets of optimal solutions of the two MSLP obtained for P and Q separately; notice that $\varphi_{PQ}(0) = \varphi(P)$, $\varphi_{PQ}(1) = \varphi(Q)$.

Proposition 1 *Assume that the sets* $\mathscr{X}^*_{PQ}(\lambda)$ *are nonempty for all* $\lambda \in [0, 1]$ *and* $\mathscr{X}^*_{PQ}(0)$ *is bounded. Then the optimal value function* $\varphi_{PQ}(\lambda)$ *is concave on* $[0, 1]$ *and contamination bounds* (8) *follow with*

$$\varphi'_{PQ}(0^+) = \min_{\boldsymbol{x} \in \mathscr{X}^*_{PQ}(0)} F(\boldsymbol{x}, 1) - \varphi_{PQ}(0).$$

The proof is an adaptation of results on existence and form of directional derivatives of the optimal value function of perturbed linear programs, cf. Chapter 3.5 of Gol'štejn and Yudin (1966), to the parametric *linear program* (16), (13)–(15).

To get upper bound (10) for the derivative means to evaluate $F(\boldsymbol{x}(0), 1)$ at an arbitrary optimal solution $\boldsymbol{x}(0)$ of the contaminated problem (16), (13)–(15) with $\lambda = 0$. Optimal solutions consist of components $\boldsymbol{x}_{k_t}(P)$, $k_t \in \mathscr{K}_t$, $t = 1, \ldots, T$, of an optimal solution of (3)–(4) complemented by components of an arbitrary *feasible* solution, say, $\boldsymbol{x}^*_{h_t}$, $t > t_1$, of the related Q-system (15), i.e. $\boldsymbol{x}(0) = \{\boldsymbol{x}_{k_t}(P)\ \forall k_t,\ \boldsymbol{x}^*_{h_t}\ \forall h_t\}$.

As we mentioned above, for $t = t_1 + 1$ the ancestors $a(h_{t_1+1})$ and the corresponding decision variables $\boldsymbol{x}_{a(h_{t_1+1})}$ in the Q-system come from the common constraints.

For $\boldsymbol{x}(0)$ and $\lambda = 1$, the sought value $F(\boldsymbol{x}(0), 1)$ of the objective function is

$$\boldsymbol{c}_1^\top \boldsymbol{x}_1(P) + \sum_{t=2}^{t_1} \sum_{k_t \in \mathscr{K}_t} p_{k_t} \boldsymbol{c}_{k_t}^\top \boldsymbol{x}_{k_t}(P) + \sum_{t=t_1+1}^{T} \sum_{h_t \in \mathscr{H}_t} q_{h_t} \boldsymbol{c}_{h_t}^\top \boldsymbol{x}_{h_t}^*$$

and the optimal value $\varphi(P) = \varphi_{PQ}(0)$ equals

$$\boldsymbol{c}_1^\top \boldsymbol{x}_1(P) + \sum_{t=2}^{t_1} \sum_{k_t \in \mathscr{K}_t} p_{k_t} \boldsymbol{c}_{k_t}^\top \boldsymbol{x}_{k_t}(P) + \sum_{t=t_1+1}^{T} \sum_{k_t \in \mathscr{K}_t} p_{k_t} \boldsymbol{c}_{k_t}^\top \boldsymbol{x}_{k_t}(P).$$

The difference $F(\boldsymbol{x}(0), 1) - \varphi_{PQ}(0)$ is the upper bound for the derivative $\varphi'_{PQ}(0^+)$ and we have, cf. (10),

$$\varphi'_{PQ}(0^+) \le \sum_{t=t_1+1}^{T} \left[\sum_{h_t \in \mathscr{H}_t} q_{h_t} \boldsymbol{c}_{h_t}^\top \boldsymbol{x}_{h_t}^* - \sum_{k_t \in \mathscr{K}_t} p_{k_t} \boldsymbol{c}_{k_t}^\top \boldsymbol{x}_{k_t}(P) \right]. \tag{17}$$

To get a tighter upper bound, one may insert for $\boldsymbol{x}_{h_t}^*$ the minimizers of

$$\sum_{t=t_1+1}^{T} \sum_{h_t \in \mathscr{H}_t} q_{h_t} \boldsymbol{c}_{h_t}^\top \boldsymbol{x}_{h_t} \tag{18}$$

subject to constraints (15) linked with the corresponding decisions $\boldsymbol{x}(P)$ through ancestors k_{t_1} of $h_{t_1+1} : \boldsymbol{x}_{a(h_{t_1+1})} = \boldsymbol{x}_{k_{t_1}}(P)$. Under assumption of an identical stages topology for P, Q the tighter bound (17)–(18) corresponds then to evaluation of the objective function at the optimal solution $\boldsymbol{x}(P)$ but for the probability distribution Q. Moreover, the same type of upper bounds appears also in the context of contaminated two-stage multiperiod SLP. It is obtained for $t_1 = 1$ in (17)–(18).

Similar theorems hold true for other instances of scenario-based MSLP and for more complex changes of their structure. They may be obtained also by application of the contamination technique to the nested form (1)–(2) of the stochastic program, see section 3.2, and can be extended to certain scenario-based nonlinear problems, e.g. to multistage stochastic programs with convex risk objectives, see section 3.1. The suggested bounds aim at reduction of computational efforts in comparison with solving the full parametric program (13)–(16) for $\lambda \in (0, 1)$.

Notice that working with the scenario-splitted form (6) is convenient only under special circumstances. In general it would mean to accept changes of the system $\boldsymbol{x} = \boldsymbol{U}\boldsymbol{x}$ if the topology of stages varies which is a substantial change of the resulting deterministic program.

3 Applications

3.1 Multistage polyhedral risk objective

Consider now a multiperiod decision process over a discrete time horizon $t = 1, \ldots, T$, and assume that we are supposed to valuate and manage the related risks, say z_j, at certain prescribed time instances $1 = t_0 < t_1 < \cdots < t_J = T$. The sequence of risks $\{z_{t_j}(\omega), j = 1, \ldots, J\}$ is nonanticipative and $z_j(\omega)$, the risk at t_j, is measurable with respect to the filtration $\mathscr{F}_{t_j-1}$ of $(\Omega, \mathscr{F}, P)$ generated by events preceding t_j.

Definition 1 (**Eichhorn and Römisch** (**2005**)) A risk measure R on $\times_{j=1}^{J} L_p(\Omega, \mathscr{F}_{t_j-1}, P)$, $p \in [1, +\infty)$, is called a **multiperiod polyhedral risk measure** if there exist numbers $k_j \in \mathbb{N}$ and vectors $\boldsymbol{d}_j \in \mathbb{R}^{k_j}$, $j = 1, \ldots, J$, $\boldsymbol{w}_{j\tau} \in \mathbb{R}^{k_{j-\tau}}$, $j = 1, \ldots J$, $\tau = 0, \ldots, j-1$, a polyhedral set $\mathscr{Y}_1 \subseteq \mathbb{R}^{k_1}$ and polyhedral cones $\mathscr{Y}_j \subseteq \mathbb{R}^{k_j}$, $j = 2, \ldots, J$, such that

$$R(\boldsymbol{z}, P) = \inf E_P \sum_{j=1}^{J} \boldsymbol{d}_j^\top \boldsymbol{y}_j(\omega) \tag{19}$$

subject to

$$\boldsymbol{y}_j \in L_p(\Omega, \mathscr{F}_{t_j-1}, P), \boldsymbol{y}_j(\omega) \in \mathscr{Y}_j \quad \text{a.s. for } j = 1, \ldots, J, \tag{20}$$

$$\sum_{\tau=0}^{j-1} \boldsymbol{w}_{j\tau}^\top \boldsymbol{y}_{j-\tau}(\omega) = z_j(\omega) \quad \text{a.s. for } j = 2, \ldots, J. \tag{21}$$

Thus the polyhedral risk measure equals the optimal value of a multistage stochastic program which does no more keep the staircase structure of (1)–(2). Such multistage stochastic programs can be transformed to those of the staircase form, see Kall and Mayer (2005) for details. The random parameters $z_j(\omega)$ occur only on the right-hand sides of constraints (21) and they can be interpreted as accumulated losses at certain stages of the multistage stochastic linear program, i.e.

$$z_j(\omega) := \boldsymbol{c}_1^\top \boldsymbol{x}_1 + \sum_{t=2}^{t_j} \boldsymbol{c}_t(\omega^{t-1,\bullet})^\top \boldsymbol{x}_t(\omega^{t-1,\bullet}).$$

The losses need not be monitored at every stage t, but only on a subset $\{t_j, j = 1, \ldots, J\} \subseteq \{2, \ldots, T\}$.

A fixed filtration is assumed. Nevertheless, through the choice of $\boldsymbol{d}_j, \mathscr{Y}_j, \boldsymbol{w}_{j\tau}$ in (19)–(21) various polyhedral risk measures can be obtained. The average CVaR, $R(\boldsymbol{z}, P) = \sum_j \mu_j \text{CVaR}_{\alpha_j}(z_j, P)$ with weights $\mu_j \geq 0\, \forall j$, $\sum_j \mu_j = 1$ is one of them; see Eichhorn and Römisch (2005) for other examples.

For minimization of a multiperiod polyhedral risk objective $R(\boldsymbol{z}, P)$, the expectation in the initial stochastic program, such as (1)–(2), is replaced by the multiperiod polyhedral risk measure (19)–(21) with

$$z_j(\omega) = \boldsymbol{c}_1^\top \boldsymbol{x}_1 + \sum_{t=2}^{t_j} \boldsymbol{c}_t(\omega^{t-1,\bullet})^\top \boldsymbol{x}_t(\omega^{t-1,\bullet})$$

or a shortcut procedure is applied. The minimal risk $\varphi_R(P)$ is then equal to the *optimal value of multistage stochastic linear program*

$$\min E_P \sum_{j=1}^{J} \boldsymbol{d}_j^\top \boldsymbol{y}_j(\omega^{t_j-1,\bullet}) \tag{22}$$

subject to

$$\boldsymbol{A}_{t1}(\omega^{t-1,\bullet})\boldsymbol{x}_1 + \sum_{\tau=2}^{t} \boldsymbol{A}_{t\tau}(\omega^{t-1,\bullet})\boldsymbol{x}_\tau(\omega^{\tau-1,\bullet}) = \boldsymbol{b}_t(\omega^{t-1,\bullet}), \quad t = 2, \ldots, T,$$

$$\boldsymbol{x}_1 \in \mathscr{X}_1,\ \boldsymbol{x}_t(\omega^{t-1,\bullet}) \in \mathscr{X}_t, \quad t = 2, \ldots, T, \text{a.s.}$$

$$\boldsymbol{y}_1 \in \mathscr{Y}_1,\ \boldsymbol{y}_j(\omega^{t_j-1,\bullet}) \in \mathscr{Y}_j, \quad j = 2, \ldots, J, \text{a.s.}$$

$$\sum_{\tau=0}^{j-1} \boldsymbol{w}_{j\tau}^\top \boldsymbol{y}_{j-\tau}(\omega^{t_j-1,\bullet}) = z_j(\omega^{t_j-1,\bullet}) \quad \text{a.s. for } j = 2, \ldots, J, \tag{23}$$

$$z_t(\omega^{t-1,\bullet}) := \boldsymbol{c}_1^\top \boldsymbol{x}_1 + \sum_{\tau=2}^{t} \boldsymbol{c}_\tau(\omega^{\tau-1,\bullet})^\top \boldsymbol{x}_\tau(\omega^{\tau-1,\bullet})\ \forall t. \tag{24}$$

The optimal decision $\boldsymbol{x}_{1R}(P)$ is the first part of the *risk-minimizing first-stage decision* $[\boldsymbol{x}_{1R}(P), \boldsymbol{y}_{1R}(P)]$ of (22)–(24).

For a fixed filtration in the scenario-based form of the multiperiod polyhedral risk measure (19)–(21) the results of Dupačová (1995) may be used in a straightforward way to stress testing the obtained value of the polyhedral risk measure with respect to additional, out-of-sample scenarios or branches of the scenario tree. The same applies also to the *minimal* multiperiod polyhedral risks, i.e. to the minimal values of multistage stochastic programs with polyhedral risk objectives if neither the filtration $\mathscr{F}_t,\ t = 1, \ldots, T-1$, related with the original stochastic program nor the filtration $\mathscr{F}_{t_j},\ j = 2, \ldots, J$, appearing in definition of the multiperiod risk measure are changed.

For illustration analyze the changes due to inclusion of one additional scenario $\omega^* = (\omega_1^*, \ldots, \omega_{T-1}^*)$ from the root of the scenario tree, i.e. the contaminating probability distribution is degenerated, $Q = \delta\{\omega^*\}$; see Figure 2. Assume that there are two stages in the T-stage stochastic linear program (3)–(4) in which the risk objectives are applied: $t_2 = 2,\ t_3 = T$. It means that the objective function in (22) is

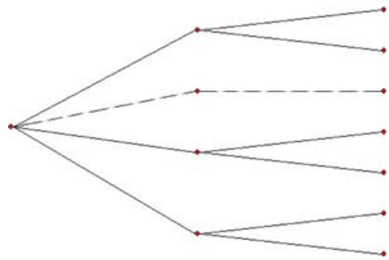
$$\boldsymbol{d}_1^\top \boldsymbol{y}_1 + E_P[\boldsymbol{d}_2^\top \boldsymbol{y}_2(\omega_1) + \boldsymbol{d}_3^\top \boldsymbol{y}_3(\omega)],$$

the coupling constraints (23) are specified as

$$\boldsymbol{w}_{20}^\top \boldsymbol{y}_2(\omega_1) + \boldsymbol{w}_{21}^\top \boldsymbol{y}_1 = z_2(\omega_1), \quad \boldsymbol{w}_{30}^\top \boldsymbol{y}_3(\omega) + \boldsymbol{w}_{31}^\top \boldsymbol{y}_2(\omega_1) + \boldsymbol{w}_{32}^\top \boldsymbol{y}_1 = z_T(\omega) \text{ a.s.}$$

and $z_t(\omega^{t-1,\bullet})$ are defined as in (24).

Fig. 2 Additional scenario



For simplicity of notation assume that only the right-hand sides $\boldsymbol{b}_t$ are random and let $\boldsymbol{b}_t^*$ consist of the right-hand sides generated by the additional scenario ω^*. The problem is solved for the initial probability distribution P which provides the optimal value $\varphi_R(P)$ and optimal first-stage decisions $\boldsymbol{x}_{1R}(P)$ and $\boldsymbol{y}_{1R}(P)$. To solve it for the degenerated probability distribution Q means to solve the *deterministic linear program*

$$\min \boldsymbol{d}_1^\top \boldsymbol{y}_1 + \boldsymbol{d}_2^\top \boldsymbol{y}_2 + \boldsymbol{d}_3^\top \boldsymbol{y}_3$$

subject to

$$\begin{aligned}
&\boldsymbol{A}_{t1}\boldsymbol{x}_1 + \sum_{\tau=2}^{t} \boldsymbol{A}_{t\tau}\boldsymbol{x}_\tau = \boldsymbol{b}_t^*, \quad t = 2, \dots, T,\\
&\boldsymbol{x}_1 \in \mathscr{X}_1, \quad \boldsymbol{x}_t \in \mathscr{X}_t, \quad t = 2, \dots, T, \quad \boldsymbol{y}_1 \in \mathscr{Y}_1, \quad \boldsymbol{y}_j \in \mathscr{Y}_j, \quad j = 2, 3\\
&\boldsymbol{w}_{20}^\top \boldsymbol{y}_2 + \boldsymbol{w}_{21}^\top \boldsymbol{y}_1 = z_2, \quad \boldsymbol{w}_{30}^\top \boldsymbol{y}_3 + \boldsymbol{w}_{31}^\top \boldsymbol{y}_2 + \boldsymbol{w}_{32}^\top \boldsymbol{y}_1 = z_T\\
&z_t = \boldsymbol{c}_1^\top \boldsymbol{x}_1 + \sum_{\tau=2}^{t} \boldsymbol{c}_\tau^\top \boldsymbol{x}_\tau \ \forall t.
\end{aligned}$$

The resulting optimal value $\varphi_R(Q)$ enters the lower bound in (8).

To get the derivative, it is necessary to evaluate the performance of the optimal first-stage decisions $\boldsymbol{x}_{1R}(P)$, $\boldsymbol{y}_{1R}(P)$ along the additional scenario ω^*, i.e. to solve

$$\boldsymbol{d}_1^\top \boldsymbol{y}_{1R}(P) + \min_{\boldsymbol{x}^*, \boldsymbol{y}^*} [\boldsymbol{d}_2^\top \boldsymbol{y}_2^* + \boldsymbol{d}_3^\top \boldsymbol{y}_3^*]$$

subject to

$$\begin{aligned}
&\boldsymbol{A}_{t1}\boldsymbol{x}_{1R}(P) + \sum_{\tau=2}^{t} \boldsymbol{A}_{t\tau}\boldsymbol{x}_\tau^* = \boldsymbol{b}_t^*, \quad t = 2, \dots, T,\\
&\boldsymbol{x}_t^* \in \mathscr{X}_t, \quad t = 2, \dots, T, \ \boldsymbol{y}_j^* \in \mathscr{Y}_j, \ j = 2, 3\\
&\boldsymbol{w}_{20}^\top \boldsymbol{y}_2^* + \boldsymbol{w}_{21}^\top \boldsymbol{y}_{1R}(P) = z_2^*, \quad \boldsymbol{w}_{30}^\top \boldsymbol{y}_3^* + \boldsymbol{w}_{31}^\top \boldsymbol{y}_2^* + \boldsymbol{w}_{32}^\top \boldsymbol{y}_{1R}(P) = z_T^*
\end{aligned}$$

and with

$$z_t^* = \mathbf{c}_1^\top \mathbf{x}_{1R}(P) + \sum_{\tau=2}^{t} \mathbf{c}_\tau^\top \mathbf{x}_\tau^* \; \forall t.$$

When testing the possibility of an additional stage, which means to change the filtration $\mathscr{F}_t$, $t = 1, \ldots, T-1$, the approach discussed in section 2.2 and Proposition 1 have to be used.

3.2 Bond portfolio management problem

The bond portfolio management problem was formulated initially as a two-stage multiperiod stochastic program (see e.g. Bertocchi et al. (2006a), Dupačová et al. (1998) or Chapter 6 of Dupačová et al. (2002)) and applied to bond portfolios containing bonds of different maturities; none of the bonds expires within the investment horizon. In addition, puttable bonds with European option at a specific time $t = t_1$ with a given exercise price were included into portfolio. The main random factor was the time evolution of interest rates ω. Scenarios of interest rates were generated according to the Black-Derman-Toy model Black et al. (1990) and the fair prices of bonds and of put options were computed accordingly. The purpose was to preserve the value of the portfolio over time, up to the given horizon T; no liabilities were considered.

To exercise the put option or not is a managerial decision and to reflect it, additional decision variables may be included. However, as no liabilities are involved and the considered bond portfolio management problem focuses on the wealth maximization at the horizon, the exercise strategy may be based solely on a data-based heuristics related with the interest rates scenarios, without using any additional decision variables. Hence, all second-stage decisions for all time periods up to the horizon are made at once, in dependence on the considered scenarios of interest rates.

The situation is different if the portfolio contains a bond j with a European *call* option at a given time $t = t_1 < T$. Similarly as in Example 1, there is an additional random factor—the exercise of call option or not—whose realization is not observed before $t = t_1$. Hence, there are two different probability distributions P, Q based on identical interest rates scenarios, which differ by the outcome of a $0-1$ random variable: P corresponds to no exercise whereas Q assumes the exercise. The two possible outcomes of option are reflected by different cash flows for bond j for $t > t_1$. Moreover, the number of variables and constraints of the Q- system differs from those of P-system: After the exercise of call, changes in holdings of bond j are no more possible. We may analyze differences of results obtained for the two two-stage multiperiod SLP, one based on scenarios ω^k, $k = 1, \ldots, K$, with prices and cash flows computed without option—probability distribution P with $\lambda = 0$ and the second one for probability distribution Q carried by identical scenarios of interest rates ω^k, $k = 1, \ldots, K$, but with cash flows reflecting the certain exercise of call option at $t = t_1$. Contamination leads then to a two-stage multiperiod SLP depending on contamination parameter $\lambda \in [0, 1]$ interpreted as the probability or just the "belief" that the option will not be exercised.

To obtain the contamination bounds for contaminated two-stage SLP is a standard task.

To model the problem correctly, nonanticipativity condition of the form

"decisions at $t < t_1$ cannot rely on the outcome of the call option at t_1"

should be observed. It suggests to reformulate the problem as a *3-stage stochastic linear program.* Nevertheless, without any liabilities the 3-stage reformulation might be superfluous. We will test its effect by the contamination technique.

The optimal value, say $\tilde{\varphi}(P_\lambda)$ of the two-stage contaminated problem is a natural bound for the optimal value $\varphi(P_\lambda)$ of the contaminated 3-stage problem which involves the nonanticipativity constraints; hence, for minimization, we have

$$\varphi(P_\lambda) \geq \tilde{\varphi}(P_\lambda) \geq (1-\lambda)\tilde{\varphi}(P) + \lambda\tilde{\varphi}(Q).$$

To get this lower bound, neither construction of common constraints nor inclusion of nonanticipativity condition are needed. What remains is to compute the upper bounds, such as (17).

To this end, let us follow firstly the procedure explained in section 2.2. For $t < t_1$ the prices of all bonds including those of the callable bond j (i.e., the coefficients in (13)) are obtained from the Black-Derman-Toy model and the common constraints consist of the corresponding balance and cash flow constraints on portfolio composition. At $t = t_1$, the branches of the scenario tree split according to the two possible outcomes of the option leading to the P and Q systems which must be tied with the common constraints through the ancestors at $t_1 - 1$. We can think of pairs of scenario subtrees rooted in nodes $k_{t_1} \in \mathscr{K}_{t_1}$ with "first-stage" variables $\boldsymbol{x}_{k_{t_1}}$, identical interest rate scenarios ω_t^s for $t \geq t_1$ but with differences in the form of cash flow constraints and with the set of variables reduced if the exercise of the option occurs (Q-constraints). The two cost-to-go functions, say $\varphi^P_{k_{t_1}}(\boldsymbol{x}_{k_{t_1}}) := \varphi^P_{t_1-1}(\boldsymbol{x}^{t_1-1,\bullet}, \omega^{t_1-1,\bullet})$ and $\varphi^Q_{k_{t_1}}(\boldsymbol{x}_{k_{t_1}}) := \varphi^Q_{t_1-1}(\boldsymbol{x}^{t_1-1,\bullet}, \omega^{t_1-1,\bullet})$ in the nested form (2) correspond to recourse functions in multiperiod two-stage stochastic linear programs and can be computed separately for all continuations $\omega_{t_1}, \ldots, \omega_{T-1}$ of the paths of interest rates of $\omega^{t_1-1,\bullet}$. Under assumption that the exercise of call option is independent of the interest rates, the cost-to-go function (2) for $t = t_1 - 1$ is based on the contaminated probability distribution $(1-\lambda)P + \lambda Q$:

$$\begin{aligned}
&\varphi_{a(k_{t_1})}(\boldsymbol{x}_{a(k_{t_1})}, \lambda) \\
&= \min_{\boldsymbol{x}_{k_{t_1}}} [\boldsymbol{c}_{k_{t_1}}^\top \boldsymbol{x}_{k_{t_1}} + \sum_{k_{t_1}} \pi_{a(k_{t_1}),k_{t_1}} \{(1-\lambda)\varphi^P_{k_{t_1}}(\boldsymbol{x}_{k_{t_1}}) + \lambda\varphi^Q_{k_{t_1}}(\boldsymbol{x}_{k_{t_1}})\}].
\end{aligned}$$

Recall that indices $k_t \in \mathscr{K}_t$ correspond to realizations of $\omega^{t-1,\bullet}$ and notice that the conditional expectation is under probability distribution P only—a simplification due to the fact that there are identical interest rates scenarios for P and Q. Minimization is carried with respect to constraints of (13) for $t = t_1$. Evidently, $\varphi_{a(k_{t_1})}(\boldsymbol{x}_{a(k_{t_1})}, \lambda)$ is a concave function of λ and its directional derivative follows the scheme (9), (10). To get the directional derivative of cost-to-go functions φ_{k_t} for $t < t_1 - 1$ and of the optimal value function, formula (12) can be applied. Of course, the resulting upper bounds follow the pattern (17) with the minimizers of the Q-problem at the place of $\boldsymbol{x}^*_{h_t}$.

Inclusion of another European call option at $t_1 < t_2 < T$ for a different bond j' may be modeled as a 4-stage stochastic program. It may be split into two 3-stage subtrees rooted at t_1 which differ by the outcome of the call option for bond j. Evaluation of the two cost-to-go functions means now to solve 3-stage stochastic linear progams. Notice that the structure of the tree is different when the call options at $t = t_1$ and $t = t_2$ concern the same bond j.

3.3 Selected numerical results

The numerical experiments for the bond portfolio management (cf. Section 3.2) and stochastic bond dedicated portfolio (cf. Example 1) were solved for one year investment horizon and monthly discretization. Scenarios for short-term interest rates were selected according to the so-called Part(8) pattern mentioned in Bertocchi et al. (2006a,b). The input data were based on the bond portfolio described in detail in our earlier papers e.g. Bertocchi et al. (2006a,b), Dupačová et al. (1998) but with two artificial *callable* bonds substituted for the original puttable ones. Their market prices are given for the date of October 3rd, 1994 and are reported in the Table 1 in comparison with prices of the original puttable bonds.

Table 2 includes the characteristics of the bonds.

Among various results of numerical experiments, we choose to comment below the results for the bond portfolio problem of Section 3.2 and for the stochastic dedicated bond portfolio selection, Example 1, for the case that only one of the two CBT is included - that of the shortest maturity (CBT13212).

Table 1 Market prices for callable bonds with respect to puttable bonds

Bonds	Market prices
CBT13212	101.6534
CBT36608	103.4933
CTO13212	103.7500
CTO36608	106.4390

Table 2 Portfolio Composition on October 3rd 1994

Bonds	Qt	Coupon	Payment dates	Exercise	Redemp.	Maturity
BTP36658	10	3.9375	01 Apr & 01 Oct		100.187	01 Oct 96
BTP36631	20	5.0312	01 Mar & 01 Sep		99.531	01 Mar 98
BTP12687	15	5.2500	01 Jan & 01 Jul		99.231	01 Jan 02
BTP36693	10	3.7187	01 Aug & 01 Feb		99.387	01 Aug 04
BTP36665	5	3.9375	01 May & 01 Nov		99.218	01 Nov 23
CBT13212	20	5.2500	20 Jan & 20 Jul	20 Jan 95	100.000	20 Jan 98
CBT36608	20	5.2500	19 May & 19 Nov	19 May 95	99.950	19 May 98

3.3.1 Bond portfolio management problem

When generating the prices along scenarios it turns out that for the writer of the callable it may be convenient to call the bond at the exercise date $t_1 = 4$ along scenarios numbered 0 to 3, i.e. the full nominal value plus coupon and premium get paid to the portfolio's owner in $t_1 = 4$, and zero cash flows follow in subsequent periods. We get optimal value 9079.147 by solving the corresponding *two-stage multiperiod optimization problem*. This case can be regarded as if the owner of the portfolio is forced to give back the callable bond if certain evolution of interest rates makes the writer to call the bond.

Recall that the original bond portfolio management problem does not involve any liabilities. Let us examine the case with only CBT13212 included by viewing the exercise of the option at $t_1 = 4$ as an additional random factor. The scenarios consist now of the initial interest rates scenarios ω^k augmented by an additional component, say α with two possible realizations.

Denote Q the distribution related to the certain occurrence of exercise. The optimal value of the corresponding two-stage problem for distribution Q is $\tilde{\varphi}(Q) = 9079.147$.

Let P denote the distribution related to problem without occurrence of exercise. The optimal value of the two-stage problem for distribution P is $\tilde{\varphi}(P) = 9241.688$—the best possible outcome for the portfolio owner. Unfortunately the bond may be called and this produces lower final wealth corresponding to Q distribution.

The contaminated distribution

$$P_\lambda = (1-\lambda)P + \lambda Q$$

consists of 16 scenarios, 8 corresponding to no exercise (P distribution) and 8 corresponding to the exercise (Q distribution), whose influence, as we already noticed, can be opportunily weighted by the parameter λ. Solving the two-stage optimization problem for P_λ with different values of λ we get the values shown in column 2 of Table 3.

From the same data a three-stage problem is obtained by including nonanticipativity constraints up to the exercise date. Solving it for different values of λ we get the values shown in column 1 of Table 3. Notice that in this example, the optimal values of the three-stage problem are always equal to the corresponding two-stage optimal values.

The purpose of the proposed contamination technique is to avoid solving the two problems repeatedly for changing values of λ but to provide an evidence whether an application of the three stage structure improves the results. Having a maximization problem, the formulas for bounds, e.g. (8), (10) or (17) are valid with reverse inequalities and max occurs in the formula for directional derivatives (9) at the place of min.

The obtained bounds are quite tight, see Figure 3 for the left and lower bound given by the approximation (dashed lines) and the exact bounds (solid lines). Accordingly it should be sufficient to persist in the two-stage formulation.

To analyse the differences in the results due to *inclusion of liabilities* we introduce a liability of 9000 Euros at month 12. We consider two models, one excluding the

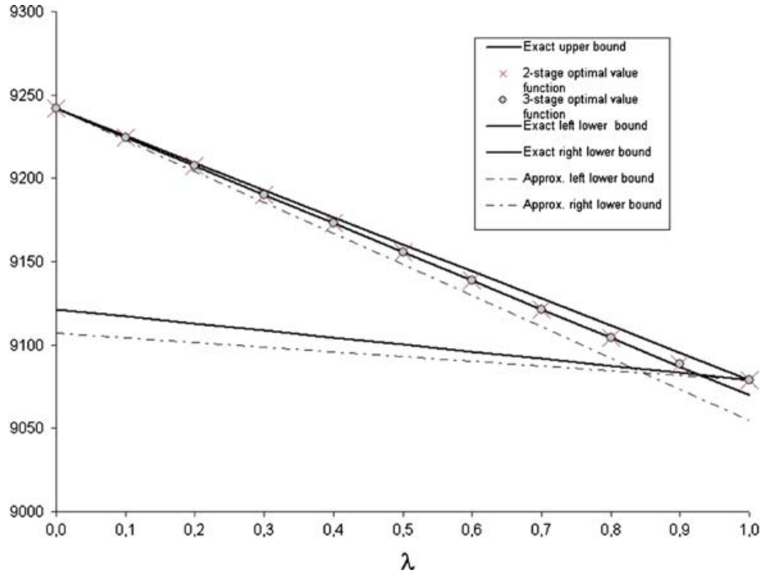


Fig. 3 Exact and approximated bounds—no liabilities

Table 3 Three-stage optimal values – no liabilities

	3-stage optimal value function	2-stage optimal value function	lambda	Exact upper bound	Exact left lower bound	Exact right lower bound	Exact first derivative contamin.	$F(Q,x(P))$	Approx. first derivative contamin.	Approx. left lower bound	Approx. right lower bound
P	9241.688	9241.688	0.0	9241.69	9241.69	9121.27	-171.984	9069.7	-186.9284	9241.69	9107.11
	9224.489	9224.489	0.1	9225.43	9224.49	9117.05				9222.99	9104.32
	9207.291	9207.291	0.2	9209.18	9207.29	9112.84				9204.3	9101.52
	9190.093	9190.093	0.3	9192.93	9190.09	9108.63				9185.61	9098.72
	9172.894	9172.894	0.4	9176.67	9172.89	9104.42				9166.92	9095.93
	9155.696	9155.696	0.5	9160.42	9155.7	9100.21				9148.22	9093.13
	9138.497	9138.497	0.6	9144.16	9138.5	9095.99				9129.53	9090.33
	9121.299	9121.299	0.7	9127.91	9121.3	9091.78				9110.84	9087.54
	9104.100	9104.100	0.8	9111.65	9104.1	9087.57				9092.15	9084.74
	9088.495	9088.495	0.9	9095.4	9086.9	9083.36				9073.45	9081.94
Q	9079.147	9079.147	1.0	9079.15	9069.7	9079.15	42.1198	9121.27	27.9681	9054.76	9079.15
								$F(P,x(Q))$			

possibility of borrowing and another one which allows for borrowing. For *no possibility of borrowing,* see Table 4 for details on the results and Figure 4 for various bounds. We can notice that the bounds given by formula (17) are quite loose.

When we allow for *borrowing* the optimal values for the two-stage and the three-stage model do not differ.

In case the callable bond has the *exercise time* at the end of the first month, which is reflected also in its market price changed to 104.6554, the optimal values of 2-stage and 3-stage problems are different even for the case without liabilities. Hence, the position of the branching point is important. The contamination results are reported in Table 5 and Figure 5 for the 2-stage and in Table 6 and Figure 6 for the 3-stage model.

We have also run various experiments with *different market prices* for the callable bonds: in case that purchasing the callable is an optimal first stage solution both with

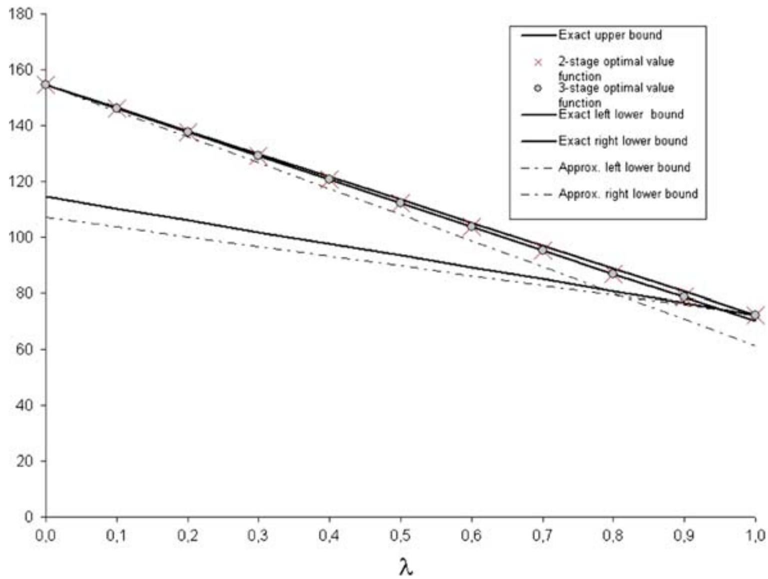


Fig. 4 Exact and approximated bounds – with liabilities

Table 4 Two-stage and three-stage optimal values – w. liabilities

	3-stage optimal value function	2-stage optimal value function	lambda	Exact upper bound	Exact left lower bound	Exact right lower bound	Exact first derivative contamin.	$F(Q,x(P))$	Approx. first derivative contamin.	Approx. left lower bound	Approx. right lower bound
P	154.533	154.533	0.0	154.533	154.533	114.497	-84.4844		-93.0293	154.533	107.112
	146.072	146.085	0.1	146.316	146.085	110.283				145.23	103.636
	137.625	137.636	0.2	138.098	137.636	106.069				135.927	100.161
	129.178	129.188	0.3	129.881	129.188	101.856				126.625	96.6857
	120.731	120.740	0.4	121.663	120.74	97.6417				117.322	93.2104
	112.284	112.291	0.5	113.446	112.291	93.4278				108.019	89.735
	103.837	103.843	0.6	105.228	103.843	89.2139				98.7157	86.2597
	95.390	95.394	0.7	97.0108	95.3942	84.9999				89.4128	82.7843
	86.943	86.946	0.8	88.7932	86.9458	80.786				80.1099	79.3089
	78.496	78.497	0.9	80.5757	78.4973	76.5721				70.807	75.8336
Q	72.358	72.358	1.0	72.3582	70.0489	72.3582	42.1391	114.497	34.7536	61.504	72.3582
								$F(P,x(Q))$			

distribution P and Q (i.e. its price is convenient with respect to other bonds under both distributions), the upper and lower bounds are very close so that optimal values for different λ move along those bounds.

3.3.2 Stochastic dedicated bond portfolio selection

Again, we consider the portfolio with only one (CBT13212) of the two CBTs included and with exercise at month 4. As mentioned in Example 1, liabilities are an important ingredient of this model. We choose one liability at month 12 equal exactly to the value we got as final wealth in the previous model for the P distribution, i.e. 9241.688, see Table 3. The minimal acquisition price 8582.687 of portfolio is obtained for

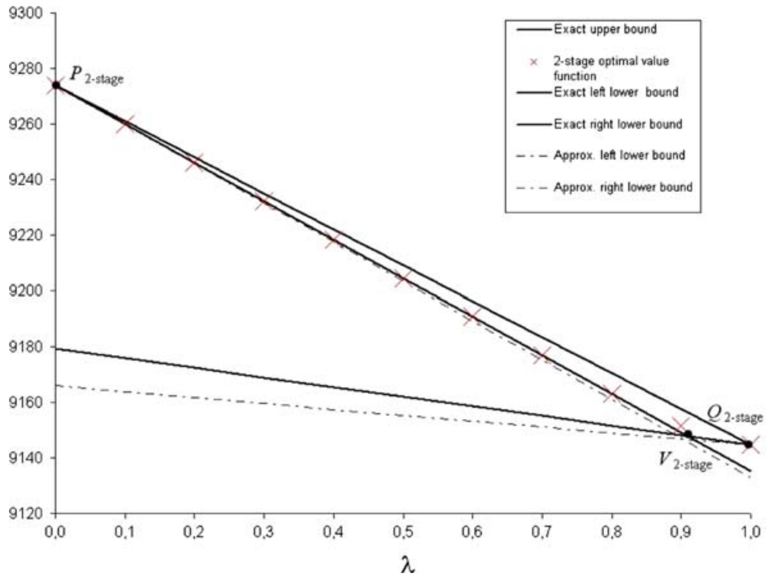


Fig. 5 Bounds for two-stage optimal values, changed exercise date

Table 5 Two-stage optimal values, changed exercise date

	2-stage optimal value function	lambda	Exact upper bound	Exact left lower bound	Exact right lower bound	Exact first derivative contamin.	$F(Q,x(P))$	Approx. first derivative contamin.	Approx. left lower bound	Approx. right lower bound
P	9273.817	0.0	9273.82	9273.82	9179.37	-138.519	9135.3	-140.856	9273.82	9165.97
	9259.965	0.1	9260.9	9259.96	9175.9				9259.73	9163.84
	9246.113	0.2	9247.99	9246.11	9172.43				9245.65	9161.71
	9232.261	0.3	9235.08	9232.26	9168.97				9231.56	9159.58
	9218.409	0.4	9222.17	9218.41	9165.5				9217.47	9157.46
	9204.557	0.5	9209.26	9204.56	9162.03				9203.39	9155.33
	9190.705	0.6	9196.34	9190.71	9158.56				9189.3	9153.2
	9176.854	0.7	9183.43	9176.85	9155.1				9175.22	9151.08
	9163.002	0.8	9170.52	9163	9151.63				9161.13	9148.95
	9151.374	0.9	9157.61	9149.15	9148.16				9147.05	9146.82
Q	9144.694	1.0	9144.69	9135.3	9144.69	34.6746	9179.37	21.2712	9132.96	9144.69
							$F(P,x(Q))$			

investment of 1.1 in CBT13212 and 8470.276 in cash. If the callable bond is exercised, no borrowing takes place due to a high cost of borrowing.

Introducing a constraint on a minimal representation of securities in the portfolio, the minimal acquisition price of the portfolio increases to 13643.312 and is attained by purchasing each of considered bonds on the minimal level of 10 and an investment of 7963.918 in cash. We also forced the cash y_0^+ to be zero which turned out to be a very expensive strategy, because the liabilities could be covered only by the cashflows that became available along the months.

If we consider again the 16 scenarios of prices as described in the previous section and run the two-stage stochastic model using a contaminated distribution P_λ, we get the minimal portfolio acquisition prices 8582.687 when the callable bond is called

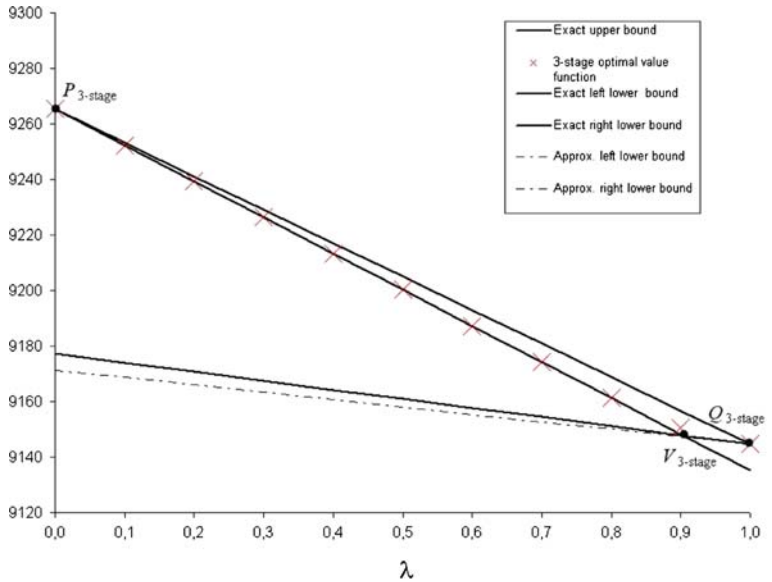


Fig. 6 Bounds for three-stage optimal values, changed exercise date

Table 6 Three-stage optimal values, changed exercise date

	3-stage optimal value function	lambda	Exact upper bound	Exact left lower bound	Exact right lower bound	Exact first derivative contamin.	$F(Q,x(P))$	Approx. first derivative contamin.	Approx. left lower bound	Approx. right lower bound
P	9265.267	0.0	9265.27	9265.27	9177.23	-129.969	9135.3	-130.053	9265.27	9171.31
	9252.270	0.1	9253.21	9252.27	9173.98				9252.26	9168.65
	9239.273	0.2	9241.15	9239.27	9170.72				9239.26	9165.99
	9226.276	0.3	9229.1	9226.28	9167.47				9226.25	9163.33
	9213.279	0.4	9217.04	9213.28	9164.21				9213.25	9160.67
	9200.282	0.5	9204.98	9200.28	9160.96				9200.24	9158
	9187.286	0.6	9192.92	9187.29	9157.71				9187.24	9155.34
	9174.289	0.7	9180.87	9174.29	9154.45				9174.23	9152.68
	9161.292	0.8	9168.81	9161.29	9151.2				9161.22	9150.02
	9150.677	0.9	9156.75	9148.29	9147.95				9148.22	9147.36
Q	9144.694	1.0	9144.69	9135.3	9144.69	32.5344	9177.23	26.6191	9135.21	9144.69
							$F(P,x(Q))$			

(i.e. $\lambda = 1$ in which case no borrowing occurs), respectively 8560.071 (for $\lambda = 0$ and with borrowing of 24.9183 at time $t = 11$ along scenario 0) when the callable bond is not called. For any other value of λ the minimal acquisition price is a linear function of λ—a weighted average of these two extremal values. No difference appears when we run the three-stage model.

If we add a constraint requiring that the sum of borrowing is less than 24.9183, we get a different optimal value for the P distribution, 8569.155, which is smaller than that for the Q distribution. It comes out that having the callable bond in the portfolio, when called, allows to cover the liabilities without borrowing, which is not the case when it is not called.

4 Conclusions

Contamination technique helps to quantify the influence of changes in the structure of multistage stochastic programs on the optimal value. To derive the results, a reformulation of the problem to the form (7) is necessary; then the contamination bounds follow by known properties of concave functions. For evaluating them, one solves stochastic programs of lower dimension and of a reduced number of stages.

For scenario-based multistage stochastic linear programs, results based on parametric linear programming applied to the arborescent form of the contaminated stochastic program cover various possibilities, including contamination bounds for problems with polyhedral risk objectives. Parallel results may be based also on nested form (1)–(2) of multistage stochastic programs whereas the scenario-splitted form (6) is of a limited use.

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