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Sensitivity of Bond Portfolio's Behavior with Respect to Random Movements in Yield Curve: A Simulation Study

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Abstract. The bond portfolio management problem is formulated as a stochastic program based on interest rate scenarios. The coefficients of the resulting program are subject to errors of various kind. In this paper, we complement the theoretical stability results of [10] by simulation experiments. Adapting the approach of [16] to problems based on perturbed yield curves, we then provide bounds for the optimality gap for various candidate first-stage solutions.

Keywords: bond portfolio management, sensitivity to term structure changes, simulations, error bounds

1. Introduction

The main purpose of the considered bond portfolio management problem is to maximize the expected value of a bond portfolio of a risk averse or risk neutral institutional investor over time. There are various options concerning the choice of an appropriate model, starting with duration based immunization models or dedicated bond portfolio management models, cf. [19], up to multistage stochastic programs which can be used for complex asset/liability management problems, cf. [21]. In this paper, we model the problem, similarly as in [15] or [12], as a multiperiod two-stage scenario based stochastic program with random recourse. The main random element is the evolution of the short interest rate over time which is regarded as the only factor that drives the prices of the considered government bonds.

Given a sequence of equilibrium future short term interest rates r_t valid for the time interval $(t, t+1]$, $t = 0, \dots, T-1$, the fair price of the jth bond at time t just after the coupon was paid equals the total value of the cashflows $f_{j\tau}$, $\tau = t+1, \dots, T$, generated by this bond in subsequent time instances discounted to t:

$$P_{jt}(\mathbf{r}) = \sum_{\tau=t+1}^{T} f_{j\tau} \prod_{h=t}^{\tau-1} (1 + r_h)^{-1}, \tag{1.1}$$

where T is greater than or equal to the time to maturity.

To build the stochastic programming model, one assumes a suitable probability distribution of the T – dimensional vector $\mathbf{r}$ of the short rates r_t, $t = 0, \ldots, T-1$, where r_0 (the rate valid in the first period) is supposed to be known. We intend to work with a discrete approximation of this distribution whose support consists of finitely many atoms, called *scenarios*; we shall index them as $\mathbf{r}^s$, $s = 1, \ldots, S$, and assign them probabilities $p_s > 0$, $s = 1, \ldots, S$, $\sum_s p_s = 1$.

We denote

$j = 1, \ldots, J$ indices of the considered bonds and T_j the dates of their maturities;

$T = \max_j T_j$;

$t = 0, \ldots, T_0$ the considered discretization of the planning horizon;

$b_j \geqslant 0$ the initial holdings (in face value) of bond j;

b_0 the initial holding in riskless asset;

f^s_{jt} cashflow generated under scenario s from bond j at time t expressed as a fraction of its face value;

ξ^s_{jt} and ζ^s_{jt} are the selling and purchasing prices of bond j at time t for scenario s obtained from the corresponding fair prices (1.1) based on the scenario s rates $\mathbf{r}^s$ by adding the accrued interest A^s_{jt} and by subtracting or adding scenario independent transaction costs and spread; the initial prices ξ_{j0} and ζ_{j0} are known, i.e., scenario independent;

L_t is an external cashflow, e.g., a scenario independent liability, at time t;

x_j/y_j are face values of bond j purchased/sold at the beginning of the planning period, at $t = 0$; x^s_{jt}/y^s_{jt} are the corresponding values for period t under scenario s.

z_{j0} is the face value of bond j held in portfolio after the initial decisions x_j, y_j have been made; z^s_{jt} are the corresponding holdings for period $(t, t+1]$ under scenario s.

The first-stage decision variables x_j, y_j, z_{j0} are nonnegative,

$$y_j + z_{j0} = b_j + x_j \quad \forall j, \tag{1.2}$$

$$y_0^+ + \sum_j \zeta_{j0} x_j = b_0 + \sum_j \xi_{j0} y_j, \tag{1.3}$$

where the nonnegative variable y_0^+ denotes the surplus in the riskless asset (cash) after the first-stage decisions. Notice that in the first stage no borrowing is allowed.

Provided that an initial trading strategy determined by feasible scenario independent first-stage decision variables x_j, y_j, y_0^+ (and z_{j0}) for all j has been accepted, the subsequent second-stage scenario dependent decisions have to be made in an optimal way regarding the goal of the model, i.e., to maximize the final wealth subject to constraints on conservation of holdings and rebalancing the portfolio:

$$\text{maximize} \quad W^s_{T_0} := \sum_j \xi^s_{jT_0} z^s_{jT_0} + y^{+s}_{T_0} - \alpha y^{-s}_{T_0} \tag{1.4}$$

$$\text{subject to} \quad z^s_{jt} + y^s_{jt} = z^s_{j,t-1} + x^s_{jt} \quad \forall j,\ 1 \leqslant t \leqslant T_0, \tag{1.5}$$

$$\sum_j \xi^s_{jt} y^s_{jt} + \sum_j f^s_{jt} z^s_{j,t-1} + \left(1 - \delta_1 + r^s_{t-1}\right) y^{+s}_{t-1} + y^{-s}_t$$
$$= L_t + \sum_j \zeta^s_{jt} x^s_{jt} + \left(1 + \delta_2 + r^s_{t-1}\right) y^{-s}_{t-1} + y^{+s}_t, \quad 1 \leqslant t \leqslant T_0, \quad (1.6)$$

$$x^s_{jt} \geqslant 0,\ y^s_{jt} \geqslant 0,\ z^s_{jt} \geqslant 0,\ y^{-s}_t \geqslant 0,\ y^{+s}_t \geqslant 0 \quad \forall j,\ 1 \leqslant t \leqslant T_0, \quad (1.7)$$

with $y^{-s}_0 = 0,\ y^{+s}_0 = y^+_0,\ z^s_{j0} = z_{j0}\ \forall s, j$. The variables y^{+s}_t / y^{-s}_t describe the (unlimited) lending/borrowing possibilities for period t under scenario s. The variable δ_1 accounts for the difference between the returns of bonds and cash while δ_2 accounts for the positive cost of borrowing. The multiplier $\alpha \geqslant 1$ is fixed according to problem area; in general, it can be interpreted as a penalty for a debt outstanding at the end of the planning horizon.

Denote by $W_{T_0}(\mathbf{r}^s; \mathbf{x}, \mathbf{y}, \mathbf{z}_0, y^+_0)$ the corresponding maximal value of (1.4), a concave and piece-wise linear function in $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}_0$, y^+_0. The full stochastic program can be now written as

$$\text{maximize} \quad \sum_{s=1}^{S} p_s U\left(W_{T_0}\left(\mathbf{r}^s; \mathbf{x}, \mathbf{y}, \mathbf{z}_0, y^+_0\right)\right) \quad (1.8)$$

subject to nonnegativity constraints on all variables and subject to (1.2)–(1.3), with U a concave nondecreasing utility function.

There are various models of evolution of interest rates; we consider here interest rate scenarios sampled from the binomial lattice obtained according to the Black–Derman–Toy (BDT) model [7] which, according to [3] "is currently close to an industry standard".

The uncertainty concerning the numerical values of the components of the interest rate scenarios, prices and of the resulting optimal solution of the portfolio management problem stems mostly from the input information used for calibration and fitting the binomial lattice, namely, the initial *term structure* obtained from the existing market data, and the applied strategy for selection of a modest number of scenarios from the 2^{T-1} scenarios available from the lattice.

According to [10], the scenario subproblems (1.4)–(1.7) are stable linear programs in the sense of Robinson [18] provided that certain acceptably weak and natural conditions on the model coefficients (e.g., $\zeta^s_{jt} > \xi^s_{jt}\ \forall j, t, s$) are met. This means, i.a., that the optimal value function $W_{T_0}(\mathbf{r}^s; \mathbf{x}, \mathbf{y}, \mathbf{z}_0, y^+_0)$ is *jointly* continuous in $\mathbf{r}^s$, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}_0$, y^+_0. Hence, also the objective function (1.8) is jointly continuous in $\mathbf{r}^s$, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}_0$, y^+_0 and in p_s $\forall s$ and concave in the first-stage decision variables on the compact convex set described by (1.2), (1.3) and nonnegativity of all variables. This implies that the optimal value function of the full problem is continuous in scenarios and their probabilities and that for small changes of coefficients, also the sets of optimal first-stage solutions display certain continuity properties.

However it is important to quantify the meaning of "small errors" in relation to the input market data. To this purpose we design suitable framework for simulations which help to identify the magnitude of the "small errors". Moreover, we exploit probabilistic bounds of Mak et al. [16] to test the quality of the first-stage solutions obtained for the binomial lattice calibrated by means of the estimated yield curve, from the point of view of the hypothetical true problem for which the calibration of the binomial lattice does not involve errors due to replacing the yield curve by its estimate. Such bounds provide an evidence about robustness of alternative first-stage solutions with respect to perturbations of the input yield curve.

Another important question is what changes can be expected in connection with inclusion of additional *out-of-sample scenarios*. With respect to changes of the optimal value function, this problem was studied via the contamination technique, see [13,14], which has proved to be a useful and numerically tractable technique.

In the next section, we shall detail the procedure for scenario generation which is the core of the simulation studies described in section 3. Numerical results are presented in section 4.

2. Generation of scenarios

According to the Black–Derman–Toy [7] model, at each time point t there are $t+1$ possible stages on the binomial lattice. The resulting one-period rates r_t^s for scenario s and for the time interval $(t, t+1]$ are given as

$$r_t^s = r_{t0} k_t^{i_t(s)}, \quad i_t(s) = \sum_{\tau=1}^{t} \omega_\tau^s, \tag{2.1}$$

where r_{t0} are the base rates, k_t the lattice volatilities and $i_t(s)$ is the number of the "up" moves, coded by $\omega_\tau = 1$, for the given scenario s which occur at time points $1, \ldots, t$.

To *calibrate the Black–Derman–Toy model*, i.e., to get all base rates r_{t0} and lattice volatilities k_t, means to use the yield and volatility curve related to yields to maturity of zero coupon government bonds of all maturities corresponding to the chosen time steps of the lattice. Such bonds are rare in the market and have to be replaced by synthetic zero coupon bonds whose yields correspond to yields of fixed coupon government bonds that do not contain any special provision such as call or put options. Out of several numerical procedures for fitting the lattice parameters, see [1], we apply the forward Bjerksund and Stensland [6] algorithm.

The calibration of the binomial lattice of the BDT model, in agreement with the (estimated) today's market term structure, provides 2^{T-1} interest rate scenarios $\mathbf{r}^s$ whose common first component equals r_0 and the subsequent components r_t^s (valid for the interval $(t, t+1]$, where $t = 1, \ldots, T-1$), depend on scenario s. A smaller, manageable number of scenarios has to be *selected or sampled* from this large set. An example is the nonrandom sampling technique of Zenios and Shtilman [20] which aims at a uniform approximation of the expected utility of the final wealth (1.8) computed for the *full* set of the 2^{T-1} scenarios of the lattice by an expected value over a *subset* of these scenarios. For a procedure based on ideas of importance sampling we refer to Nielsen [17]. The resulting scenarios $\mathbf{r}^s$ are identified by the *full* set of the lattice parameters $r_{t0},\ k_t,\ t = 1, \ldots, T-1$, and by the scenario dependent trajectories through the lattice identified by the sequences $i_t(s),\ t = 1, \ldots, T-1$. This allows us to study separately the influence of errors in the lattice parameters, the main topics of the present paper, and the impact of the sampling strategy.

Various regression models have been used to *estimate the yield curve* from the existing market data on yields of fixed coupon government bonds at the given day. Having tried different parametric nonlinear models as well as nonparametric ones, as reported in [11], we chose to use a simple form of the yield curve applied already by Bradley and Crane [9]

$$g(t; \alpha, \beta, \gamma) = \alpha t^{\beta} \mathrm{e}^{\gamma t} \tag{2.2}$$

and we applied its linearized form to the logarithms of yields: for the market information consisting of yields $u_i, i = 1, \ldots, n$, of various fixed coupon government bonds (without option) characterized by their maturities t_i, the postulated model is

$$\lg u_i = \lg \alpha + \beta \lg t_i + \gamma t_i + e_i, \quad i = 1, \ldots, n, \tag{2.3}$$

where the random errors $e_i, i = 1, \ldots, n$, are independent, normal $\mathcal{N}(0, \sigma^2)$.

The least squares estimates $\lg \widehat{\alpha},\ \widehat{\beta},\ \widehat{\gamma}$ of parameters $\lg \alpha,\ \beta,\ \gamma$ are approximately normal, with the mean values equal the true parameter values and the covariance matrix

$$\sigma^2 \Sigma^{-1}, \quad \Sigma = \mathbf{G}^{\top}\mathbf{G},$$

where $\mathbf{G}$ is the matrix of gradients of the function $\lg g(t_i; \theta)$ in (2.3) with respect to the parameters and σ^2 is estimated by

$$s^2 = \frac{1}{n-3} \min_{\lg \alpha, \beta, \gamma} \sum_{i=1}^{n} (\lg u_i - \lg \alpha - \beta \lg t_i - \gamma t_i)^2.$$

To estimate the yields of zero coupon bonds of all required maturities, which are not directly observable, $\tilde{t} \neq t_i$, we replace the unobservable logarithm of yield by the corresponding value on the already estimated log-yield curve. Such estimates are subject to additional error.

We assume that the logarithm of the yield $\tilde{u} = u(\tilde{t})$ for maturity $\tilde{t}$ is

$$\lg \tilde{u} = \lg \widehat{\alpha} + \widehat{\beta} \lg \tilde{t} + \widehat{\gamma} \tilde{t} + \tilde{e}$$

with $\tilde{e}$ normal, independent of $e_i, i = 1, \ldots, n,\ E\tilde{e} = 0$, $\operatorname{var} \tilde{e} = \sigma^2$ and with the true parameter values denoted by asterisks. Then $\lg \tilde{u}$ is approximately normal,

$$\lg \tilde{u} - \lg \widehat{\alpha} - \widehat{\beta} \lg \tilde{t} - \widehat{\gamma} \tilde{t} \sim \mathcal{N}\big(0, \sigma^2\big(1 + Q^2(\tilde{t})\big)\big), \tag{2.4}$$

where

$$Q^2(t) = [1, \lg t, t] \Sigma^{-1} [1, \lg t, t]^{\top}. \tag{2.5}$$

We assume in this paper that volatilities of yields or log-yields are not a subject of perturbation, which of course is a simplification. We use here "volatility of errors" which is the (time dependent) standard deviation of the normal distribution (2.4).

3. Simulation studies

The simulation experiments described below aim at quantification of errors that are present in the optimal first-stage solution of the stochastic programming model (1.8) when a perturbation in the yield curve is introduced. As we have observed in the previous section, the perturbations will propagate in the interest rates lattice, in the prices and finally in the coefficients of model (1.4), (1.6). The method of estimating the yield curve by parametric regression presented briefly in section 2 provides a basis for simulation of log-yields at individual points t which are needed for fitting the binomial lattice; we keep the initial volatility curve.

We devise three approaches for simulation, all requiring repeated solution of the scenario based program (1.8) for various sets of coefficients. The former allows to construct simple statistics about the optimal values from simulation while the latest two result in probabilistic bounds on the "optimality gap" which is the difference in objective values between a candidate solution and the optimal solution.

The first simulation approach requires two initial steps:

(i) At each point t of the discretization of the time horizon T generate the random error e by sampling from the normal distribution $\mathcal{N}(0, \sigma^2(1 + Q^2(t)))$ and put $\lg u_t = \lg \widehat{\alpha} - \widehat{\beta} \lg t - \widehat{\gamma} + e$. Let $\mathbf{e}$ be the vector of the independent normally distributed components e obtained in the described way.

(ii) For each vector of log-yields simulated according to (i), get the vector of simulated yields $\mathbf{u}$ together with the original volatility curve, fit the lattice and evaluate the interest rates r_t^s according to (2.1), prices P_{jt}^s using (1.1) and the coefficients ξ_{jt}^s, ζ_{jt}^s of (1.4), (1.6) for $s = 1, \ldots, S$ according to the chosen sampling strategy.

By repeated solution of the scenario based programs (1.8) for various sets of coefficients obtained by the simulation procedure (i), (ii) one gets repeated "observations" φ^k, $k = 1, \ldots, K$, of the optimal value and of the optimal initial trading strategy $\widehat{\mathbf{x}}^k$, $\widehat{\mathbf{y}}^k$, $\widehat{\mathbf{z}}_0^k$, $\widehat{y}_0^{+k}$ which allows to construct an empirical distribution of the maximal expected utility of the final wealth, a useful information for subsequent, sample-based statistical inference, and to discover how the bonds in the given portfolio cluster with respect to the buy, sell or do nothing options.

The second simulation procedure adapts the approach suggested by [16] to test the quality of the optimal first-stage solution, $\widehat{\mathbf{x}}$, $\widehat{\mathbf{y}}$, $\widehat{\mathbf{z}}_0$, $\widehat{y}_0^+$ based on the estimated yield curve. Notice that the technique can be used to test the quality of any ad hoc candidate first-stage solution. More specifically, the quality testing is based on the construction of probabilistic bounds on the "optimality gap" of the objective function values which will be detailed below.

Let us consider the first-stage (optimal) solution $\widehat{\mathbf{x}}$, $\widehat{\mathbf{y}}$, $\widehat{\mathbf{z}}_0$, $\widehat{y}_0^+$ of the unperturbed problem as our candidate solution. Denote by $\widehat{f}(\mathbf{e}^k)$ the value of the objective function in (1.8) evaluated at this candidate solution for scenarios obtained in the kth simulation experiment and by φ^* the "true" optimal expected wealth based on a complete stochastic information, i.e., the true yield curve, which should have been used for calibration of the lattice.

The simulation procedure requires the two initial steps (i) and (ii) and the following ones:

(iii) Let K_l be the number of simulation experiments; for i.i.d. $\mathbf{e}^k$ the average value

$$L(K_l) = \frac{1}{K_l} \sum_k \widehat{f}(\mathbf{e}^k) \tag{3.1}$$

is a *stochastic* lower bound for the "true" optimal value φ^*; it is an asymptotically normal estimate of the true expectation of the (random) value $\widehat{f}(\mathbf{e})$ of the objective function evaluated at the decision $\widehat{\mathbf{x}}$, $\widehat{\mathbf{y}}$, $\widehat{\mathbf{z}}_0$, $\widehat{y}_0^+$ (that need not equal the "true" optimal decision). According to the central limit theorem, we have asymptotically $\sqrt{K_l}[L(K_l) - E\widehat{f}(\mathbf{e})] \sim \mathcal{N}(0, \sigma_l^2)$, for $K_l \to \infty$, where $\sigma_l^2 = \text{var}\,\widehat{f}(\mathbf{e})$.

(iv) To get a *stochastic upper bound*, one has to use K_u i.i.d. batches of K simulated vectors $\mathbf{e}^{kh}$, $h = 1, \ldots, K_u$, $k = 1, \ldots, K$, to compute K_u optimal values $f^{h*} := \max \frac{1}{K} \sum_{k=1}^{K} f(\mathbf{e}^{kh})$ of the objective functions in (1.8) that correspond to simulated data $\mathbf{e}^{kh}$ $\forall k$ and their averages

$$U(K_u) = \frac{1}{K_u} \sum_h f^{h*}. \tag{3.2}$$

Now, $U(K_u)$ is a stochastic upper bound for the "true" optimal value φ^*, cf. [16] and, once more it is an average of i.i.d. values f^{h*} so that the central limit theorem implies its asymptotically normal distribution. This bound does not depend on the candidate solution and to evaluate it means to solve K_u stochastic programs based on *KS* scenarios to get the values f^{h*}.

(v) An approximate $(1-2\alpha)$-confidence interval for the *optimality gap* $\varphi^* - E\widehat{f}(\mathbf{e})$ of the obtained first-stage decision $\widehat{\mathbf{x}}$, $\widehat{\mathbf{y}}$, $\widehat{\mathbf{z}}_0$, $\widehat{y}_0^+$ is

$$\big[0, \big(U(K_u) - L(K_l)\big)^+ + \varepsilon_l + \varepsilon_u\big], \tag{3.3}$$

where

$$\varepsilon_u = \frac{u(1-\alpha)s_u(K_u)}{\sqrt{K_u}}, \qquad \varepsilon_l = \frac{u(1-\alpha)s_l(K_l)}{\sqrt{K_l}}, \tag{3.4}$$

$u(1-\alpha)$ is the $100(1-\alpha)\%$ quantile of distribution $\mathcal{N}(0, 1)$, $s_u^2(K_u)$ is the sample variance of f^{h*}, $h = 1, \ldots, K_u$, and $s_l^2(K_l)$ is the sample counterpart of the variance σ_l^2.

The third approach improves the upper bound of (3.3). The simulation procedure again requires step (i) and (ii) and the following one:

(vi) Estimate the optimality gap, say $\widehat{G}$, at the first-stage solution $\widehat{\mathbf{x}}$, $\widehat{\mathbf{y}}$, $\widehat{\mathbf{z_0}}$, $\widehat{y}_0^+$ directly. The empirical counterpart of $\widehat{G} := \varphi^* - E\widehat{f}(\mathbf{e})$ based on the sample of size K is

$$\widehat{G}_K = \max \frac{1}{K}\sum_{k=1}^{K} f\left(\mathbf{e}^k\right) - \frac{1}{K}\sum_{k=1}^{K} \widehat{f}\left(\mathbf{e}^k\right), \tag{3.5}$$

where, similarly as in (iv), $f(\mathbf{e}^k)$ denotes the objective function in (1.8) which corresponds to simulated perturbations $\mathbf{e}^k$ and the optimality gap $\widehat{G} \leqslant E\widehat{G}_K$, see [16]. Using K_g i.i.d. batches of K simulated vectors $\mathbf{e}^{kh}$, $h = 1, \ldots, K_g$, $k = 1, \ldots, K$, we compute K_g values of empirical optimality gaps $\widehat{G}_K^h$, $h = 1, \ldots, K_g$. According to the central limit theorem we have asymptotically $\sqrt{K_g}[B(K_g) - E\widehat{G}_K] \sim \mathcal{N}(0, \sigma_g^2)$, for $K_g \to \infty$, where $B(K_g) = \frac{1}{K_g}\sum_{h=1}^{K_g} \widehat{G}_K^h$ and $\sigma_g^2 = \operatorname{var} \widehat{G}_K$. Hence, with $\varepsilon_g = u(1-2\alpha)s_g(K_g)/\sqrt{K}_g$ where $s_g^2(K_g)$ is the sample variance of $\widehat{G}_K^h$, $h = 1, \ldots, K_g$, we have an approximate confidence interval for $\widehat{G}$

$$\left[0, B(K_g) + \varepsilon_g\right], \tag{3.6}$$

which is again valid with probability greater or equal to $1 - 2\alpha$.

These simulation experiments, namely, the repeated evaluation of the optimal values f^{h*}, are demanding both for the computing time and memory point of view and a high performance environment can be of some help.

4. Numerical results

The numerical experiments were done for portfolio of bonds whose composition has been suggested by a local bank and it was considered a good representative of an investment strategy including some short, medium, long bonds together with puttable bonds; see table 1.

The quantities in the portfolio composition are expressed in lots of million items. The initial value of this portfolio in market prices of September 1st 1994 is $W_0 = 10465.86$ million Liras which includes also cash of 500 million Liras. No liability is

Table 1
Portfolio composition on September 1st, 1994.

Bonds	Qt	Coupon	Payment dates	Exercise	Redemp.	Maturity
BTP36658	10	3.9375	01Apr & 01Oct		100.187	01Oct96
BTP36631	20	5.0312	01Mar & 01Sep		99.531	01Mar98
BTP12687	15	5.2500	01Jan & 01Jul		99.231	01Jan02
BTP36693	10	3.7187	01Aug & 01Feb		99.387	01Aug04
BTP36665	5	3.9375	01May & 01Nov		99.218	01Nov23
CTO13212	20	5.2500	20Jan & 20Jul	20Jan95	100.000	20Jan98
CTO36608	20	5.2500	19May & 19Nov	19May95	99.950	19May98

included, the value of α is set to 1 and the model parameters δ_1 and δ_2 are fixed respectively to 0.0005 and 0.0016. Only the linear utility function was applied.

The yield curve for September 1, 1994 and the volatility of errors were estimated from the market data of the day according to the linearized Bradley–Crane model, see (2.3)–(2.5). To generate normally distributed perturbations, normally distributed random numbers have been generated according to [2]. The perturbed yields vectors and the initial volatility of errors were used to build the lattice using the Bjerksund–Stensland [6] approximation.

One characteristic which is common to all experiments is that all perturbations to the initial yield curve are maintained in a limited range. More precisely, the error vector **e** is built to satisfy the property that its components belong to a normal distribution with zero mean and standard deviation equal to $h10^{-2}\sigma[1 + Q^2(t)]^{1/2}$, where h is in the range $(0, 1]$. It has to be noted that, according to our experience, all numerical procedures used to generate the binomial lattice are quite sensitive to big and irregular fluctuations in the initial input yield curve. This observation confirms the comment of [8] and it in fact *gives an upper bound to* "*small perturbations*" of the data in the considered problem. Figure 1 shows graphical examples of perturbed yield curves that we used in our simulation.

The parameters of the binomial lattice have been computed with a monthly discretization along 5 years. This choice guarantees that the perturbations do not destroy the lattice construction. To evaluate the bonds with longest maturities, interest rates have been kept constant after the 5th year and equal to the last computed interest rate value. Bond prices along the lattice were computed by backwardation using formula (1.1) while for CTO prices we took into account the possibility of exercise at the strike date depending on interest rate scenarios.

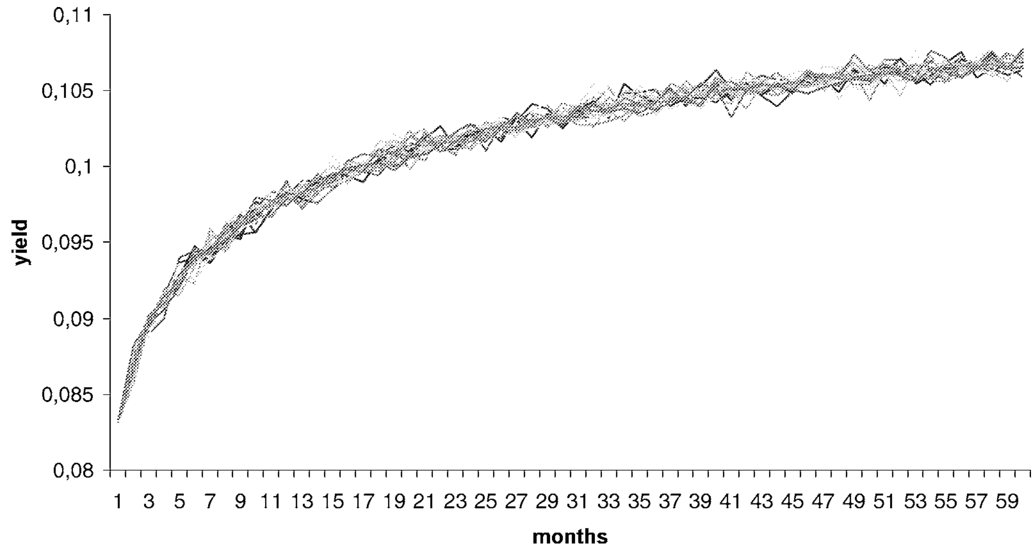


Figure 1. Perturbed yield curves (Sept. 1, 1994).

The sampling strategy was chosen between different alternatives:

– use of Zenios and Shtilman [20] nonrandom sampling strategy with different length in covering fully the beginning of the lattice, and proceeding with alternating up-down movements, see [4]; the acronyms are ZS(No. of scenarios);

– use of 8 particular scenarios along the planning horizon ($t = 1, \dots, 12$), as reported in exhibit 1 in appendix, and proceed with alternating up-down movements, cf. [4]; the acronym is Part(8).

The first alternative, tested for 8, 16, 32 and 64 scenarios corresponding to fully covering the beginning of the lattice till discretization times 3, 4, 5, 6 gives identical first-stage optimal solutions with slightly different optimal values for 8 and 32 scenarios and also for 16 and 64 scenarios; however some parts of the lattice were completely ignored in the simulation. For this reason we decided to check a second alternative where the scenarios were constructed in such a way to cover better the lattice up to the model horizon $T_0 = 12$. Again, we obtained a first-stage optimal solution different from the previous ones, but with an optimal value close to the previous ones. Table 2 shows in details the values obtained.

All the numerical tests were run on a PC IBM ThinkPad 380, based on Intel Pentium 266 Mhz, 64 Mbytes RAM, under Windows 95 operating system. Visual C++ language (v. 4.0 compiler) was used on estimating the yield curve, on fitting the lattice and on generating the perturbed yield curve. The optimization problem was solved by the MINOS solver available in the GAMS package.

The expected value of perfect information (EVPI) and the value of stochastic solution (VSS), see [5], were computed for the problems Part(8), ZS(8) and ZS(16), see table 3. As expected, the Part(8) sampling strategy seems to represent better uncertainty due to a large EVPI value associated to it in comparison to the remaining two sampling strategies.

For the first part of the simulation study (i)–(ii) suggested in the previous section, the number of simulations, i.e., the value of K has been fixed to 100 and then extended to 1500 to obtain a better statistics concerning possible clustering. During this first experiment, we chose the 8 particular scenarios as described in Part(8) alternative.

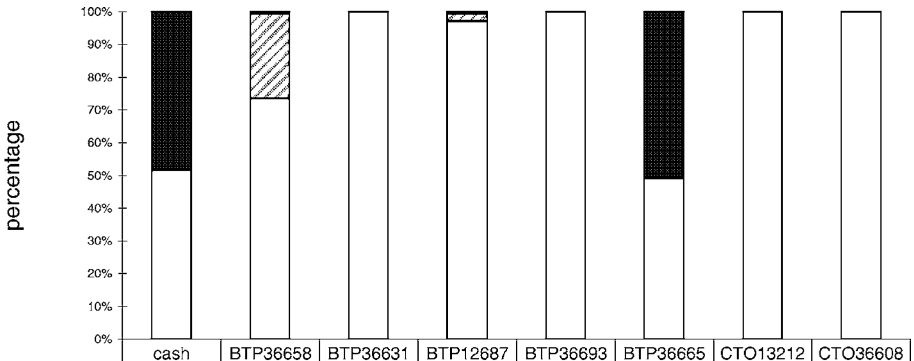
Table 2
First-stage optimal solutions.

Portfolio	Initial	Part(8)	ZS(8)	ZS(16)	ZS(32)	ZS(64)
Cash	500	0	0	7947	0	7947
BTP36658	10	0	0	10	0	10
BTP36631	20	0	0	0	0	0
BTP12687	15	15	103.4	15	103.4	15
BTP36693	10	0	0	0	0	0
BTP36665	5	114.4	0	0	0	0
CTO13212	20	0	0	0	0	0
CTO36608	20	0	0	0	0	0
Optimal value		11499	11560	11472	11559	11470

Table 3
EVPI and VSS values for different sampling strategies.

Uncertainty measures	Part(8)	ZS(8)	ZS(16)
EVPI	689.8031	73.1475	104.8809
VSS	2096.5764	1677.1441	2754.7234

The initial composition of the portfolio on September 1st, 1994 together with a survey on how the considered bonds are distributed with respect to the strategies of selling, holding and buying, is displayed in figure 2 for $K = 1500$ (results are very similar for $K = 100$); the optimal first-stage solutions together with the simulated optimal values φ^k for the first 100 simulation runs are displayed in exhibit 2. These exhibits and figures suggest that there are only a few typical optimal strategies which, however, are far from being similar. The reason is that the *short rates and the prices obtained for perturbed yield curves differ essentially*. Even when the progression of errors in the Black–Derman–Toy lattice was limited as the interest rates were kept constant after the fifth year, we found out that for the perturbed problems the rates can differ up to 80% (e.g., 78.8169% in the month 56 for the simulation run marked by * in exhibit 2) and, consequently, the prices, especially for the long bond, exhibit relatively large differences, too (e.g., for −14.3325% of the price of the long bond in the last period for the simulation run * in exhibit 2). These errors differ for different simulation runs, e.g., for that marked by ** in exhibit 2, the maximal difference of rates is 43.7424% and the maximal difference in prices is 4.4136% and they depend on the magnitude of perturbations of the yield curve which was less than 1.2472% in simulation **, and attained maximum of 1.5459% in simulation *. The low price of the long bond at the planning horizon for simulation * results in investment into cash only, whereas the increase of the price of the long bond in simulation ** is reflected by investing solely into this bond.



	cash	BTP36658	BTP36631	BTP12687	BTP36693	BTP36665	CTO13212	CTO36608
■ Buy	725	9	0	10	0	756	0	0
▨ Do Nothing	0	389	0	34	0	7	1	0
□ Sell	775	1102	1500	1456	1500	737	1499	1500

bond

Figure 2. First-stage investment strategies.

Another observation comes from exhibit 2 by comparison of the performance of expected values for the buy & hold strategy at the termination T_0. The expected values are computed again for the same 100 simulations of the particular 8 scenarios selected according to exhibit 1 which are used in the simulated stochastic programs, and their evaluation takes into account expiration of the CTOs and the cashflows due in the considered time period of September 1, 1994–August 31, 1995. Comparing the results for the expected values of the buy & hold strategies, see buy-hold column of exhibit 2, with those for optimal values for the 100 simulated stochastic programs, it is clear that the distribution of the stochastic programming values is shifted to higher function values, is nonsymmetrical, see exhibit 3, case C, and provides possibilities of rather large values.

Statistics for optimal values that correspond to particular strategies, which can be verbally described for instance as "cash only" – case A, and "long bond only" – case B, is given on exhibit 3. Clearly, it is the investment in the long bond which provides a very skewed distribution with possibilities of large optimal values. The collection of the first-stage optimal solutions of perturbed problems says that, under the fitted yield curve, *the longest bond is a dominant investment* in the most of perturbed cases (see exhibits 2 and 4).

For 100 simulations, the mean of the (empirical) optimal values φ^k is 13041 with the standard deviation of 2479, for 1500 simulations this changes to the mean value 12911 with the standard deviation of 2461. The resulting 2σ confidence interval around the true optimal value is relatively large and covers also very low values which is in agreement with the displayed skewness, and nonnormal behavior in general, of these optimal values; see exhibit 3, case C.

These preliminary results are not satisfactory concerning the length of confidence intervals for the optimal value. Moreover, due to the reported relatively large differences in coefficients of the linear program for some of coefficients and some perturbations, the theoretical stability properties of optimal solutions for the perturbed data cannot be expected for the chosen magnitude of perturbations of the yield curve. The stability properties can be demonstrated after the magnitude of the perturbations is decreased to $h10^{-5}\sigma[1 + Q^2(t)]^{1/2}$. In that case, the optimal first-stage solutions are equal to that for the unperturbed case in all simulation runs.

To get information about the robustness of the obtained candidate solutions with respect to the perturbed input data we test the quality of the optimal solutions coming from stochastic programs based on the Part(8), ZS(8) (and also for the ZS(32)) and ZS(16) (and on the ZS(64)) beds of scenarios, see table 2, and also of the buy & hold strategy. This can be approached by the method of Mak et al. [16] described in the second part of section 3.

The sample sizes, i.e., K_l, K_u and K, have been respectively fixed to 250, 25 and 10, and increased to 350, 35, 10 and 500, 50, 10.

The stochastic lower bounds (3.1) were evaluated for the first-stage optimal solutions of the unperturbed stochastic programs based of scenario beds Part(8), ZS(8), ZS(16) and ZS(32) and for the buy & hold strategy with a feasibility tolerance of E-12 in the optimization solver. Due to the relatively large size of the optimization problems which have to be computed repeatedly, the upper bound (3.2) was evaluated only with precision of E-5 and only for the two stochastic programs ZS(8) and Part(8) based on 8 scenarios. Notice that in this case, evaluation of the upper bound requires a repeated solution of stochastic programs based on 80 scenarios to get f^{h*}. The lower and upper bounds $L(K_l)$ and $U(K_u)$ computed according to (3.1), (3.2), the tolerances ε_l, ε_u, see (3.4), and the optimality gaps obtained as the upper limit of the interval (3.3) are reported in exhibit 4 for the ZS(8) problem; results for Part(8) problem are similar.

The lowest upper limit for the optimality gap was obtained for the Part(8) based optimal first-stage strategy and the value of the optimality gap decreases essentially with the sample size, the worst upper limit for the optimality gap, as expected, is obtained for the buy & hold investment strategy. We want to stress once more the meaning of these results: they aim at analysis of the quality of a *candidate* first-stage solution for a scenario-based stochastic program (1.8) with coefficients which are influenced by perturbations of the input data, in our case, by small random movements of the yield curve. The recommendation is to use the optimal first stage solution which is based on the particular choice of the 8 scenarios as the obtained bounds indicate its most robust behavior.

The optimality gaps (3.6), evaluated for the first-stage optimal solutions of the unperturbed stochastic programs based of scenario beds Part(8), ZS(8), ZS(16) and ZS(32) and for the buy & hold strategy, are shown in exhibit 5 for the ZS(8); again, the results for Part(8) sampling strategy show no substantial differences. What is remarkable is that the optimality gaps based on (3.6) are much better than those based on (3.3); compare with exhibit 4.

5. Conclusions

We have proposed a framework which allows to quantify the behavior of our model with respect to random movements in the initial term structure. It turns out that errors due to small perturbations of the yield curve propagate into large errors in the short rate interest rates and, consequently, in relatively large difference in the coefficients of the linear program so that stability of optimal solutions for perturbed problems cannot be in general expected. For the considered problem, we found out that the required magnitude of perturbations for which stability of the optimal solution is achieved is very small.

Therefore we applied the simulation technique based on [16] to test the robustness of the obtained candidate first-stage solutions. The technique is of a general applicability; it only requires i.i.d. replicas of the problem. It can be thus, without any problem, applied to analysis of the influence of a random sampling strategy for selection of limited number of scenarios from the whole population.

The reported simulation studies are computationally expensive and require repeated replicas of the same kind of computations on different input data. This is exactly the case when a parallel environment can help in speed-up the overall computing time.

Acknowledgements

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Appendix. Exhibits

	Investment time horizon													
Scenario	1	2	3	4	5	6	7	8	9	10	11	12	13	continue
0	down	down	down	down	down	down	down	down	down	down	down	down	down	up&down
1	down	down	down	up	down	down	down	down	down	up	down	down	down	up&down
2	down	up	down	down	down	up	down	up	down	down	down	up	down	up&down
3	up	down	up	down	up	down	up	down	up	down	up	down	up	up&down
4	up	down	up	down	up	up	down	up	up	down	up	down	up	up&down
5	up	down	up	up	up	down	up	down	up	up	up	down	up	up&down
6	up	up	up	down	up	up	up	up	up	down	up	up	up	up&down
7	up	up	up	up	up	up	up	up	up	up	up	up	up	up&down

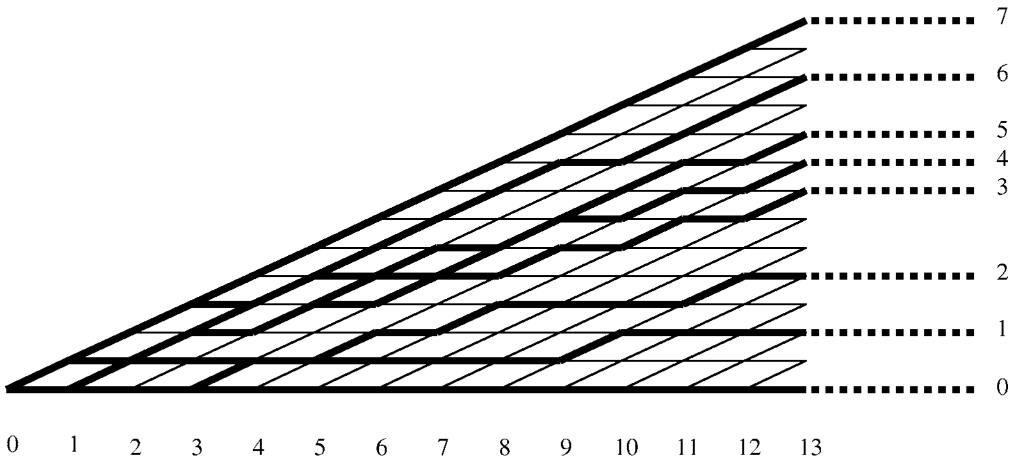

Exhibit 1. Part(8) sampling strategy.

	Buy-hold value	Optimal value	Cash	BTP36658	BTP36631	BTP12687	BTP36693	BTP36665	CTO13212	CTO36608
Unperturbed	11.344	11.499	0	0	0	15	0	114,44	0	0
*	11.171	11.478	10455,86	0	0	0	0	0	0	0
	11.257	11.476	10455,86	0	0	0	0	0	0	0
	11.246	11.502	10455,86	0	0	0	0	0	0	0
**	11.383	11.910	0	0	0	0	0	133,82	0	0
	11.849	17.328	0	0	0	0	0	133,82	0	0
	11.674	15.083	0	0	0	0	0	133,82	0	0
	11.906	18.293	0	0	0	0	0	133,82	0	0
	11.279	11.462	10455,86	0	0	0	0	0	0	0
	11.196	11.479	10455,86	0	0	0	0	0	0	0
	11.150	11.481	10455,86	0	0	0	0	0	0	0
	11.503	13.141	0	0	0	0	0	133,82	0	0
	11.464	12.600	0	0	0	0	0	133,82	0	0
	11.708	15.487	0	0	0	0	0	133,82	0	0
	11.916	18.333	0	0	0	0	0	133,82	0	0
	11.124	11.481	9461,44	10	0	0	0	0	0	0
	11.155	11.480	9461,44	10	0	0	0	0	0	0
	11.029	11.473	0	104,95	0	0	0	0	0	0
	10.950	11.471	9461,44	10	0	0	0	0	0	0
	11.110	11.480	9461,44	10	0	0	0	0	0	0
	11.867	17.550	0	0	0	0	0	133,82	0	0
	11.380	11.780	0	0	0	0	0	133,82	0	0
	11.321	11.476	9461,44	10	0	0	0	0	0	0
	11.250	11.478	10455,86	0	0	0	0	0	0	0
	11.654	14.729	0	0	0	0	0	133,82	0	0
	11.630	14.395	0	0	0	0	0	133,82	0	0
	11.561	13.637	0	0	0	0	0	133,82	0	0
	11.063	11.485	9461,44	10	0	0	0	0	0	0
	11.118	11.469	9461,44	10	0	0	0	0	0	0
	11.434	12.352	0	0	0	0	0	133,82	0	0
	11.186	11.479	10455,86	0	0	0	0	0	0	0
	11.339	11.485	0	0	0	99,54	0	5	0	0
	11.257	11.476	9461,44	10	0	0	0	0	0	0
	10.913	11.480	9461,44	10	0	0	0	0	0	0
	11.384	11.946	0	0	0	0	0	133,82	0	0
	11.458	12.591	0	0	0	0	0	133,82	0	0
	11.508	13.201	0	0	0	0	0	133,82	0	0
	10.928	11.473	9461,44	10	0	0	0	0	0	0
	10.939	11.482	9461,44	10	0	0	0	0	0	0
	11.242	11.464	10455,86	0	0	0	0	0	0	0
	11.431	12.356	0	0	0	0	0	133,82	0	0
	11.403	12.025	0	0	0	0	0	133,82	0	0
	11.291	11.491	9461,44	10	0	0	0	0	0	0
	11.486	12.843	0	0	0	0	0	133,82	0	0
	12.184	22.379	0	0	0	0	0	133,82	0	0
	10.952	11.467	10455,86	0	0	0	0	0	0	0
	11.373	11.681	0	0	0	0	0	133,82	0	0
	11.169	11.474	10455,86	0	0	0	0	0	0	0
	11.805	16.726	0	0	0	0	0	133,82	0	0
	11.523	13.311	0	0	0	0	0	133,82	0	0
	11.656	14.854	0	0	0	0	0	133,82	0	0

Exhibit 2. First stage optimal solutions versus buy and hold outcome for perturbed problems.

Buy-hold value	Optimal value	Cash	BTP36658	BTP36631	BTP12687	BTP36693	BTP36665	CTO13212	CTO36608
11.932	18374	0	0	0	0	0	133,82	0	0
11145	11468	10455,86	0	0	0	0	0	0	0
11154	11480	9461,44	10	0	0	0	0	0	0
10996	11475	9461,44	10	0	0	0	0	0	0
11415	12217	0	0	0	0	0	133,82	0	0
11379	11865	0	0	0	0	0	133,82	0	0
11034	11473	9461,44	10	0	0	0	0	0	0
11224	11497	10455,86	0	0	0	0	0	0	0
11303	11490	10455,86	0	0	0	0	0	0	0
11082	11472	10455,86	0	0	0	0	0	0	0
11041	11473	9461,44	10	0	0	0	0	0	0
11547	13600	0	0	0	0	0	133,82	0	0
11715	15515	0	0	0	0	0	133,82	0	0
11527	13304	0	0	0	0	0	133,82	0	0
11553	13750	0	0	0	0	0	133,82	0	0
12217	22850	0	0	0	0	0	133,82	0	0
11610	14409	0	0	0	0	0	133,82	0	0
11452	12560	0	0	0	0	0	133,82	0	0
11218	11469	10455,86	0	0	0	0	0	0	0
11732	15754	0	0	0	0	0	133,82	0	0
11692	15262	0	0	0	0	0	133,82	0	0
11343	11487	8941,48	0	0	15	0	0	0	0
11665	14892	0	0	0	0	0	133,82	0	0
11319	11474	10455,86	0	0	0	0	0	0	0
11366	11687	0	0	0	0	0	133,82	0	0
11055	11470	9461,44	10	0	0	0	0	0	0
12084	20857	0	0	0	0	0	133,82	0	0
11209	11473	9461,44	10	0	0	0	0	0	0
11544	13411	0	0	0	0	0	133,82	0	0
11570	13703	0	0	0	0	0	133,82	0	0
11403	12092	0	0	0	0	0	133,82	0	0
11549	13547	0	0	0	0	0	133,82	0	0
11547	13494	0	0	0	0	0	133,82	0	0
11053	11474	9461,44	10	0	0	0	0	0	0
11704	15317	0	0	0	0	0	133,82	0	0
11022	11469	9461,44	10	0	0	0	0	0	0
11095	11468	9461,44	10	0	0	0	0	0	0
11350	11490	7947,06	10	0	15	0	0	0	0
11448	12467	0	0	0	0	0	133,82	0	0
11200	11487	10455,86	0	0	0	0	0	0	0
11237	11494	10455,86	0	0	0	0	0	0	0
11538	13471	0	0	0	0	0	133,82	0	0
11346	11498	0	10	0	89,71	0	5	0	0
11147	11475	9461,44	10	0	0	0	0	0	0
11878	17712	0	0	0	0	0	133,82	0	0
11867	17561	0	0	0	0	0	133,82	0	0
10971	11476	9461,44	10	0	0	0	0	0	0
11041	11471	9461,44	10	0	0	0	0	0	0
11306	11474	9461,44	10	0	0	0	0	0	0
11055	11474	10455,86	0	0	0	0	0	0	0
Initial composition:		**500**	**10**	**20**	**15**	**10**	**5**	**20**	**20**

Exhibit 2. Continued.

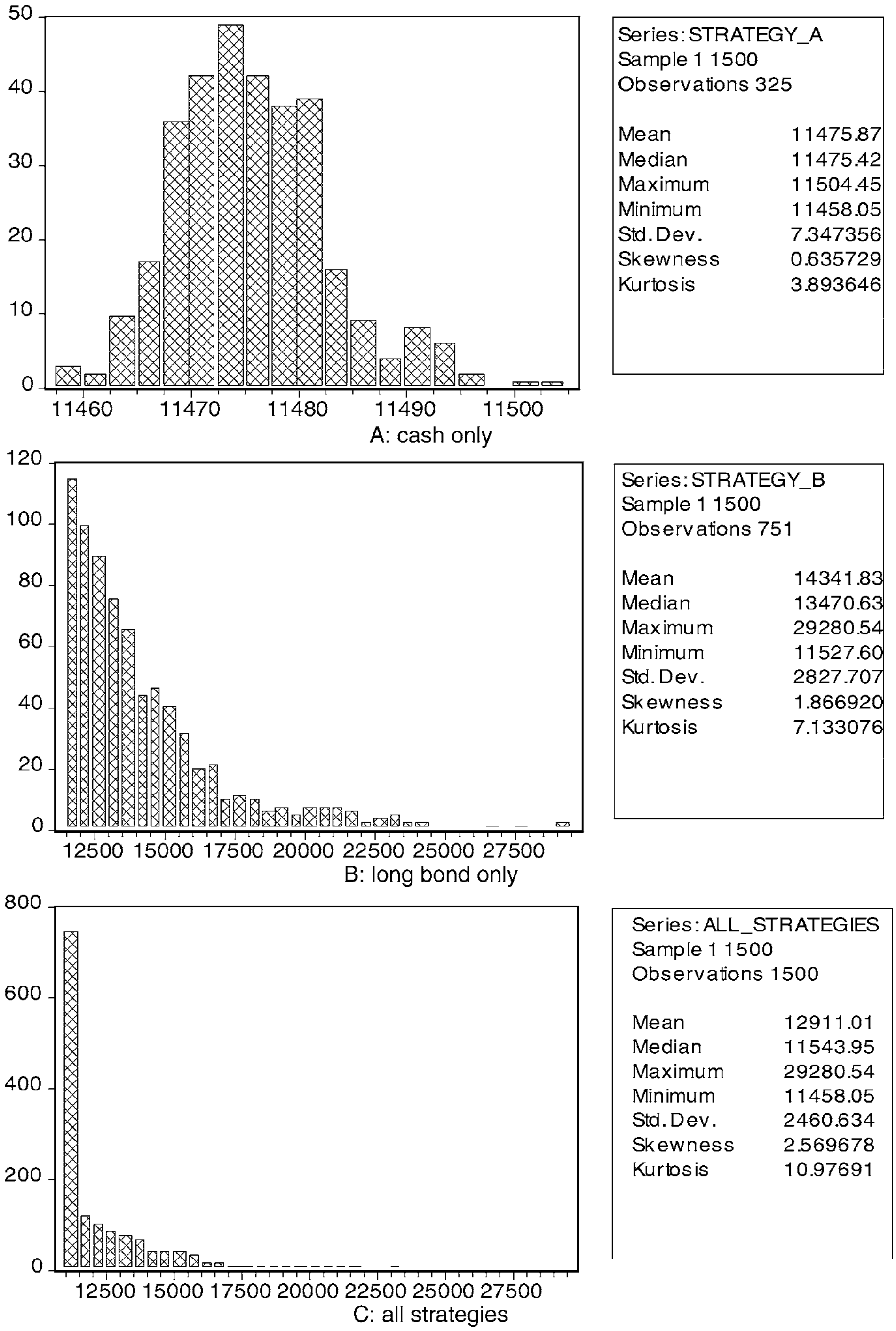

Exhibit 3. Statistics for optimal values.

Bounds

	Lower bounds for optimal strategies of problems					Upper bound
Initial strategy	Part(8)	ZS(8)	ZS(16)	ZS(32)	Buy-hold	
α	1%	1%	1%	1%	1%	1%
$u(1-\alpha)$	2,326342	2,326342	2,326342	2,326342	2,326342	2,326342
Sample size	250	250	250	250	250	25
Mean	12137,20	11580,38	11481,84	11580,38	11412,32	12342,58
Std. dev.	2741,76	562,89	83,19	562,89	286,51	1152,37
Error term	403,40	82,82	12,24	82,82	42,15	536,16
Optimality gap	1144,94	1381,19	1409,14	1381,19	1508,58	–
Sample size	350	350	350	350	350	35
Mean	12145,16	11586,09	11482,55	11586,09	11414,38	12342,80
Std. dev.	2655,86	558,72	82,55	558,72	281,52	1052,83
Error term	330,25	69,48	10,27	69,48	35,01	414,00
Optimality gap	941,89	1240,18	1284,51	1240,18	1377,42	–
Sample size	500	500	500	500	500	50
Mean	12165,72	11590,07	11483,39	11590,07	11420,87	12361,18
Std. dev.	2636,76	559,49	82,95	559,49	294,80	1071,14
Error term	274,32	58,21	8,63	58,21	30,67	352,40
Optimality gap	822,19	1181,72	1238,82	1181,72	1323,39	–

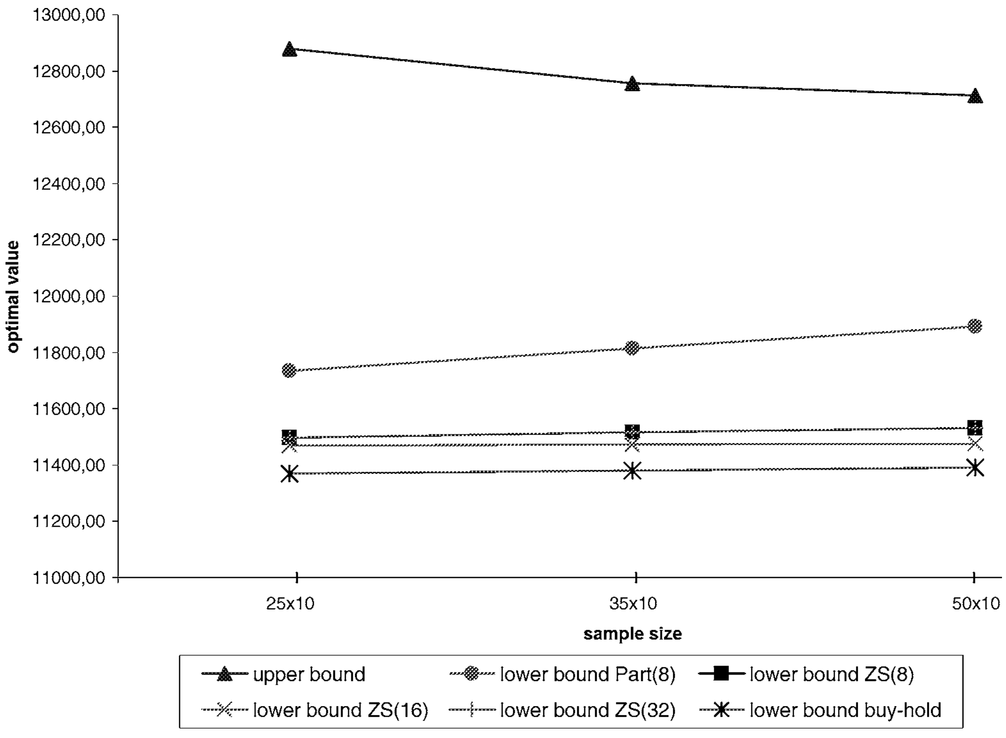

Exhibit 4. Lower and upper bounds under ZS(8) sampling strategy.

Bounds

	Gap for optimal strategies of problems				
Initial strategy	Part(8)	ZS(8)	ZS(16)	ZS(32)	Buy-hold
α	1%	1%	1%	1%	1%
$u(1-2\alpha)$	2,053748	2,053748	2,053748	2,053748	2,053748
Sample size	25	25	25	25	25
Mean	205,38047	762,21	860,74	762,21	930,26
Std. dev.	232,06555	952,57	1122,42	952,57	1043,19
Error term	95,32	391,27	461,03	391,27	428,49
Optimality gap	300,70	1153,48	1321,77	1153,48	1358,76
Sample size	35	35	35	35	35
Mean	197,64	756,71	860,25	756,71	928,41
Std. dev.	213,24	864,24	1024,81	864,24	950,99
Error term	74,03	300,02	355,76	300,02	330,13
Optimality gap	271,66	1056,73	1216,01	1056,73	1258,55
Sample size	50	50	50	50	50
Mean	195,47	771,12	877,80	771,12	940,32
Std. dev.	196,93	874,56	1041,80	874,56	969,79
Error term	57,20	254,01	302,58	254,01	281,67
Optimality gap	252,67	1025,13	1180,38	1025,13	1221,99

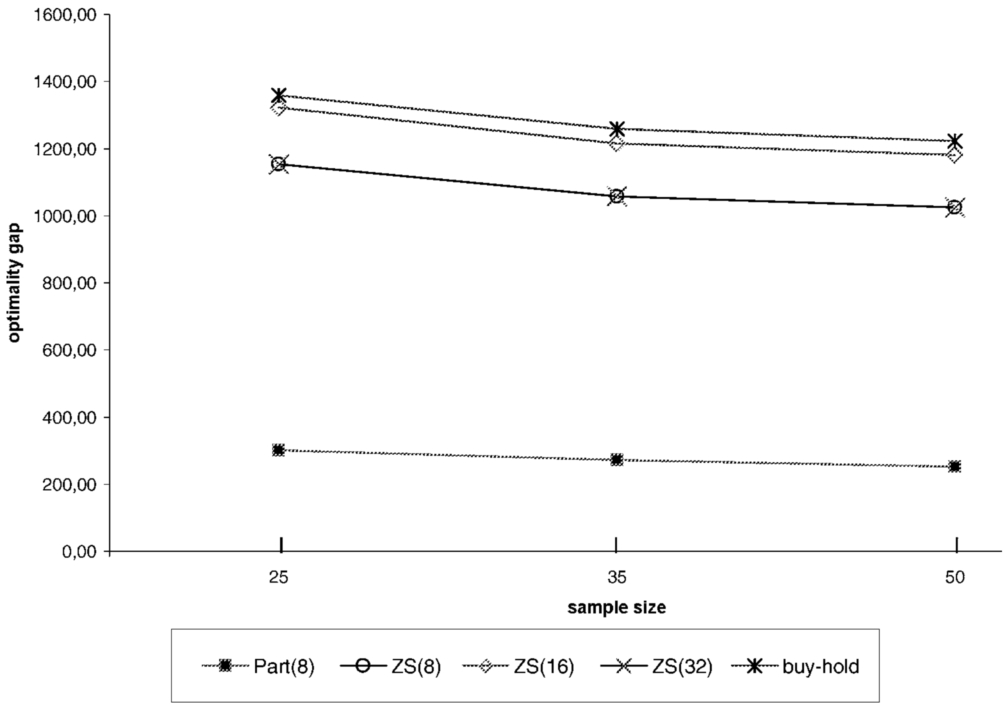

Exhibit 5. Optimality gap under ZS(8) sampling strategy.

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