Convex Optimization 2025/26

Practical session # 6

November 6, 2025

1. (Portfolio optimization): We would like to optimize an investment portfolio. Let $x = (x_1, \dots, x_n)$, such that $x_i \geq 0$ is our investment into some asset $i \in \{1, 2, 3, \dots, n\}$. Moreover, let us assume that we have a total budget of $\sum_{i=1}^n x_i = 1$.

   The price change of our assets is modeled by a random variable $p \in \mathbf{R}^n$, with expected value $\bar{p} \in \mathbf{R}^n$ and covariance matrix $\Sigma \in S_+^n$. Therefore, our portfolio has the expected return $\bar{p}^T x$, and the variance $x^T \Sigma x$.

   (a) Describe a problem that maximizes the expected return $\bar{p}^T x$.

   (b) Describe a problem that minimizes the 'risk' of our investment (i.e the variance $x^T \Sigma x$).

   (c) Are the problems (a) and (b) convex/LPs/QPs/...? Solve them for $n = 2$ and

   $$\bar{p} = (0.1, 0.5),\ \Sigma = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

   (d) general discussion: Try to come up with (convex) optimization problems that consider *both* the expected return and risk at the same time.

2. Show that the norm approximation problem

   $$\text{minimize } \|Ax - b\|_4 = (\sum_{i=1}^m (a_i^T x - b_i)^4)^{1/4}$$

   is equivalent to a QCQP.
   Hint: try to argue similarly as in the proof that $\|\cdot\|_1$-approximation is equivalent to and LP.

3. Robust LPs:
   Let us consider an LP, but for a given inequality constraints $a^T x \leq b$ there is some level of uncertainty in how to pick the parameter a. We just know that a must lie in a given confidence ellipsoid $E = \{\bar{a} + Pu \mid \|u\|_2 \leq 1\}$, for $\bar{a} \in \mathbf{R}^n$, $P \in \mathbf{R}^{n \times n}$.

   The corresponding *robust LP* requires that the constraint $a^T x \leq b$ holds for *all* $a \in E$. Show that this problem is equivalent to a second order cone problem (SOCP).

4. SDPs

   (a) Show that every LP is equivalent to an SDP.

   (b) Show that every SOCP is equivalent to an SDP.
   Hint: Show first that $\|x\|_2 \leq t$ if and only if

   $$\begin{bmatrix} tI & x \\ x^T & t \end{bmatrix} \succeq 0$$