Convex Optimization 2025/26

Practical session # 5

October 30, 2025

Recall the definition of the norms on $\mathbf{R}^n$:

- $\|x\|_p = (|x_1|^p + |x_2|^p + \ldots + |x_n|^p)^{1/p}$, for $p \geq 1$
- $\|x\|_\infty = \max(|x_1|, |x_2|, \ldots, |x_n|)$

1. Discuss the unit balls $B_p = \{x \in \mathbf{R}^n \mid \|x\|_p \leq 1\}$, for both $p \in \{1, \infty\}$. Show that, in both cases, they are convex polytopes. Determine their vertices, and give a description by linear inequalities.

2. Let $A \in \mathbf{R}^{m \times n}, b \in \mathbf{R}^m$. The norm approximation problem for a system of linear equation $Ax \approx b$ is the problem of minimizing $\|Ax - b\|$, for a given norm $\|\cdot\|$. Show that

   (a) minimize $\|Ax - b\|_\infty$

   (b) minimize $\|Ax - b\|_1$

   are both equivalent to linear programs (that are of polynomial size in n and m).

3. Describe explicit solutions of the linear program of minimizing $c^T x$ for some $c \in \mathbf{R}^n$, subject to

   (a) box constraints: $l \preceq x \preceq r$, for some $l \preceq r$ in $\mathbf{R}^n$.

   (b) probability constraints: $\sum_{i=1}^n x_i = 1$ and $x \succeq 0$.

   (c) equational constraints: $Ax = b$.
   (Hint: If feasible, distinguish between the case where c is in the image of A^T, or not)

4. Show that the norm approximation problem

   $$\text{minimize } \|Ax - b\|_4$$

   is equivalent to a QCQP.