Convex Optimization

Practical session # 2

October 9, 2025

1. Show that the halfspace $\{x \in \mathbf{R}^n \mid a^T x \leq b\}$ for given $a \in \mathbf{R}^n, b \in \mathbf{R}$ is indeed a convex set.

2. Are the following functions convex, concave or neither?

   (a) $f(x_1, x_2) = x_1 x_2$

   (b) $f(x_1, x_2) = \log(x_1 + x_2)$, $\mathrm{dom}(f) = \mathbf{R}^2_{++}$

   (c) $f(x_1, x_2) = \sqrt{x_1 x_2}$, $\mathrm{dom}(f) = \mathbf{R}^2_{++}$

3. Consider the optimization problem:

$$\begin{aligned} \text{minimize } & f(x, y) = \max(y - x, 3x - y) \\ \text{subject to } & x, y \leq 3 \\ & x, y \geq 0 \end{aligned}$$

   (a) Is f convex?

   (b) Can you describe an equivalent linear problem?

   (c) Generalize (a) and (b) to objective functions $f(x) = \max(c_1^T x, c_2^T x, \ldots, c_n^T x)$

4. The perspective function is defined as $p(x, t) = \frac{x}{t}$, for $x \in \mathbf{R}^n, t > 0$. Show that, if $C \subseteq \mathbf{R}^n \times \mathbf{R}_{++}$ is convex, then $p(C)$ is convex.

5. Let $x_0, x_1, \ldots, x_k \in \mathbf{R}^n$ be a set of distinct points. The *Voronoi region* around x_0 with respect to $x_1, x_2, \ldots, x_k$ is the set of point that are closer to x_0 than other x_i:

$$P_0 = \{x \in \mathbf{R}^n \mid \|x - x_0\|_2 \leq \|x - x_i\| \text{ for } i = 1, 2, \ldots, k\}.$$

   (a) Show that P_0 is a (convex) polyhedron - express it as $P_0 = \{x \in \mathbf{R}^n \mid Ax \preceq b\}$.

   (b) Conversely, given a polyhedron, describe it as a Voronoi region of some points.

   (*) The Voronoi regions $P_0, P_1, \ldots, P_k$ for points $\{x_0, x_1, \ldots, x_k\}$ form a decomposition of $\mathbf{R}^n$ into finitely many polyhedra. Conversely, is every decomposition of $\mathbf{R}^n$ into polyhedra a Voronoi diagram?